6.02 Fall 2014
Lecture #16

- Modulation/Demodulation
- Frequency Division Multiplexing
Complementary/dual behavior of a signal in the time and frequency domains

- Wider in time, narrower in frequency; and vice versa.
  - This is actually the basis of the uncertainty principle in physics!

- Smoother in time, sharper in frequency; and vice versa

- Rectangular pulse in time is a (periodic) sinc in frequency, while rectangular pulse in frequency is a sinc in time; etc.
A shaped pulse $x'[n]$ versus a rectangular pulse $x[n]$

Slightly round the transitions from 0 to 1, and from 1 to 0, by making them sinusoidal, just 30 samples on each end.
In the spectral domain:

\[ |\text{DTFT}| \text{ of rectangular pulse } x[n] \]

\[ -|\text{DTFT}| \text{ of shaped pulse } x'[n] \]

Frequency content of shaped pulse only extends to here, around 1500 Hz.

\[ \Omega = \frac{f_s}{2\pi} \]
Modulation/Demodulation

• You have: a signal $x[n]$ at baseband (i.e., spectrum centered around 0 frequency, and zero for $|\Omega| \geq \Omega_m$)
• You want: the same signal spectrum, but centered around some specific frequency pair $\pm \Omega_c$
• Modulation: convert from baseband out to $\pm \Omega_c$, to get $t[n]$
• Demodulation: convert from $\pm \Omega_c$ down to baseband
Modulation by **Heterodyning** or Amplitude Modulation (AM)

\[ x[n] \times \cos(\Omega_c n) \rightarrow t[n] \]

i.e., just replicate baseband signal at ±\(\Omega_c\), and scale by \(\frac{1}{2}\).

To get this nice picture, the baseband signal needs to be **band-limited** to some range of frequencies \([-\Omega_m, \Omega_m]\), where \(\Omega_m < \Omega_c < \pi - \Omega_m\).

** Reginald Fessenden’s invention:
http://www.ewh.ieee.org/reg/7/millennium/radio/radio_unsung.html
At the Receiver: Demodulation

• In principle, this is (as easy as) modulation again:
  
  If the received signal is \( r[n] \),
  
  \[ r[n] = x[n] \cos(\Omega_c n) = t[n] \text{ when there is no distortion or noise}, \]
  
  then simply compute
  
  \[ d[n] = r[n] \cos(\Omega_c n) \]

  i.e., modulation of \( r[n] \) onto the same carrier, so in the case of no distortion/noise:
  
  \[ d[n] = x[n] \cos^2(\Omega_c n) \]
  
  \[ = 0.5 \{ x[n] + x[n] \cos(2\Omega_c n) \} \]
  
  \[ = 0.5 x[n] + 0.5x[n] \cos(2\Omega_c n) \]

• What does the spectrum of \( d[n] \), i.e., \( D(\Omega) \), look like?

• More generally, \( r[n] \neq t[n] \), because of distortion and noise
Demodulation Frequency Diagram

\[ R(\Omega) = T(\Omega) \]

What we want

Note combining of signals around 0 results in doubling of amplitude

[Diagram with labels and annotations]
Example: Demodulation (time)

Showing idealized signals ---
no bandwidth limit on channel

Baseband signal $x[n]$

$t[n] = x[n]\cos(\Omega_c n)$

$z[n] = t[n]\cos(\Omega_c n) \rightarrow \text{NOT absolute value } |t[n]|$

Note: lowpass filtering of this signal will yield $x[n]/2$!
Example: Modulation (time)

Shaped pulses! Chosen because we know the channel is bandlimited.

