MIT 6.02 **DRAFT** *Lecture Notes*
Fall 2010 (Last update: November 9, 2010)
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LECTURE 14
Frequency-Domain Sharing and Fourier Series

In earlier lectures, we learned about medium access (MAC) protocols for allowing a set of users to share a single communication medium by well-controlled turn-taking, a form of **time-domain** sharing (we used the term "time sharing" earlier). In these three lectures, we will focus on using P sinusoids of P different frequencies to simultaneously "carry" P different messages (from one or more transmitters) over a common, shared communication medium. This form of sharing is termed *frequency-domain sharing* (aka "frequency sharing") or *spectral-domain sharing*.[1] Frequency sharing eliminates contention, so there are no collisions, but at the same time, dedicates frequencies to different transmissions whether or not they are used, so when traffic loads are skewed, the peak data transfer rate is generally lower than with contention protocols. In practice, wireless networks use a combination of time and frequency sharing, as we will see in a case study on 802.11 (WiFi) networks later in the course.

To understand these trade-offs, we will need a new mathematical tool, the **Fourier series**. Even though we will focus on using the Fourier series to analyze only frequency-domain sharing, Fourier series appear in an enormous variety of applications including quantum mechanics, electromagnetics, video, audio, and image compression, semiconductor transport, magnetic-resonance imaging (MRIs), and crystallography, to name just a few.

■ 14.1 Spectral-Domain Channel Sharing, Once over "Lightly"

A good way to understand some of the issues in spectral-domain sharing is to consider the visible spectrum, colors from red to violet, corresponding to frequencies roughly in the 4×10^{14} to 8×10^{14} hertz (400 to 800 *terahertz*) range. If two users want to simultaneously send different messages over some distance using high powered lamps, they can use different colors. The first transmitter could send a message by turning on and off a red lamp, and the second transmitter could send a message by turning on an off a green lamp. Over

[1] We will use the two terms interchangeably.

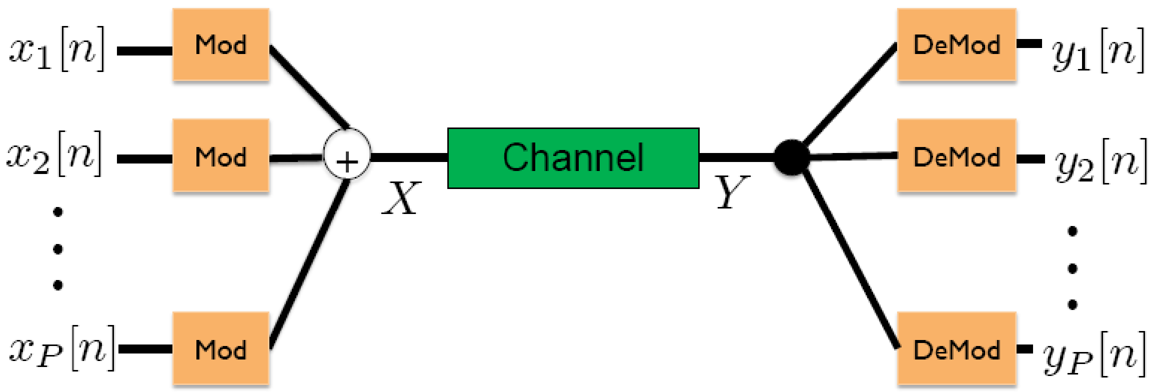

Figure 14-1: A Diagram of frequency-domain sharing.

time, a distant receiver will see a changing mixture of red, green, and yellow, but will be able to untangle the messages from the two transmitters by "pulling out" the frequency of interest. If there are many transmitters, all using different colors, then the receiver may see what looks like white light, but with the aid of a simple glass prism, the receiver will be able to separate out each of the colors and determine each of the messages.

In our "colorful" example, one of the transmitters digitally **modulates** the information they wish to send over the red light by turning on an off a red lamp, where "on" can indicate a logical 1, and "off" a logical 0. The red light **carries** the modulated digital message and is said to be the **carrier**. When a receiver uses a prism to separate out the different colors, and then converts a specific color's intensity changes back in to the digital data, we refer to the process as **demodulation**. Each other transmitter uses a different colored light to carry its modulated signal. Figure 14-1 summarizes, in diagram form, the process of modulation, transmission across a channel, and demodulation, for the case of P transmitters and receivers sharing a single physical channel.

Spectral-domain sharing using different colors (wavelengths of light) is actually a commonly used approach to transfer data over optical fibers. The method is referred to as wavelength-division multiplexing (WDM), and modern fiber-optic communication systems often use a hundred different wavelengths, corresponding to a hundred different frequencies or a hundred different colors. With current technology, each color can carry data through the optical fiber at a rate of 100 gigabits per second, yielding a net transfer rate of 10 *terabits* per second.

