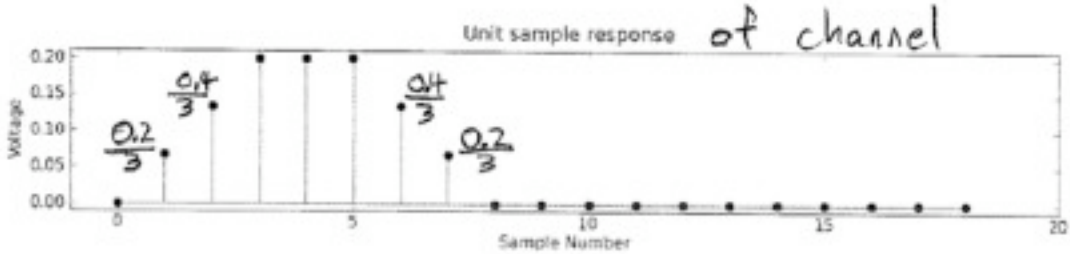
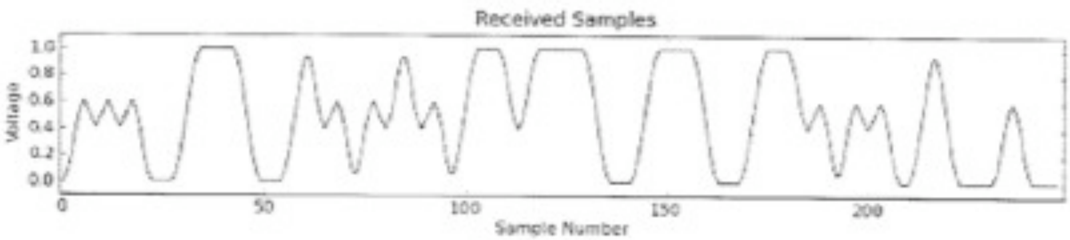
Example of Bit Error Rate (BER) Calculation with Intersymbol Interference (ISI)

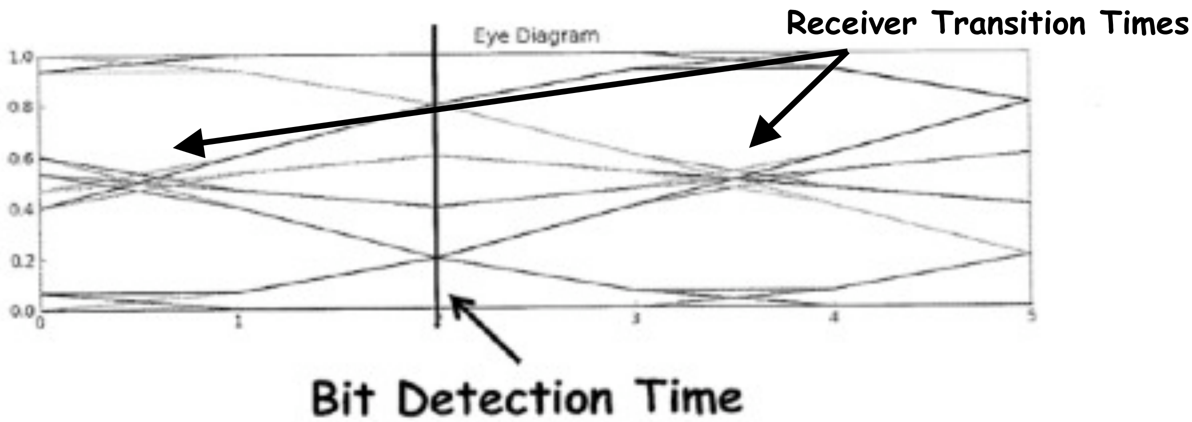
Suppose we have a channel with the following unit sample response, h[n]:



If we look at the output for an input stream of 3 Samples/bit, we see the following at the receiver:



By looking at this output, we can clearly see the intersymbol interference (ISI) apparent by the received samples not able to reach the min or max voltage value before transitioning to the next sample value. And if we look at the eye diagram,

