

LECTURE 12

Channel Sharing Using Sinusoids: Convolution Analysis

The next five lectures starting from this one address an important problem faced in many communication systems at the physical layer: how to divide up a single communication channel (wired or wireless) amongst more than one communicating entity? We will develop a solution that divides up the bandwidth of the communication channel into non-overlapping frequencies and arrange for the different transmitters to send their data *concurrently*. The receivers can each then pick off the frequency range of interest to them and retrieve data. Applications of this idea, termed *frequency division multiplexing*, include broadcast television and radio as well as cellular wireless communications (cellular networks use a combination of techniques, including frequency division). A pictorial representation of this problem is shown in Figure 12-1.

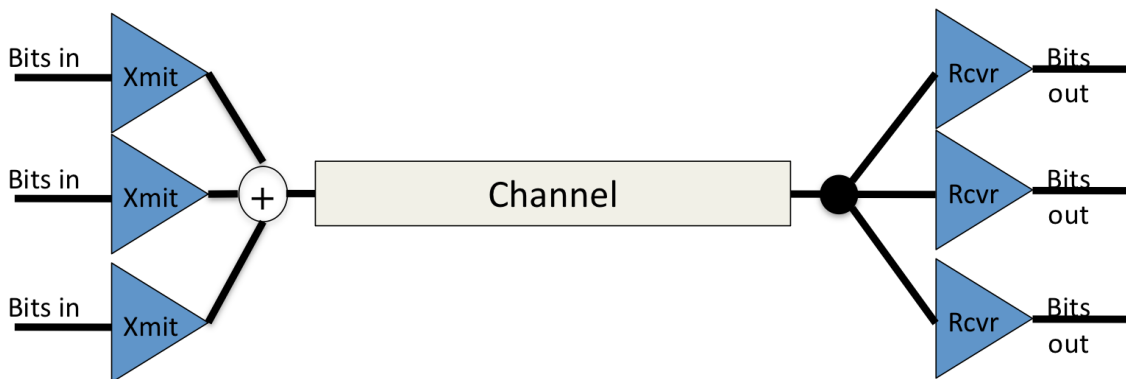


Figure 12-1: The channel sharing problem, which we will solve using frequency division multiplexing.

This lecture focuses on one of the building blocks toward our grand goal. We will study what happens when a sinusoid is sent over a linear time-invariant (LTI) system.

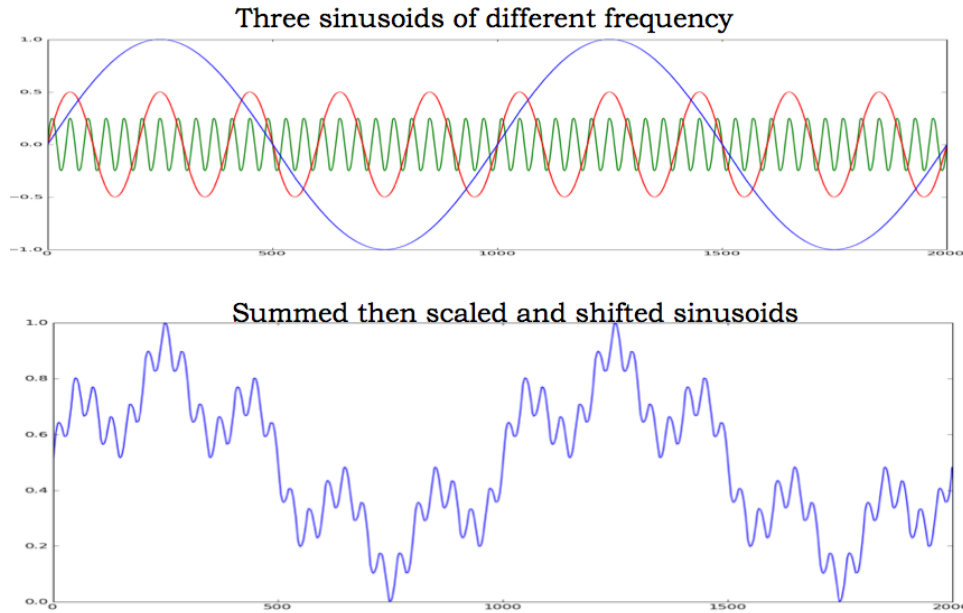


Figure 12-2: Sum of sinusoids of different frequencies.