In the case of fiber, it is tempting to suggest that one should just keep adding channels, to make the net data transfer rate approach infinity. Unfortunately, for fiber, whose carrier frequencies are on the order of 10^{14} hertz, there are technological problems that limit the number of carriers. For wireless transmission systems, whose carrier frequencies are on the order of 10^9 hertz (i.e., one to a few gigahertz), current technology can easily achieve a more fundamental limit: for wireless systems, there is a fundamental trade-off between the distance between carrier frequencies, and the maximum data rate per carrier.

To understand these limitations, we will need a new tool, the Fourier series. The Fourier series is a way to express any periodic sequence as a weighted sum of cosines and sines or, equivalently, as a weighted sum of complex exponentials (taking advantage of the familiar identities relating sines and cosines to complex exponentials).

Before describing the Fourier series, we discuss the notion of discrete frequencies and how they arise from a continuous waveform.

■ 14.2 Discrete Frequency

If we have some periodic function in continuous time, t, say, $x(t) = cos(2\pi f t)$, we say that it has a frequency, f, and a period, $T = \frac{1}{f}$. The notion of frequency for such a periodic continuous waveform is easy to interpret in part because it is well-defined for all values of t. But how do we go about defining frequency for a discretized function?

To discretize this continuous function into a set ot discrete voltage samples, we sample it at some other *sampling frequency*, f_s, resulting in a discrete sequence

$$x[n] = cos(2\pi f t)\mid_{t=nT_s} = cos(2\pi f T_s n), \tag{14.1}$$

where $T_s = \frac{1}{f_s}$. The discrete sequence, X defined by the $x[n]$ values, is said to have a discrete frequency, Ω, defined as

$$\Omega = 2\pi f T_s = \frac{2\pi f}{f_s}.$$

We can therefore relate the continuous frequency f to its discrete equivalent Ω, in terms of the sampling frequency used to discretize the continuous waveform, f_s.

Furthermore, it suffices to consider discrete frequencies Ω in the range $[-\pi, \pi)$. The reason is that for any frequency outside this range, there is an equivalent frequency within this range. To see why, suppose $x[n] = e^{j(\pi+\phi)n}$ and $\phi \in (0, 2\pi)$, which defines a sequence with frequency outside the range. Because $e^{j\pi n} = e^{-j\pi n}, x[n] = e^{j(\phi-\pi)n}$, which is a frequency in the range $[-\pi, \pi)$. If $\phi > 2\pi$ or $\phi < 0$, we can simply substract out the largest integer multiple of 2π and apply the same argument, because $e^{j2\pi n} = 1$ for all integers n.

■ 14.3 Periodicity and the Fourier Series

The Fourier series can represent any periodic sequence, that is, any sequence for which there is some finite N such that

$$x[n + N] = x[n], \;\; \forall n \in (-\infty, \infty). \tag{14.2}$$

The assumption of periodicity is not as limiting as it seems. One can make a periodic sequence out of a any finite-length sequence, just by repeating the sequence, as shown for $N = 400$ in Figure 14-2. In addition, we can assume that N is even (if N were odd, we can just double it to produce an even N for which Eq. (14.2) holds).

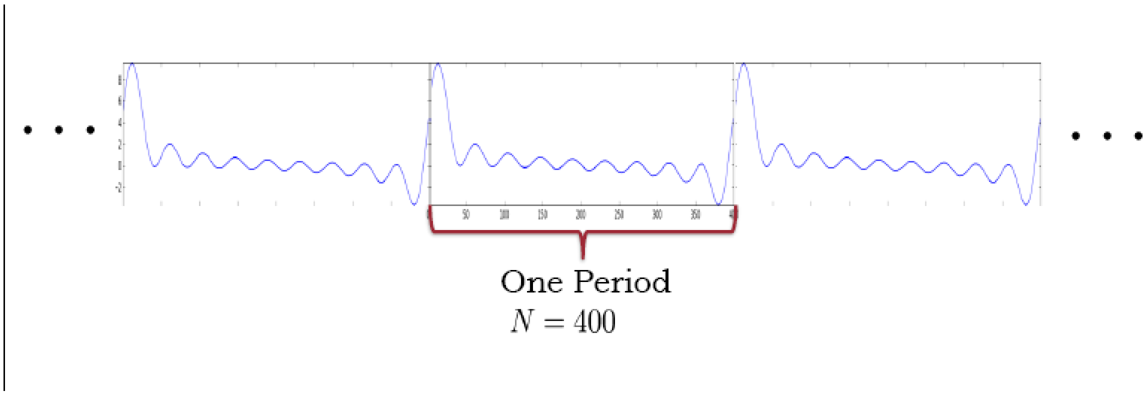

Figure 14-2: Making a finite sequence periodic.