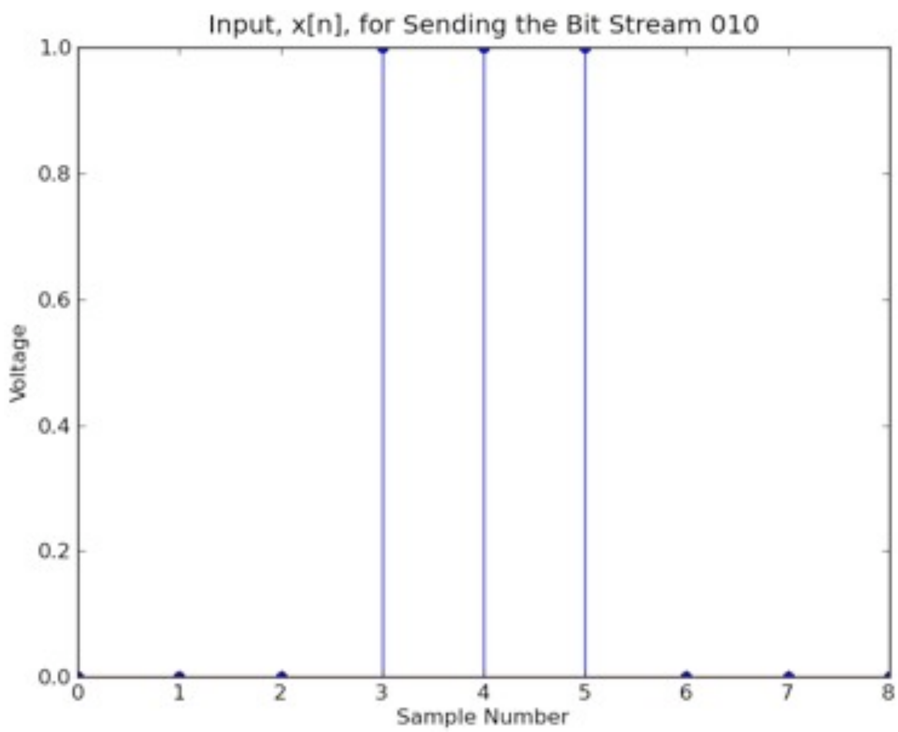


we can see that at the bit detection time, the received sample voltage can take on 6 different values. Remember, thus far we have not included any noise. We are looking at the noise free channel. The voltage value of the received sample is dependent on the previously transmitted bit, the current bit, and the next bit.

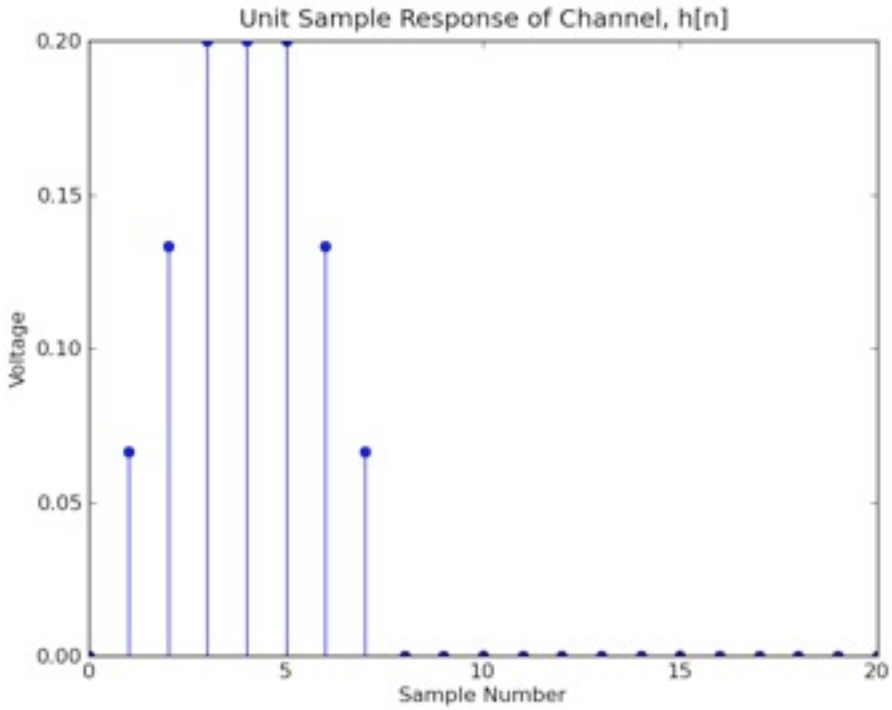
We can build a table describing the sampled voltages as such:

