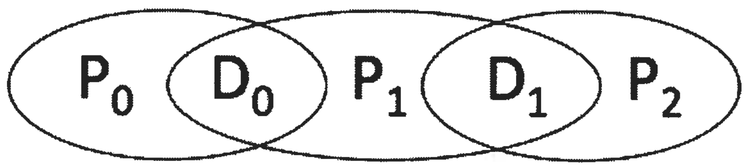
 NAME: SOLUTIONS

Problem 1 (15 points)

Two-bit Communications (TBC), a slightly suspect ISP, uses the following block code to protect transmissions over its point-to-point links.



where $P_0 = D_0$, $P_1 = (D_0 + D_1) \bmod 2$, $P_2 = D_1$.

As each 5-bit block arrives, TBC receivers compute the three syndrome bits as shown below and attempt to perform single-bit error correction.

$$
\begin{aligned}
E_0 &= (D_0 + P_0) \bmod 2 \\
E_1 &= (D_0 + D_1 + P_1) \bmod 2 \\
E_2 &= (D_1 + P_2) \bmod 2
\end{aligned}
$$

1A. (8 points) For the eight possible combinations of syndrome values shown in the table below, indicate the error detected (none, a particular data or parity bit, or multiple errors), making your choice based on the smallest number of errors that's consistent with the given syndrome.

E_0	E_1	E_2	**Error detected**
0	0	0	no errors
1	0	0	error in P_0
0	1	0	error in P_1
1	1	0	error in D_0
0	0	1	error in P_2
1	0	1	multiple errors
0	1	1	error in D_1
1	1	1	multiple errors

1B. (2 points) We often characterize block codes with a (n, k) designation. What are n and k for this code?

n = # of bits in block = 5

k = # of message bits in block = 2

(n, k): (5,2)

1C. (2 points) Suppose that the 5-bit blocks arrive at the receiver in the following bit order: D_0, D_1, P_0, P_1, P_2. If the block 11011 arrives, what will the TBC receiver report as the received data after error correction has been performed?

corrected values for D_0, D_1: 0, 1

$E_0 = (1+0) \bmod 2 = 1$

$E_1 = (1+1+1) \bmod 2 = 1$ } error in D_0 → flip bit to correct.

$E_2 = (1+1) \bmod 2 = 0$

1D. (3 points) TBC would like to improve download speeds by switching to a shorter code while still maintaining single-bit error correction. Each block in the new code would still have two data bits (D_0, D_1) but only two parity bits (P_0, P_1). Please give the formulas for P_0 and P_1 that will allow single-bit error correction, or briefly explain why no such code is possible.

no such code exists.

for single-bit error correction we require

$$n \le 2^{n-k} - 1$$

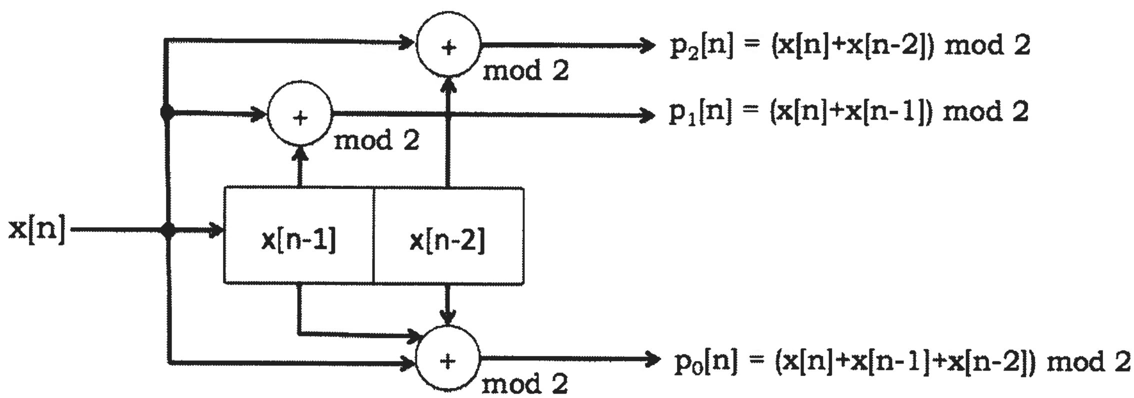
with $n=4$ and $k=2$:

$$4 \not\le 2^{4-2} - 1$$

so this proposed code can't do single-bit error correction.