14.3.1 Discretized Frequencies

There are some limitations imposed by assuming periodicity. In particular, the frequencies of sines and cosines are restricted to discrete values. We already know that for cosine and sine sequences with frequency Ω, we can limit consideration to $-\pi \leq \Omega < \pi$. If the sequence is now also periodic with period N, Ω must satisfy an additional constraint. To see this, consider

$$e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}, \;\; \forall n \in (-\infty, \infty) \tag{14.3}$$

and therefore $e^{j\Omega N} = 1$. There are only certain values of Ω for which $e^{j\Omega N} = 1$. We can now say that an N-periodic sine or cosine must have a frequency in the set

$$\Omega \in \{0, \pm\frac{2\pi}{N}, \pm 2\frac{2\pi}{N}, \ldots, \pm\left(\frac{N}{2} - 1\right)\frac{2\pi}{N}, \pm\pi\}. \tag{14.4}$$

14.3.2 The Fourier Series

Any periodic function can be exactly represented with a Fourier series, a statement we will not prove here (the proof is given in the annotated slides of lecture 14). Instead, we will state the Fourier Series Theorem in the form we will find most useful. For any periodic sequence, X, with period N, there exists a representation of that sequence as a sum of complex exponentials. That is,

$$x[n] = \sum_{k=-K}^{K-1} X[k] e^{j\frac{2\pi}{N}kn} \;\; K = \frac{N}{2}, \tag{14.5}$$

or to simplify notation,

$$x[n] = \sum_{k=-K}^{K-1} X[k] e^{j\Omega_k n} \;\; \Omega_k = \frac{2\pi}{N} k, \tag{14.6}$$

where the complex number, $X[k]$, is the referred to as the Fourier coefficient associated with Ω_k.

Note that the Fourier series has $2K = N$ *complex* coefficients, $X[k]$, $-\frac{N}{2} \leq K \leq \frac{N}{2} - 1$, but there are only N unique *real* values for X, $x[0], ..., x[N-1]$. It may seem like there are too many Fourier coefficients (because the Fourier coefficients have a real and an imaginary part), but that is not the case. As shown in the Lecture 14 annotated notes, the real part of the Fourier coefficients are an even function of k, and the imaginary part of the Fourier coefficients is an odd function of k,

$$real(X[k]) = real(X[-k]) \quad imag(X[k]) = -imag(X[-k]). \tag{14.7}$$

■ 14.4 Modulation and Demodulation

Modulating is defined as multiplying our input X (whose nth value is $x[n]$) by a cosine (or sine) sequence. If we multiply $x[n]$ by a cosine of frequency Ω_m, we can observe what happens to the Fourier coefficients. The product can be represented by two copies of the Fourier series representation for $x[n]$, one shifted up by Ω_m and one shifted down by Ω_m, and each scaled by $\frac{1}{2}$. Mathematically,

$$x[n]\cos\Omega_m[n] = \left(\sum_{k=-K}^{K-1} X[k]e^{j\Omega_k n}\right)\left(\frac{1}{2}e^{j\Omega_m n} + \frac{1}{2}e^{-j\Omega_m n}\right), \tag{14.8}$$

which can be simplified to

$$= \frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k+\Omega_m)n} + \frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k-\Omega_m)n}. \tag{14.9}$$

Equation 14.9 shows that the Fourier coefficients of the product are exactly the Fourier coefficients of $x[n]$ ($X[K]$), scaled and at the new frequencies of Ω_k+Ω_m and Ω_k-Ω_m.

If the channel is ideal, so that $Y = X$, then demodulation by multiplying by $\cos\Omega_m n$ is given by

$$\left(\frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k+\Omega_m)n} + \frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k-\Omega_m)n}\right)\left(\frac{1}{2}e^{j\Omega_m n} + \frac{1}{2}e^{-j\Omega_m n}\right), \tag{14.10}$$

which can be simplified to

$$\frac{1}{4}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k+2\Omega_m)n} + \frac{1}{4}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k-2\Omega_m)n} + \frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k)n}. \tag{14.11}$$

As is clear from (14.11), the process of multiplying by a cosine to modulate, and then a cosine to demodulate, results in a version of the original Fourier series for X, scaled by $\frac{1}{2}$, and two copies of the Fourier series representation for X, one shifted up by $2\Omega_m$ and one shifted down by $2\Omega_m$, with each scaled by $\frac{1}{4}$. If X is bandlimited, so that $X[k] = 0$ whenever $|\Omega_k| \geq |\Omega_m|$, then the three sums in (14.11) have no overlapping terms (note: it must also be true that $3 * |\Omega_m| \leq \pi$ to avoid "wrap-around"). Then, X can be recovered with a low-pass filter.

If demodulation is performed by multiplying by $\sin \Omega_m n$, then

$$\left(\frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k+\Omega_m)n} + \frac{1}{2}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k-\Omega_m)n}\right)\left(-\frac{j}{2}e^{j\Omega_m n} + \frac{j}{2}e^{-j\Omega_m n}\right), \quad (14.12)$$

which can be simplified to

$$\frac{-j}{4}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k+2\Omega_m)n} + \frac{j}{4}\sum_{k=-K}^{K-1} X[k]e^{j(\Omega_k-2\Omega_m)n} \quad (14.13)$$

and there is no unshifted version of the Fourier series of X to low-pass filter.