■ 12.1 Our Goal: Understanding Sinusoids over an LTI Channel

Suppose we have a set of transmitters, each sending a sinusoid of a different frequency. When we observe the aggregate effect of these transmissions, we get a result that might look like something in Figure 12-2. After they are sent through an LTI channel, the result might look as in Figure 12-3.

There are three interesting properties that one can discern by just inspecting the pictures, without any algebra or mathematical manipulation:

1. The periodic sinusoidal behavior is still quite visible.
2. The different constituent frequencies do not seem to “mix”, and are each fairly noticeable.
3. The different sinusoidal frequencies have different gains on the output, after they have been sent through the LTI channel.

These empirical observations suggest that we might be able to take the data of each transmitter in Figure 12-1, encode them on different frequencies and send them concurrently, and somehow have the receiver extract only the frequency band of interest to it. The task of encoding on different frequencies is called **modulation**. The task of pulling out a specific frequency band is called **filtering**.

■ 12.2 Sinusoids, Complex Exponentials, and Discrete Time

As we will see in this and subsequent lectures, sinusoidal waveforms play such a key role in LTI systems that we will review a few properties of such waveforms.

First, continuous-time sinusoidal (or cosinusoidal) waveforms are functions of time, t ,

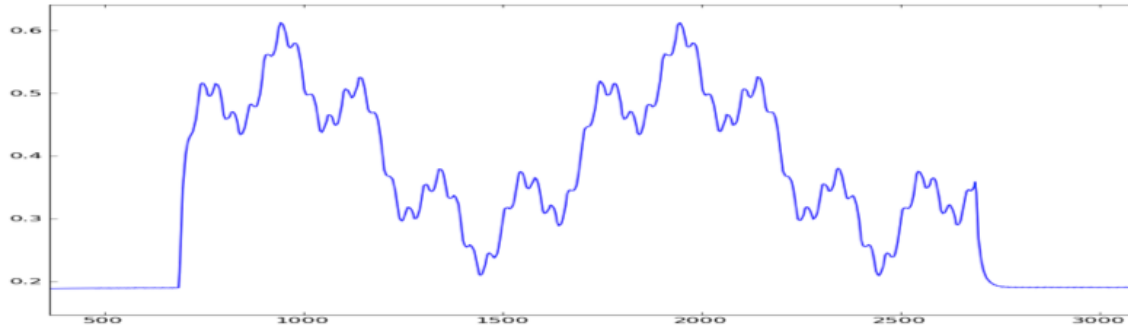


Figure 12-3: What the sum of sinusoids of different frequencies looks like after being sent through an (approximately) LTI channel.

and can be represented in several forms,

$$\sin \omega t = \sin 2\pi f t = \sin 2\pi \frac{t}{T}$$

for a sinusoidal waveform and

$$\cos \omega t = \cos 2\pi f t = \cos 2\pi \frac{t}{T},$$

for a cosinusoidal waveform, where ω denotes the sinusoidal frequency in radians per second, f denotes the sinusoidal frequency in cycles per second, and T is the period of the sinusoidal waveform in seconds. Therefore, the frequency is inversely related to the period, that is, $f = \frac{1}{T}$. In addition, there is a simple relation between sinusoidal and cosinusoidal waveforms,

$$\cos \left(\omega t - \frac{\pi}{2} \right) = \sin \omega t,$$

and

$$\sin \left(\omega t + \frac{\pi}{2} \right) = \cos \omega t,$$

where the $\frac{\pi}{2}$ term is often referred to as a *phase* difference between the sine and the cosine.

For many of the manipulations associated with analyzing sinusoidal and cosinusoidal waveforms when processed by LTI systems (such as filters) or by modulation (to be covered later), the complex exponential representation often simplifies the analysis. When using complex exponential representations of sines and cosines, the most useful identities are

$$e^a e^b = e^{a+b},$$

and

$$e^{j\omega t} = \cos \omega t + j \sin \omega t,$$

from which it follows that

$$\cos \omega t = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

and

$$\sin \omega t = \frac{-j}{2} e^{j\omega t} + \frac{j}{2} e^{-j\omega t}.$$

In the above, we used the notation $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \approx 2.718$ and $j = \sqrt{-1}$ or the imaginary number¹

As we have mentioned before, in many digital signal processing systems, the inputs are continuous-time signals that are sampled to generate discrete-time signals. For example, the infrared system in the 6.02 lab samples continuous time signals at a rate of four million samples every second, and we say that the *sampling frequency*, f_s of the IR system is 4 million samples per second. Equivalently, the *sampling period*, T_s of the IR system is 0.25 microseconds. For sampling in general, $f_s = \frac{1}{T_s}$, meaning that sampling at a frequency of f_s times a second is equivalent to taking one sample every T_s seconds.