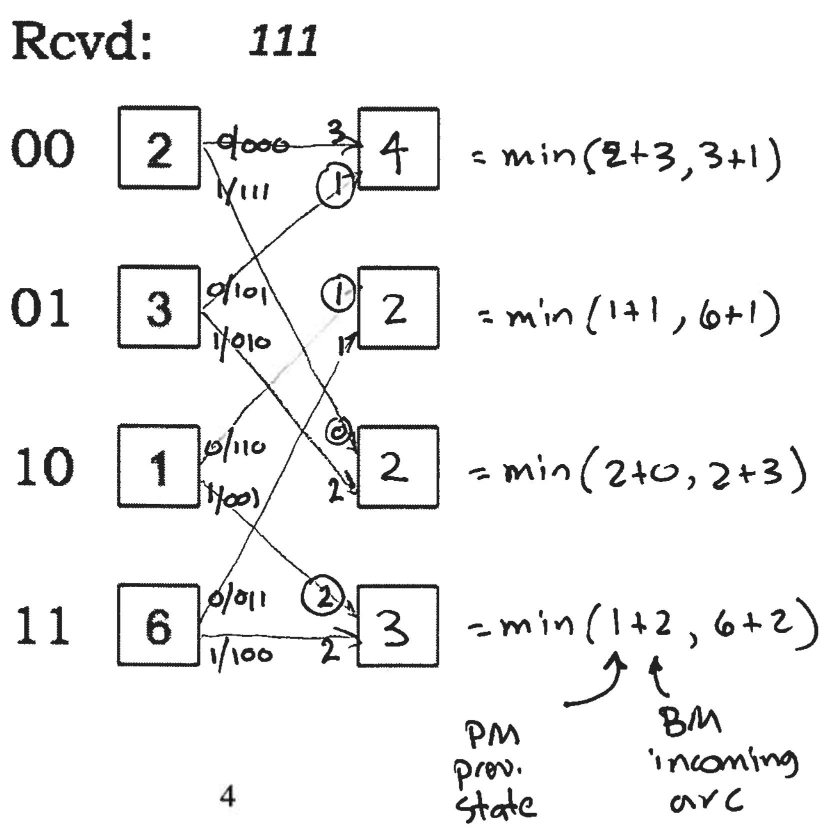
Problem 2 (30 points)

Consider the $k = 3$, $rate = \frac{1}{3}$ convolutional code generated by the following block diagram:



During transmission the three parity bits generated for each message bit are sent in the order p_0, p_1, p_2. The transmit bit stream is sent over a noisy channel that introduces occasional bit errors.

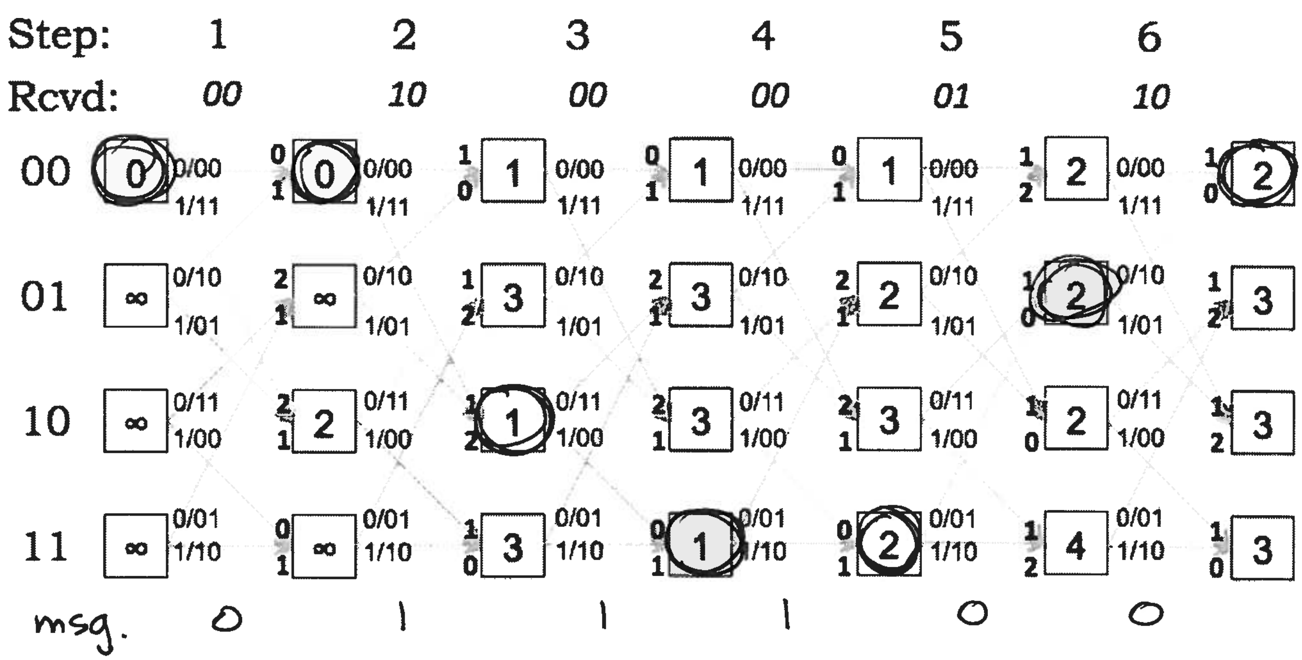
2A. (8 points) Fill in the state transition diagram for the transmitter, shown in the form of a trellis diagram below. The 4 states are labeled with $x[n-1]x[n-2]$. Add the appropriate transition arcs between the current states on the left-hand column and the destination states in the right-hand column. Label each arc showing the following information: $x[n]/p_0p_1p_2$. For now, ignore the path metrics written in the squares of the left-hand column and the "Rcvd: 111" at the top.



