6.02 Fall 2009
Lecture #13

- Untangling Frequency Division Multiplexing (FDM)
- Basic Tools: Cplx Exp, Freq Resp.
- Filters by solving equations
- Using poles for filtering (if there’s time).
Basic Tools: Frequency Response

- Complex Exponentials
  \[ \cos \Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n} \quad \sin \Omega n = \frac{1}{2j} e^{j\Omega n} - \frac{1}{2j} e^{-j\Omega n} \]

- Frequency Response
  \[ y[n] = e^{j\Omega n} \quad w[n] = \sum_{m=0}^{L} h[m] y[n - m] \]
  \[ w[n] = H\left(e^{j\Omega}\right) e^{j\Omega n} \quad H\left(e^{j\Omega}\right) = \sum_{m=0}^{L} h[m] e^{-j\Omega m} \]

- \( H(e^{j\Omega}), -\pi < \Omega < \pi, \) is the frequency response
\[
W_{CN} = \frac{1}{2} \left( H_{\text{res}} \right) \text{ or } \frac{1}{2} H_{\text{res}} + \frac{1}{2} H_{\text{res}}
\]

\[
Y_{CN} = \cos \phi = \frac{1}{2} e^{\text{num}} + \frac{1}{2} e^{-\text{num}}
\]

Superposition

\[
H(\text{freq}) = \frac{1}{2} \text{Hcm} e^{-j\omega}
\]

\[
W_{CN} = H(\text{freq}) \text{ or } \frac{1}{2} \text{Hcm} e^{-j\omega}
\]

\[
Y_{CN} = \frac{1}{2} e^{\text{num}} + \frac{1}{2} e^{-\text{num}}
\]

\[
\text{Freqency Response}
\]

\[
\sin \phi = \frac{1}{2} e^{\text{num}} + \frac{1}{2} e^{-\text{num}}
\]

\[
\cos \phi = \frac{1}{2} e^{\text{num}} + \frac{1}{2} e^{-\text{num}}
\]

Complex Exponential

Basic Tools
Slow Channel

Unit Sample Response

Frequency Response

Only low frequencies get through!
Fast Channel

Unit Sample Response

Magnitude of $|H(e^{j\omega})|$

Frequency Response

Fast channel allows high frequencies to get through.
Design $h_i$ for extracting $A_i$. 

$y[n] = w_i[n] = A_i \cos(\omega n)$
Bandpass 0.1pi

$L = 200$

Magnitude of $|H(e^{j\Omega})|$
Bandpass $0.4\pi$

$L = 200$
Three constraints notch

Magnitude of $|H(e^{i\omega})|$
Four constraints notch
100 constraints notch
\[ P_2 = \frac{e^{-1}}{e^{-1} + j2\omega} \]

Recall poles from \( 6.10 \):

\[ (1 + \alpha_1^2 R_2 + \alpha_2^2 R_2) = \frac{(1 - \alpha_1 R_2)}{(1 + \alpha_1 R_2 + \alpha_2^2 R_2)} \]

If \( 1 + \alpha_1 R_2 + \alpha_2^2 R_2 \equiv 0 \) \( \Rightarrow \) \( H(e^{j\omega}) \equiv \infty \)

\[ H(e^{j\omega}) = e^{\alpha_1 R_2} + \epsilon j \alpha_2 \epsilon j \alpha_2 \]

Substitute the values:

\[ W[\epsilon] = \epsilon R_2 \]

\[ W[\epsilon] + \alpha_1 W[\epsilon - 1] + \alpha_2 W[\epsilon - 2] = Y[\epsilon] \]

Like convolution with another type of filter.
As, \( L \to \infty \):

\[
H(e^{-2\alpha p}) = \frac{1}{(1 - \alpha p)(1 - \alpha p - 1/\alpha)^2} = \frac{1}{(1 - \alpha p)(1 - \alpha p - 1/\alpha)^2} - \frac{1}{\alpha p}
\]