The process of sampling a continuous-time signal to generate a discrete-time signal is quite straightforward, though some important subtleties arise when sampling continuous-time sinusoidal waveforms to generate discrete-time sinusoidal waveforms. To understand these subtleties, consider $x[n]$, the n^{th} sample of a continuous-time signal, $x(t)$. In this case, $x[n]$ is given by

$$x[n] = x(t)|_{t=nT_s}, \quad (12.1)$$

where $T_s = 1/f_s$ is the sampling interval, as described above.

If $x(t) = \cos 2\pi f t$, then the associated discrete-time cosine waveform, $x[n]$, is given by

$$\begin{aligned} x[n] &= \cos 2\pi f(nT_s) \\ &= \cos \Omega n, \end{aligned} \quad (12.2)$$

where Ω is used to denote the discrete-time frequency, and $\Omega \equiv 2\pi f T_s \equiv 2\pi \frac{f}{f_s}$. As one might expect, the process of sampling preserves the complex exponential relationship, so

$$\cos \Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n}$$

and

$$\sin \Omega n = \frac{-j}{2} e^{j\Omega n} + \frac{j}{2} e^{-j\Omega n}.$$

Consider the five examples in Figure 12-4, where continuous-time sinusoidal and cosinusoidal waveforms are super-imposed on the associated discrete-time sines and cosines generated by sampling. In the top example, there are sixteen samples per period of the continuous-time sinusoidal waveform, corresponding to a discrete-time frequency of $\Omega = \frac{\pi}{8}$. In the middle example, the continuous-time cosine's frequency is four times higher, and therefore there are only four samples per period. This corresponds to a discrete-time frequency of $\Omega = \frac{\pi}{2}$. In general, for a given sampling frequency (or sampling interval), we get many more samples per period for lower frequency continuous-time sinusoidal waveforms than for the higher frequency ones. Moreover, as shown in the fourth exam-

¹By convention, we will use $j = \sqrt{-1}$, rather than i , which is the notation used in math and physics. The reason is mundane because in electrical engineering (and in physics), i or I is used for current. As a further aside, I was used because current used to be called "intensity". When it was changed to current, "C" and "c" were both unavailable, the former referring to capacitance and the latter to the speed of light in vacuum!