2B. (4 points) At the right-hand end of each transition arc you entered in part (A) fill in the appropriate hard-decision branch metric assuming that the received parity bits are 111. Recall that hard-decision branch metrics are computed as the Hamming distance between the transmitted parity bits and the received parity bits. see prev. page

2C. (4 points) Please fill in the right-hand column in part (A) with the updated Viterbi path metric, using the path metrics given in the left-hand column and the branch metrics you computed in part (B). see prev. page

Consider the following Viterbi decoder trellis diagram for the $k = 3$, $rate = \frac{1}{2}$ convolutional code used in Lab 5. The trellis shows the results of decoding the received parity bits shown across the top with the path metrics entered in the squares and the two hard-decision branch metrics for the incoming arcs shown in bold just to the left of each square.



2D. (6 points) In the figure above, circle the states along the most-likely path as determined by the Viterbi algorithm.

2E. (4 points) How many bit errors were detected in the transmission? At which steps?

number of bit errors detected: 2, **step numbers:** 2, 4
↳ PM of most-likely final state ↳ where PM increments

2F. (4 points) What was the most-likely message as determined by the Viterbi decoder?

most-likely message: 011100

Problem 3 (~~10~~ 12 points)

Suppose a linear time-invariant system has a frequency response $H_A(e^{j\Omega})$ given by the formula

$$H_A(e^{j\Omega}) = \frac{1}{\left(1 - 0.95e^{-j(\Omega - \frac{\pi}{2})}\right)\left(1 - 0.95e^{-j(\Omega + \frac{\pi}{2})}\right)}$$

3A. (~~5~~ 6 points) Which frequency response plot (I, II or III) corresponds to $H_A(e^{j\Omega})$ above, and what is the numerical value of M in the plot you selected? Please justify your selection and your computation of M.

when $\Omega = \pi/2$ $\quad H(e^{j\Omega}) = \frac{1}{(1-0.95e^{j0})(1-0.95e^{-j\pi})}$

$= \frac{1}{(0.05)\,1+0.95} = \frac{400}{41} = 10.26 \simeq 10$

Only system I has $|H(e^{j\pi/2})| / |H(e^{j0})|$ so large (>10)

when $\Omega = 0$ $\quad H(e^{j\Omega}) = \frac{1}{(1-0.95e^{-j\pi/2})(1-0.95e^{+j\pi/2})}$

$= \frac{1}{1 - 0.95\underbrace{(e^{j\pi/2}+e^{-j\pi/2})}_{=0} + (0.95)^2} \simeq \frac{1}{2}$

frequency response plot (I, II, or III) = I

Numerical value of M = $\frac{400}{41}$ or 10.26 or 10

3B. (~~5~~ 6 points) For what values of a_1 and a_2 will the system described by the difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n]$$

have a frequency response given by $H_A(e^{j\Omega})$ above? Show your work.

$$H_A(e^{j\Omega}) = \frac{1}{1 + a_1 e^{-j\Omega} + a_2 e^{-j2\Omega}}$$

$$= \frac{1}{\left(1 - \frac{19}{20}e^{-j(\Omega - \pi/2)}\right)\left(1 - \frac{19}{20}e^{j(\Omega + \pi/2)}\right)}$$

$$= \frac{1}{1 \underbrace{- \frac{19}{20}\left(\cancel{e^{j\pi/2} + e^{-j\pi/2}}\right)}_{a_1} e^{-j\Omega} + \underbrace{\left(\frac{19}{20}\right)^2}_{a_2} e^{-j2\Omega}}$$

$= 0$ $\quad \cancel{\cos \pi/2}$

a_1 = 0

a_2 = $\left(19/20\right)^2 = \frac{361}{400}$

$= 0.9025$

$= (0.95)^2$

Problem 4 (~~10~~ 6 points)

Suppose the input to a linear time invariant system is the sequence

$$x[n] = 2 + \cos\frac{5\pi}{6}n + \cos\frac{\pi}{6}n + 3(-1)^n$$

4A. (~~5~~ 2 points) What is the maximum value of the sequence x, and what is the smallest positive value of n for which x achieves its maximum? Show your work.

$$\max_n\left(\cos\frac{\pi}{6}n\right) = 1 \longrightarrow n = 0,\ n = 12, \ldots$$
$$\max_n\left(\cos\frac{5\pi}{6}n\right) = 1 \longrightarrow n = 0,\ n = 12, \ldots$$
$$\max_n 3(-1)^n = 3 \longrightarrow n \text{ is even}$$

$\max_m x[m] =$ $2 + 1 + 1 + 3 = 7$

Smallest $n > 0$ for which $x[n] = \max_m x[m]$ $n = 12$

4B. (~~5~~ 4 points) Suppose the above sequence x is the input to a linear time invariant system described by one of the three frequency response plots above (I, II or III). If y is the resulting output and is given by

$$y[n] = 8 + 12(-1)^n,$$

which frequency response plot describes the system, and what is the value of M in the plot you selected? Be sure to justify your selection and your computation of M.