Previous Bit	Current Bit	Next Bit	Bit Detection Sample Voltage
0	0	0	0.0
0	0	1	0.2
1	0	0	0.2
1	0	1	0.4
0	1	0	0.6
0	1	1	0.8
1	1	0	0.8
1	1	1	1.0

How did we calculate the sample voltages? Looking at the eye diagram, we can approximate what we think these voltages could be, but we want precision. To calculate the voltage values the bit decision will have, we need to convolve the input with the channel unit sample response shown above and look at the output. For example, let's look at the case where the previous bit was a 0, the current bit is a 1, and the next bit is a 0. Our input looks like the following (remember our system specified 3 samples/bit):



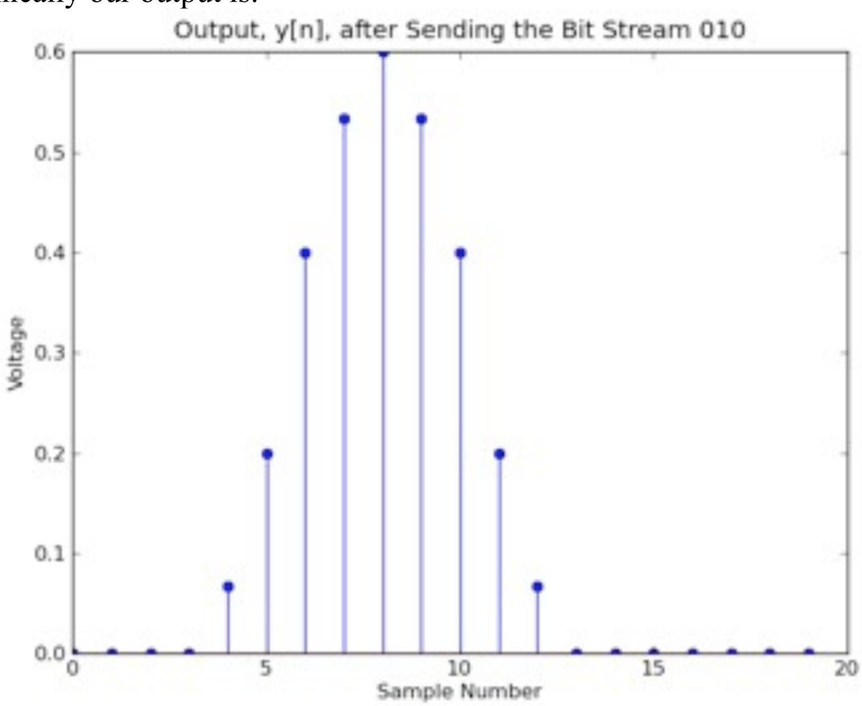
Once again, our unit sample response is shown as:



We can now convolve both of these signals to obtain the output, or in equation form,

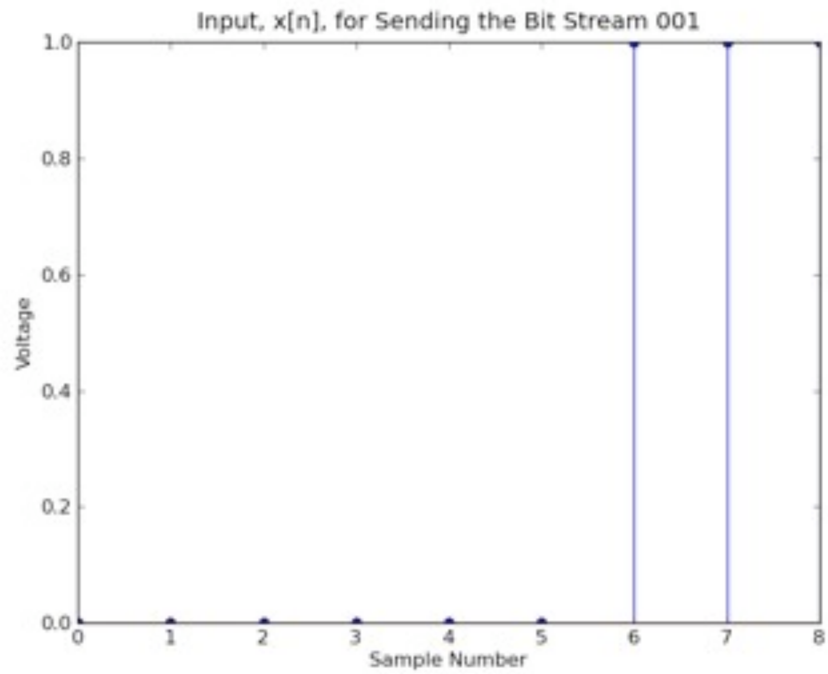
$$y[n] = x[n] * h[n] = \sum_{m=0}^{n} x[m] \cdot h[n-m].$$