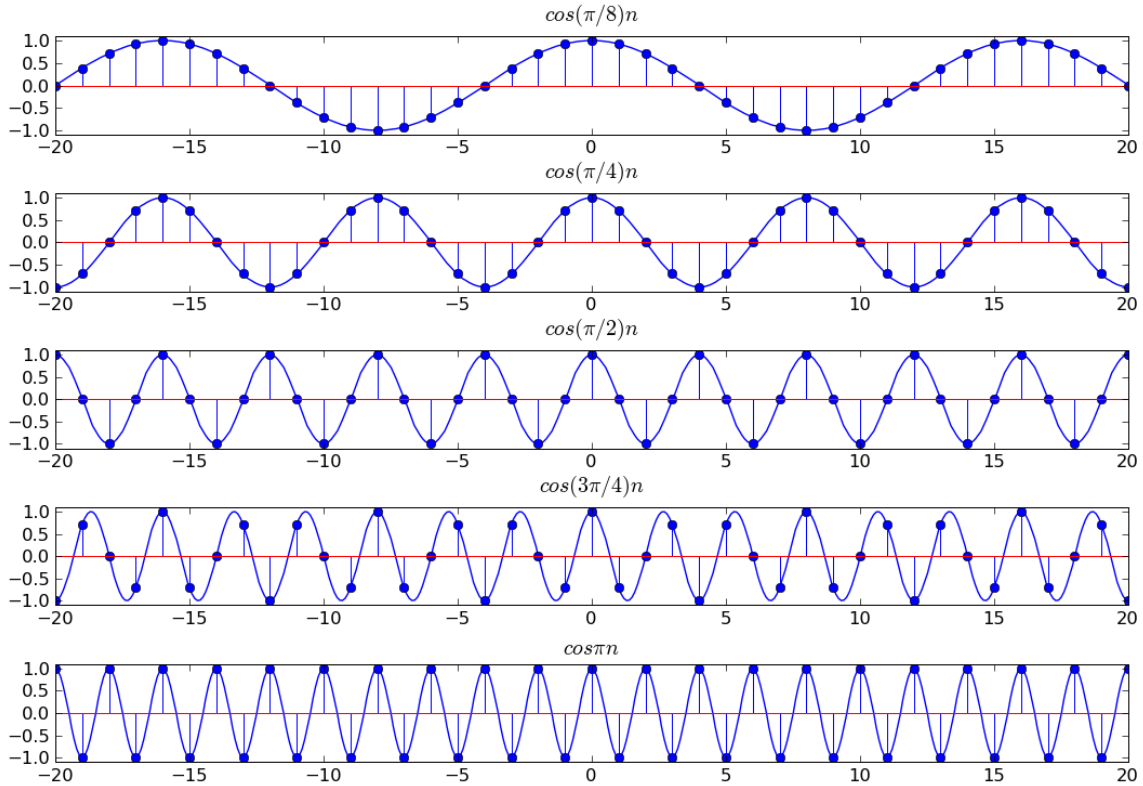


Figure 12-4: Sampling cosines of different frequencies.

ple, just because the samples are uniformly spaced in time does *not* imply that each period of the continuous-time waveform will be sampled at identical phases of each period. Finally, in the fifth (or bottom) example, there are only two samples per period. This is the highest continuous-time frequency that can be uniquely represented by sampling, and corresponds to $\Omega = \pi$.

The issue of a maximum discrete-time frequency, $\Omega = \pi$, can seem counter-intuitive as there is no maximum continuous-time frequency. There are several insights that can help clarify this curious phenomenon:

1. Any discrete-time frequency Ω outside the range $-\pi < \Omega \leq \pi$ can be “wrapped back” to be within this range, a property unique to the discrete-time case. To see this, consider an Ω larger than π . Then,

$$e^{j\Omega n} = e^{j(\Omega - 2\pi)n} e^{j2\pi n} = e^{j(\Omega - 2\pi)n}$$

where the last equality follows from the fact that $e^{j2\pi n} = 1$ as n is always an integer. If n were allowed to be any real number, t , then $e^{j2\pi t}$ could take on values other than 1.

2. If $\Omega = \pi$, like the bottom example of Figure 12-4, then the period of the continuous-time sinusoidal waveform, T , is twice the sampling interval, T_s . Or, the frequency of the continuous-time waveform, f , is half the sampling frequency, f_s . In other words, the sampling interval, T_s , must satisfy $2\pi f_{\max} = \pi$, which implies that

$f_{\max} = \frac{1}{2T_s} = \frac{f_s}{2}$. If we sample a continuous-time waveform with a frequency higher than f_{\max} , the resulting samples will be indistinguishable from samples produced from a continuous time sinusoidal waveform of frequency lower than f_{\max} (Try it and see).

3. In the bottom example of Figure 12-4, for which $\Omega = \pi$, the samples alternate as $\dots, -1, +1, -1, +1, -1, +1, \dots$. Clearly, it is not possible for a discrete-time signal to oscillate more rapidly.

■ 12.3 LTI Channels and Sinusoidal Inputs

We are now ready to understand what happens when we send sinusoids through an LTI channel. We will first examine an idealized case where we assume that the input is an eternal sine or cosine, e.g. $x[n] = \cos \Omega n$ for $-\infty < n < \infty$. Then, we will consider the more realistic case, where the input “turns on” at zero, that is, $x[n] = 0$ for $n < 0$.

For a causal LTI channel with unit sample response H , we can use convolution to compute the output for any input. So, if $x[n] = \cos \Omega n$ for $-\infty < n < \infty$, then the output, $y_{\cos}[n]$, is given by

$$\begin{aligned} y_{\cos}[n] &= \sum_{m=0}^{\infty} h[m]x[n-m] \\ &= \sum_{m=0}^{\infty} h[m] \cos \Omega(n-m). \end{aligned} \quad (12.3)$$

We have denoted the output as $y_{\cos}[n]$ to avoid confusion between the output due to an input cosine versus the output due to a complex exponential, a case we will examine below.

The lower index m starts at 0 because the LTI system is causal, and the upper index is ∞ because the input is assumed eternal. If the unit sample response, H , has a finite length, $L + 1$, then $h[i] = 0$ for $i > L$, and we get

$$y_{\cos}[n] = \sum_{m=0}^L h[m] \cos \Omega(n-m). \quad (12.4)$$

How do we simplify this formula to produce something we can both interpret and remember easily? Time to apply the *complex exponential* representation described above. As we will see below, using complex exponentials will both simplify the expression and provide an important intuition.