Frequency response at $\Omega = \pi/6$ and at $\Omega = \frac{5\pi}{6}$ must be zero. Could only be III

$$y[n] = H(e^{j0}) \cdot 2 \cdot (1)^n + H(e^{j\pi}) \cdot 3 \cdot (-1)^n$$
$$= 4 \cdot 2 \cdot (1)^n + 4 \cdot 3 \cdot (-1)^n$$

Frequency response plot (I, II, or III) = III

Numerical value of M = 4

12

Problem 5 (~~10~~ points)

Suppose the unit sample response of a linear time-invariant system has only three nonzero *real* values, $h[0]$, $h[1]$, and $h[2]$. In addition, suppose these three *real* values satisfy the three equations:

$$
\begin{aligned}
h[0]+h[1]+h[2] &= 5\\
h[0]+e^{-j\frac{\pi}{2}\cdot 1}h[1]+e^{-j\frac{\pi}{2}\cdot 2}h[2] &= 0\\
h[0]+e^{j\frac{\pi}{2}\cdot 1}h[1]+e^{j\frac{\pi}{2}\cdot 2}h[2] &= 0.
\end{aligned}
$$

4

5A. (~~3~~ points) Which of the above plots, I, II or III, is a plot of the magnitude of the frequency response of this system, and what is the value of M in the plot you selected? Be sure to justify your selection and your computation of M.

$$H(e^{j\Omega}) = \sum_{m=0}^{2} h[m]e^{-j\Omega m}$$

First eqn $\Omega = 0 \Rightarrow H(e^{j0}) = 5 = M$

Second eqn $\Omega = \pi/2 \Rightarrow H(e^{j\pi/2}) = 0$

Third eqn $\Omega = -\pi/2 \Rightarrow H(e^{-j\pi/2}) = 0$

frequency response plot (I, II, or III) = Must be II

Numerical value of M = 5

8

5B. (~~7~~ points) If

$$y[n] = \sum_{m=0}^{2} h[m]x[n-m]$$

and

$$x[n] = e^{j\frac{\pi}{6}n}$$

for all n, please determine the complex numerical value for $\frac{y[n]}{e^{j\frac{\pi}{6}n}}$.

It might be helpful to know the following numerical values: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = 0.5$, $\cos\frac{\pi}{3} = 0.5$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Show your work.

$$y[n] = H(e^{j\pi/6})x[n] \Rightarrow \frac{y[n]}{e^{j\pi/6 n}} = H(e^{j\pi/6})$$

Find $h[0]$, $h[1]$ and $h[2]$

$h[0] + h[1] + h[2] = 5$

$h[0] - jh[1] - h[2] = 0$

$h[0] + jh[1] - h[2] = 0$

$h[0] = 2.5 \quad h[2] = 2.5 \quad h[1] = 0$

$\frac{y[n]}{e^{j\frac{\pi}{6}n}} =$ $H(e^{j\pi/6}) = \frac{15}{4} - \frac{5}{4}\sqrt{3}j$

$H(e^{j\pi/6}) = h[0] + h[2]e^{-j\pi/3}$

$= 2.5 + 2.5(\cos\pi/3 + j\sin(-\pi/3))$

$= 3.75 - j(1.25 \cdot \sqrt{3})$

Problem 6 (25 points)

In this modulation problem you will be examining periodic signals and their associated discrete Fourier transform (DFT) coefficients. Recall that a signal $s[n]$ has DFT coefficients given by