When \( \alpha \neq 1 \) or \( a < b \)

\[
\sqrt{1 - 2r \cos \beta + r^2} = \sqrt{1 - 2r \cos \beta + r^2} + 1 - r \cos \beta = 1 - r \cos \beta + r^2 \to H(e^{-r}) = \frac{1}{1 - 2e^{-r} \cos \beta + e^{-2r}}
\]

\[
H(e^{-1}) = \frac{1 + a \varepsilon_{1p} + a^2 \varepsilon_{2p} + 2 \varepsilon_{1p} \varepsilon_{2p}}{1 + 2 \varepsilon_{1p} \varepsilon_{2p}}
\]

Bandpass Filter with Poles

\[14\]
Two Poles (magnitude 0.9)
Two poles (magnitude 0.97)
Two poles, three constraints notch
Audio example (in lecture)

Suppose sampling at 8000 samples/second

Frequency, in Hz,

\[
\text{Frequency} = \frac{2\pi f}{\text{samples per second}}
\]

In discrete time

\[
T_s = \frac{1}{f_s}
\]

Sample rate and "real" frequency

\[
E_{\text{samples MMAE}}
\]
Two Kinds of Filters

1) \[ h[n] = \sum_{m=0}^{L+1} w[n-m] y[c-n] \]

where \( h[n] \) is the unit sample response

Example:

\[ w[0] + a, w[-1] = y[n] \]

Unit sample response must be computed may never go to zero

\[ W[0] = 0, W[1] = 0 \]

2) \[ h[n] = \frac{1}{L} \sum_{i=0}^{L-1} y[n-i] \]

\[ W[0] = \frac{1}{L}, W[1] = \frac{1}{L}, \ldots, W[L-1] = \frac{1}{L} \]

\[ w[0] = 0, w[1] = 0 \]

W[0] = \alpha, W[1] = \alpha^2, \ldots, W[L-1] = \alpha^{L-1}

Infinite. Long unit sample response (LUT) never equals zero

\[ |\alpha| < 1 \]

\[ \lim_{n \to \infty} w[n] = 0 \]

Unstable
\[ y \in \{ h_{\text{cm}} = 0, h_{\text{cm}} = 0.5 \} \]

\[ T = \{ -1, 1 \} \]

\[ T = \{ -1, 1 \} \quad \text{Note:} \quad T \neq \emptyset \]

\[ H_{\text{cm}} = 0 \]

\[ \frac{2}{T} \text{h}_{\text{cm}} \text{e}^{-1.5m} = H_{\text{cm}}(\text{e}^{1.5}) \]

Zero frequency "Notch" Filler
Two Point Notch Filter

Magnitude of $|H(e^{j\Omega})|$
Reference Equation for Filtration Efficiency for Secondary Ion Filtration Plots

Replace 0.9 with 0.99

\[
\begin{align*}
\frac{1}{H(e, \nu)} &= 1 - 0.9 = 0.1 \\
H(e, \nu) &= \frac{1}{\nu - 0.9} e^{-\frac{e}{\nu}}
\end{align*}
\]

Example for \([W]\) - 0.9 \(W - 2\) = \(W\)

\[
\text{Example}\ \text{Response}\ \text{for} \ \text{Filtration Efficiency for Secondary Ion Filtration Plots}
\]
Pole at 0.99
Diff Eqn Filter
\[
\frac{0.1 + 0.1}{0.5} = \frac{2}{0.5} = 4
\]

For small values of \( \gamma \), consider

\[
\frac{1}{1 - \cos \theta} \quad \frac{2}{1 - \cos \theta + 0.9 \sin \theta}
\]

Expanding

\[
\frac{1}{1 - \cos \theta + 0.9 \sin \theta} = \frac{\frac{1}{1 - \cos \theta}}{1 - \cos \theta + 0.9 \sin \theta}
\]

Why does that produce a good fit?

See plots

\[ Y \rightarrow H(x) \rightarrow B_U(\Theta) \rightarrow W \]

\[ W_U \cap \{0, 9\} \cup Y = \{W\} \]

Intermediate

Composing two types of flips

III
Note U SR

Magnitude of $|H(e^{j\omega})|$}

Pole at 0.9
Composed System

Magnitude of $|H(e^{j\omega})|$