We know that $\cos \Omega n = \frac{e^{j\Omega n} + e^{-j\Omega n}}{2}$, so if we can develop a formula for the output due to $x[n] = e^{j\Omega n}$ for an arbitrary Ω , then we can use superposition to solve the problem for any sinusoidal or cosinusoidal waveform.

Understanding what happens when a complex exponential is sent through an LTI channel is straightforward because the exponential in the convolution sum “separates” easily

using the $e^a e^b = e^{a+b}$ identity mentioned above,

$$\begin{aligned} y[n] &= \sum_{m=0}^L h[m] e^{j\Omega(n-m)} \\ &= \sum_{m=0}^L (h[m] e^{-j\Omega m}) e^{j\Omega n} \\ &= \left(\sum_{m=0}^L h[m] e^{-j\Omega m} \right) e^{j\Omega n}. \end{aligned} \quad (12.5)$$

Let $H(e^{j\Omega})$ be defined as

$$H(e^{j\Omega}) \equiv \left(\sum_{m=0}^L h[m] e^{-j\Omega m} \right),$$

and therefore $H(e^{j\Omega})$ is a complex number *independent of n* . Further, if $x[n] = e^{j\Omega n}$, then

$$y[n] = H(e^{j\Omega}) e^{j\Omega n}.$$

That is, *and this is a crucial point*, **if the input to an LTI system is an eternal complex exponential waveform with frequency Ω , then the output is also an eternal complex exponential waveform with frequency Ω** . Though, the output will be scaled by a frequency-dependent complex number, $H(e^{j\Omega})$.

The easy manipulation above shows the power of using the complex exponential representation and the $e^a e^b = e^{a+b}$ identity, eliminating the need to resort to trigonometric identities (and who remembers those anyway). When it becomes necessary to derive the output for

$$x[n] = \cos \Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n},$$

we can use superposition to derive

$$y_{\cos}[n] = \frac{1}{2} H(e^{j\Omega}) e^{j\Omega n} + \frac{1}{2} H(e^{-j\Omega}) e^{-j\Omega n}. \quad (12.6)$$

■ 12.3.1 The non-eternal case

What happens when $x[n] = e^{j\Omega n}$ for $n \geq 0$ but $x[n] = 0$ for $n < 0$ (or, X “turns on” at $n = 0$)? Much of the analysis of the output a causal LTI channel is the same, except for the limits of the sums. That is,

$$\begin{aligned} y[n] &= \sum_{m=0}^n h[m] e^{j\Omega(n-m)} \\ &= \sum_{m=0}^n (h[m] e^{-j\Omega m}) e^{j\Omega n} \\ &= \left(\sum_{m=0}^n h[m] e^{-j\Omega m} \right) e^{j\Omega n}. \end{aligned} \quad (12.7)$$

Unlike the eternal case, the sum

$$\left(\sum_{m=0}^n h[m]e^{-j\Omega m} \right)$$

is not independent of n unless $n > L$. If $n > L$, the result is the same as in the eternal case.

What is the meaning of this result? If $x[n] = e^{j\Omega n}$ for $n \geq 0$ is the input to a causal LTI system with an L -length unit sample response, then $y[n] = H(e^{j\Omega})e^{j\Omega n}$ for $n > L$. Or, if the input to an LTI system is a complex exponential with frequency Ω , then if the system has a finite length unit sample response, eventually the output will be a complex exponential with frequency Ω .

■ 12.3.2 Frequency Response

Although not discussed in any detail above, we did note that if the input to an LTI system is $x[n] = e^{j\Omega n}$, then the output is $y[n] = H(e^{j\Omega})e^{j\Omega n}$, where $H(e^{j\Omega})$ is a frequency-dependent complex scale factor. This scale factor, $H(e^{j\Omega})$, is referred to as the *frequency response* of the LTI system (Recall that Ω ranges from $-\pi$ to π). The name frequency response stems from the fact that if $\|H(e^{j\Omega})\| \approx 1$, inputs of frequency Ω will pass through the LTI system, but if $\|H(e^{j\Omega})\| \approx 0$, then inputs of frequency Ω will be blocked. In the next lecture, we will examine using LTI systems, designed to have a desired frequency response, for filtering applications.