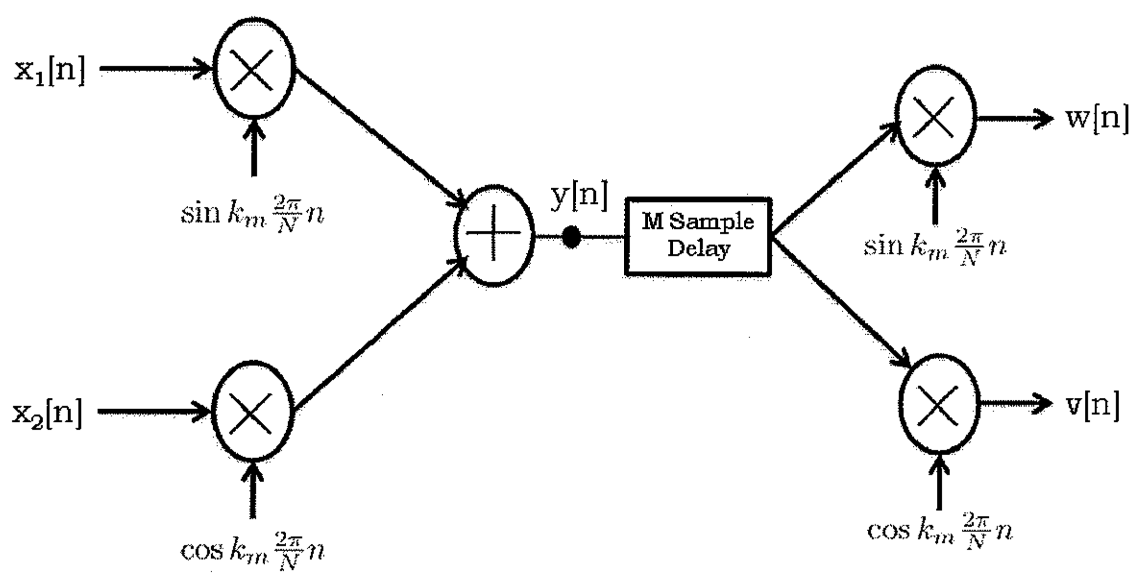
$$S[k] = \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi}{N}kn}$$

and that the signal $s[n]$ can be reconstructed from the DFT coefficients using

$$s[n] = \frac{1}{N} \sum_{k=-K}^{K} S[k] e^{j\frac{2\pi}{N}kn}$$

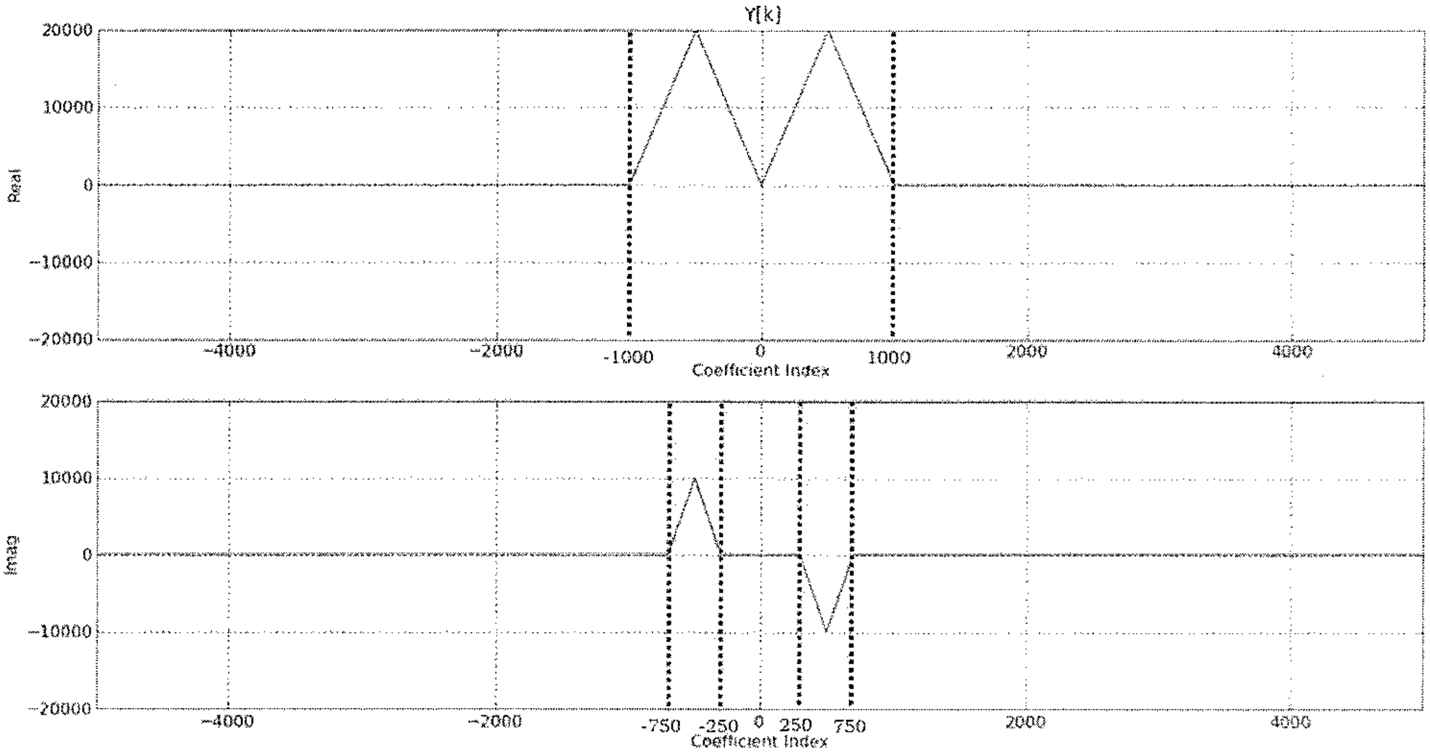
where N is the period of the signal and $-K \leq k \leq K$ with $2K + 1 = N$.