Baseband input $x[n]$

Carrier signal

Transmitted signal $t[n]$
Demodulation

\[ r[n] \rightarrow \times \rightarrow d[n] \rightarrow \text{LPF} \rightarrow y[n] \]

\[ \cos(\Omega_c n) \]

Cutoff @ ±Ω_c
Gain = 2

before LPF

after LPF
Ideal Modulation/Demodulation

**transmitter**

\[ x[n] \rightarrow t[n] \rightarrow \times \rightarrow \cos(\Omega_c n) \]

**receiver**

\[ z[n] \rightarrow \times \rightarrow \cos(\Omega_c n) \rightarrow \text{LPF} \rightarrow y[n] \]

Cutoff @ \( \pm k_{in} \)  
Gain = 2

**Graphs:**
- band-limited \( in[n] \)
- \( y[n] \) where \( \Omega = 35(2\pi/N) \)
- \( z[n] \)
- \( out[n] \)
Phase Error In Demodulation

When the receiver oscillator is out of phase with the transmitter:

\[ d[n] = r[n] \cdot \cos(\Omega_c n - \varphi) = x[n] \cdot \cos(\Omega_c n) \cdot \cos(\Omega_c n - \varphi) \]

But

\[ \cos(\Omega_c n) \cdot \cos(\Omega_c n - \varphi) = 0.5\{\cos(\varphi) + \cos(2\Omega_c n - \varphi)\} \]

It follows that the demodulated output, after the LPF of gain 2, is

\[ y[n] = x[n] \cdot \cos(\varphi) \]

So a phase error of \( \varphi \) results in amplitude scaling by \( \cos(\varphi) \).

Note: in the extreme case where \( \varphi = \pi/2 \), we are demodulating by a sine rather than a cosine, and we get \( y[n] = 0 \).
Demodulation with $\sin(\Omega_c n)$
… produces

Note combining of signals around 0 results in cancellation!
Phase Error in Demodulator

transmitter

\[ x[n] \rightarrow t[n] \rightarrow \cos(\Omega_c n) \]

receiver

\[ z[n] \rightarrow \cos(\Omega_c n - \varphi) \rightarrow \text{LPF} \rightarrow y[n] \]

Cutoff @ \( \pm k_{in} \), Gain = 2

Band-limited \( i[n] \)

\[ \text{out}[n] \text{ with } \phi = \pi/4 \]

\[ \text{out}[n] \text{ with } \phi = \pi/2 \]

\[ \text{out}[n] \text{ with } \phi = 3\pi/4 \]
Channel Delay

\[ x[n] \times \cos(\Omega_c n) \rightarrow t[n] \times d[n] \rightarrow y[n] \]

Time delay of D samples

Cutoff @ ±k\_in
Gain = 2

Very similar math to the previous “phase error” case:

\[
d[n] = t[n - D] \cdot \cos(\Omega_c n)
\]

\[
= x[n - D] \cdot \cos[\Omega_c (n - D)] \cdot \cos(\Omega_c n)
\]

Passing this through the LPF:

\[
y[n] = x[n - D] \cdot \cos(\Omega_c D)
\]

Looks like a phase error of \(\Omega_c D\)

If \(\Omega_c D\) is an odd multiple of \(\pi / 2\), then \(y[n] = 0\) !!
(e.g., with \(D=10\) at 1200 Hz in Audiocom)
Fixing Phase Problems in the Receiver

So phase errors and channel delay both result in a scaling of the output amplitude, where the magnitude of the scaling can’t necessarily be determined at system design time:

- channel delay varies on mobile devices
- phase difference between transmitter and receiver is arbitrary

One solution: *quadrature demodulation*

\[
\begin{align*}
I[n] &= x[n-D] \cdot \cos(\theta) \\
Q[n] &= x[n-D] \cdot \sin(\theta)
\end{align*}
\]

\[
\theta = \varphi - \Omega_c D
\]
Quadrature Demodulation

If we let

\[ w[n] = I[n] + jQ[n] \]

then

\[ |w[n]| = \sqrt{I[n]^2 + Q[n]^2} = |x[n-D]| \sqrt{\cos^2 \theta + \sin^2 \theta} = |x[n-D]| \]

OK for recovering \( x[n] \) if it never goes negative, as in on-off keying

Constellation diagrams:

\( x[n] = \{ 0, 1 \} \)
QPSK Modulation

We can use the quadrature scheme at the transmitter too, as in Quadrature Phase Shift Keying:

\[ I[n] \times \cos(\Omega_c n) + Q[n] \times \sin(\Omega_c n) \]