All parts of this question pertain to the following modulation-demodulation system, where all signals are periodic with period $N = 10001$ and therefore $K = 5000$. Please also assume that the sample rate associated with this system is 10001 samples per second, so that k is both a coefficient index and a frequency. In the diagram, the modulation frequency, k_m, is 500.



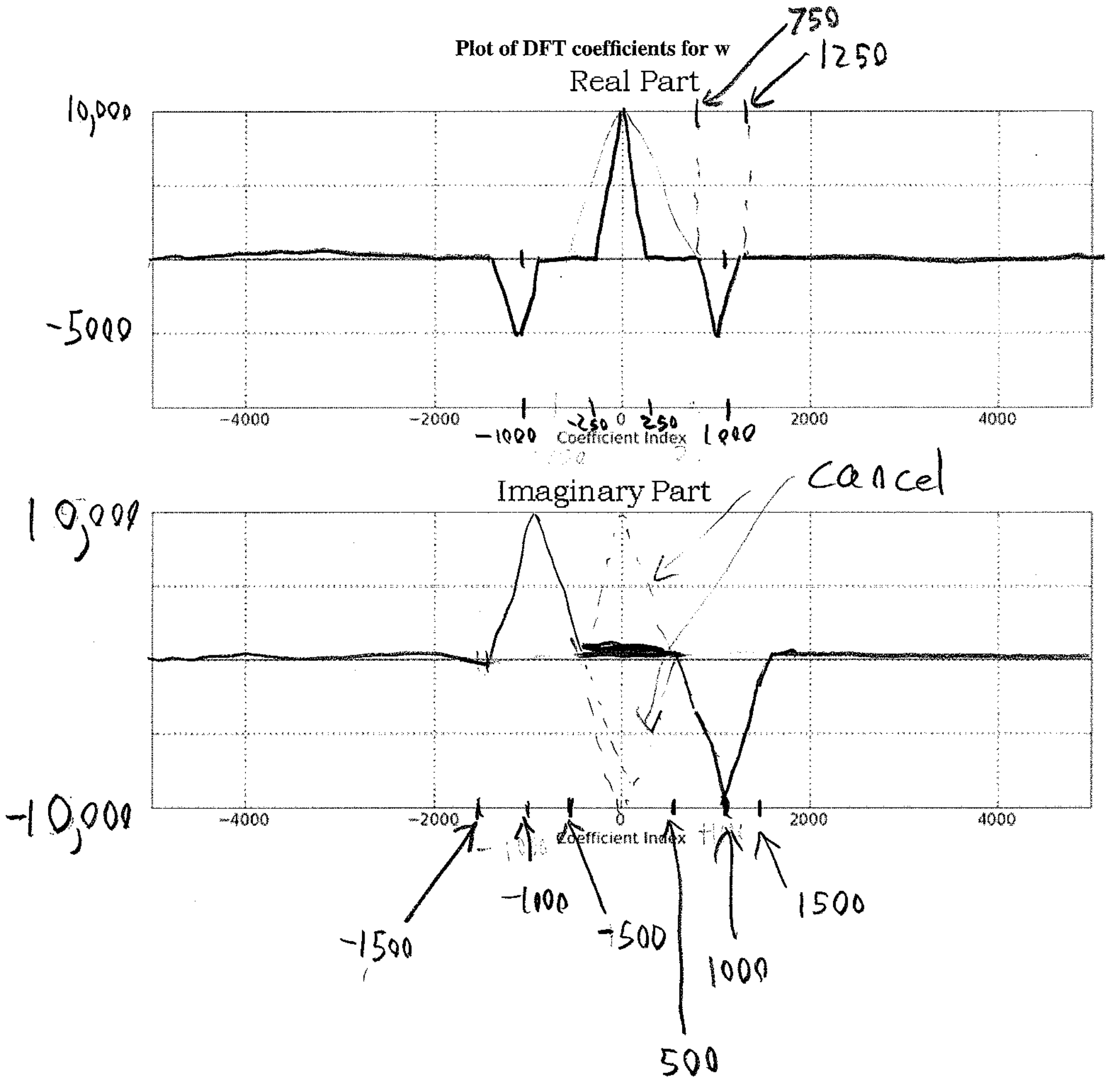
12

6A. (~~9~~ points) Suppose the DFT coefficients for the signal $y[n]$ in the modulation/demodulation diagram are as plotted below.

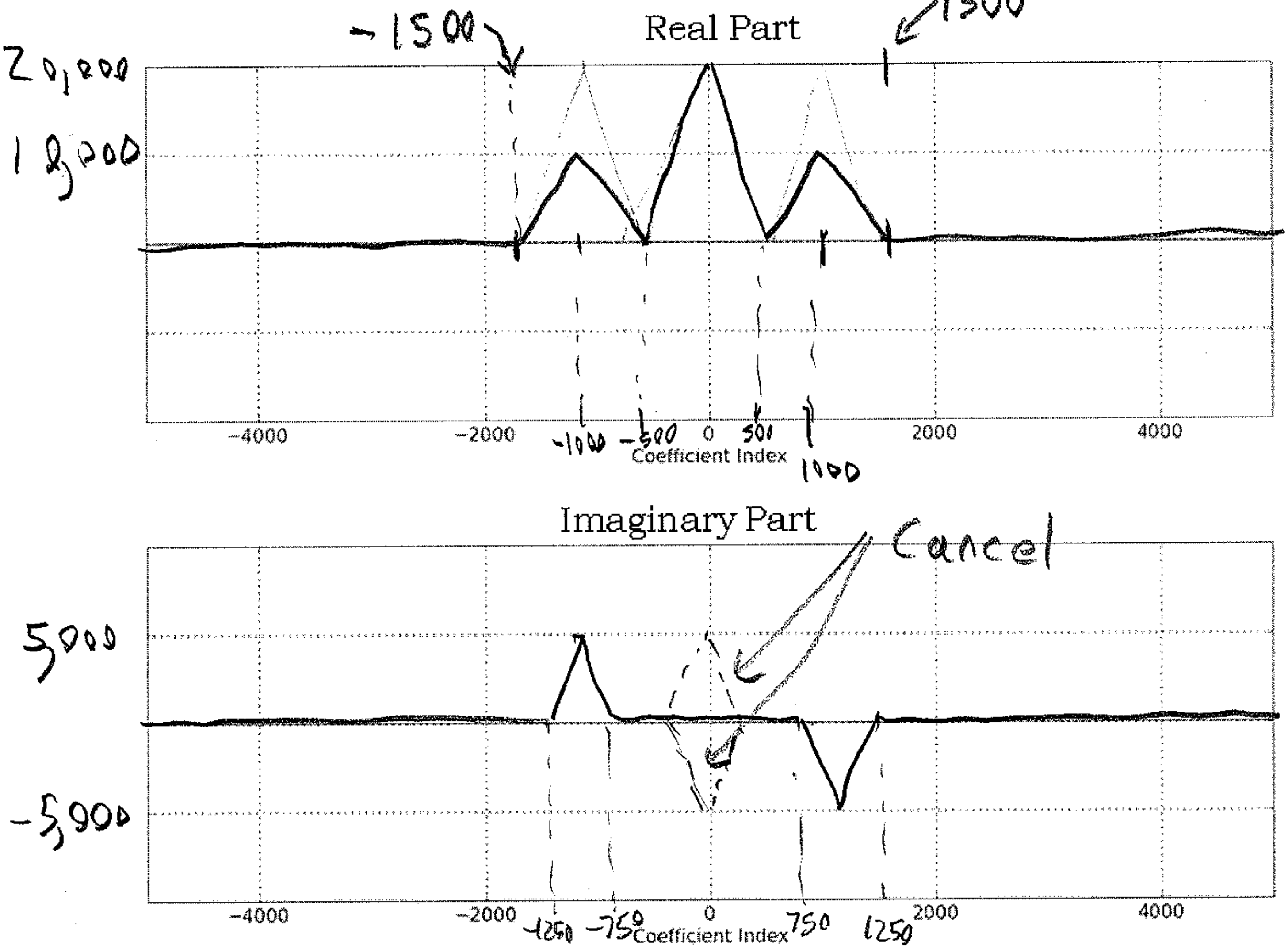


Assuming that $M = 0$ for the M-sample delay (no delay), on the two sets of axes on the next pages, please plot the DFT coefficients for the signals **w** and **v** in the modulation/demodulation diagram. Be sure to label key features such as values and coefficient indices for peaks.