Samples from first bit stream

Samples from second bit stream
The wireless LAN standard, IEEE 802.11b-1999,\textsuperscript{[1][2]} uses a variety of different PSKs depending on the data-rate required. At the basic-rate of 1 Mbit/s, it uses DBPSK (differential BPSK). To provide the extended-rate of 2 Mbit/s, DQPSK is used. In reaching 5.5 Mbit/s and the full-rate of 11 Mbit/s, QPSK is employed, but has to be coupled with complementary code keying. The higher-speed wireless LAN standard, IEEE 802.11g-2003\textsuperscript{[1][3]} has eight data rates: 6, 9, 12, 18, 24, 36, 48 and 54 Mbit/s. The 6 and 9 Mbit/s modes use OFDM modulation where each sub-carrier is BPSK modulated. The 12 and 18 Mbit/s modes use OFDM with QPSK. The fastest four modes use OFDM with forms of quadrature amplitude modulation. Because of its simplicity BPSK is appropriate for low-cost passive transmitters, and is used in RFID standards such as ISO/IEC 14443 which has been adopted for biometric passports, credit cards such as American Express’s ExpressPay, and many other applications.\textsuperscript{[4]}

Bluetooth 2 will use (p/4)-DQPSK at its lower rate (2 Mbit/s) and 8-DPSK at its higher rate (3 Mbit/s) when the link between the two devices is sufficiently robust. Bluetooth 1 modulates with Gaussian minimum-shift keying, a binary scheme, so either modulation choice in version 2 will yield a higher data-rate. A similar technology, IEEE 802.15.4 (the wireless standard used by ZigBee) also relies on PSK. IEEE 802.15.4 allows the use of two frequency bands: 868–915 MHz using BPSK and at 2.4 GHz using OQPSK.
Multiple Transmitters: 

Frequency Division Multiplexing (FDM)

Choose bandwidths and $\Omega_c$'s so as to avoid overlap! Once signals combine at a given frequency, can’t be undone... LPF cutoff needs to be half the minimum separation between carriers.

Channel “performs addition” by superposing signals (“voltages”) from different frequency bands.
Audicom results, channel “G”
Audicom results, channel “R”
Audicom results, channel “B”
Edwin Howard Armstrong**
(1890-1954)

Invented and patented first positive feedback ("regenerative") electronic oscillator and tuned amplifier while undergrad at Columbia (1914); superheterodyne reception for FDM (1918, see block diagram below, also Pset 6, Problem 2); frequency modulation (FM, 1930’s); ... (42 patents)

**http://en.wikipedia.org/wiki/Edwin_Armstrong
http://en.wikipedia.org/wiki/Superheterodyne_receiver
Superhet Reception: AM and FM Radio

AM radio receivers span 510 – 1655 kHz in the US, with carrier frequencies of different stations spaced 10 kHz apart. The usual IF frequency is 455 kHz.

FM radio receivers span 88 – 108 MHz, with carrier frequencies spaced 200 kHz apart. The usual IF frequency is 10.7 MHz.
What about Channel Distortion?

What if \( R(\Omega) \neq T(\Omega) \) ?

Then \( D(\Omega) \) is no longer:

\[
\begin{align*}
\text{Re}(T) & \quad -\Omega_d \quad +\Omega_c \\
\text{Im}(T) & \quad -2\Omega_d \\
\text{Re}(D) & \quad -2\Omega_d \quad -\Omega_m \quad \Omega_m \quad +2\Omega_c \\
\text{Im}(D) & \quad -\Omega_m \quad \Omega_m
\end{align*}
\]
Effects of Channel Distortion

- The channel frequency response in the range

\[ \pm [\Omega_c - \Omega_m, \Omega_c + \Omega_m] \]

determines what signal is recovered at baseband after demodulation and filtering. Tracking through all the details, we find:

\[ Y(\Omega) = 0.5[H(\Omega + \Omega_c) + H(\Omega - \Omega_c)] X(\Omega) \]

so the effective baseband frequency response is

\[ H_b(\Omega) = 0.5[H(\Omega + \Omega_c) + H(\Omega - \Omega_c)] \]

This is what we were actually dealing with in our exploration of LTI models at baseband!

(Noise around the carrier frequency is also reflected into baseband.)