Plot of DFT coefficients for w



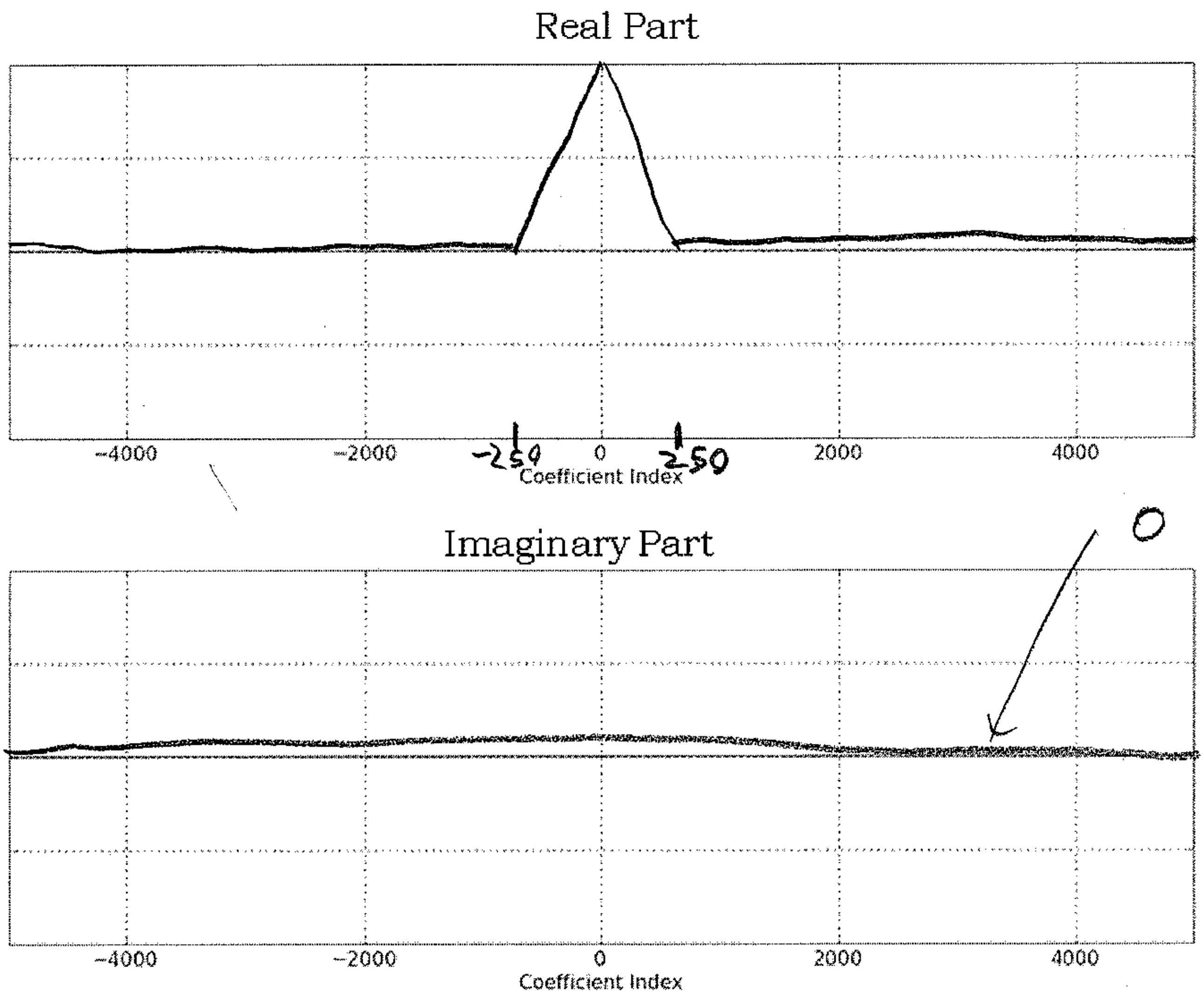
Plot of DFT coefficients for v



6B. (~~7~~ 5 points) Assuming the DFT coefficients for the signal $y[n]$ are the same as in part A, on the axes below, please plot the DFT coefficients for the signal x_1 in the modulation/demodulation diagram. Be sure to label key features such as values and coefficient indices for peaks.

Plot of DFT coefficients for x_1

20,000



8

6C. (~~6~~ points) If the M-sample delay in the modulation/demodulation diagram has the right number of samples of delay, then it will be possible to nearly perfectly recover $x_2[n]$ by low-pass filtering $w[n]$. Please determine the number of samples of delay that are needed and the cut-off frequency for the low-pass filter. Please be sure to justify your answers, using pictures if appropriate.

$$\cos\left(k_M \frac{2\pi}{N}(n-M)\right) \cong \sin\left(k_M \frac{2\pi}{N} n\right)$$

for $W[n]$ to demod x_2

$$\cos\left(500 \frac{2\pi}{10{,}001} n - \underbrace{500 \frac{2\pi}{10{,}001} M}\right) \simeq \sin\left(\frac{500 \cdot 2\pi}{10{,}001} n\right)$$

$$= \pi/2 \Rightarrow \frac{1000M}{10{,}001} \approx \frac{1}{2} \quad M=5$$

band width of x_2 is 500.

Smallest M (number of samples of delay) > 0 = $M=5$

Cutoff Frequency of Low Pass Filter = Cutoff frequency = 500

End of Quiz 2!