And graphically our output is:

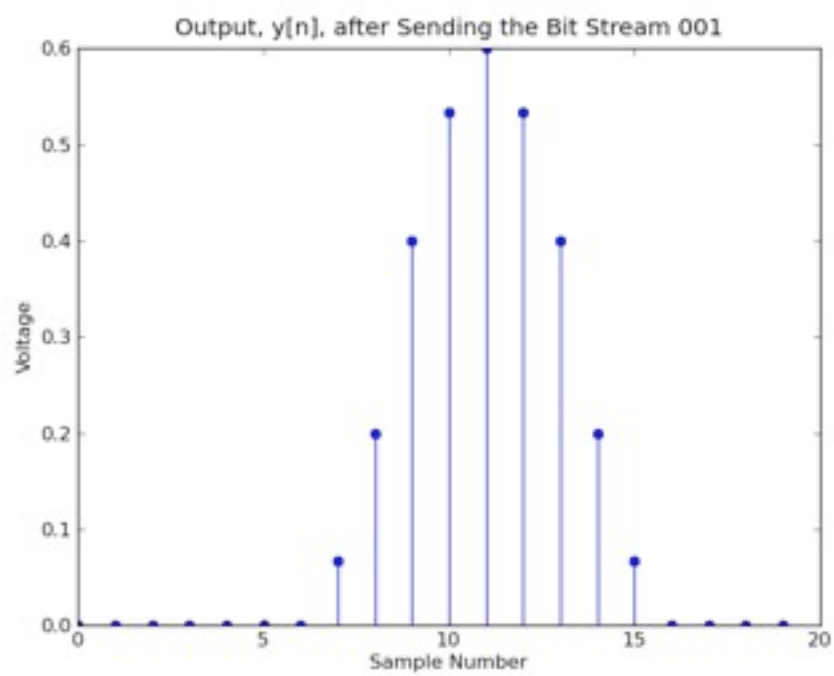


This plot shows the outputs of the input stream ‘010’ and looking at sample 8, we see that the received voltage value for the transmitted current bit ‘1’ is 0.6.

Just for fun, let’s repeat this with a different input stream, ‘001’. Here is our input:



and here is our output:



Again, if we look at sample 8, we see that our sample voltage is 0.2 V. Why are we looking at sample 8? It's really all about when your bit detection time is. Samples 4-6 represent the out at the receiver from the previous bit, 7-9 for the current bit, and 10-12 for the next bit. So, just like we did in lab, we choose to look at the middle sample in the current bit to determine what our voltage sample is. If you look at the convolution graphically, the output value at sample 8 is actually the output when the current bit overlaps exactly with the three peak voltages spikes in our unit sample response. In fact, in lab 3 you explore which voltage sample is the right choice for doing bit detection.

By doing this for every sequence of inputs we obtain the table shown here again:

Previous Bit	**Current Bit**	**Next Bit**	**Bit Detection Sample Voltage**
0	0	0	0.0
0	0	1	0.2
1	0	0	0.2
1	0	1	0.4
0	1	0	0.6
0	1	1	0.8
1	1	0	0.8
1	1	1	1.0

Can you verify these voltage values and convince yourself they are correct?

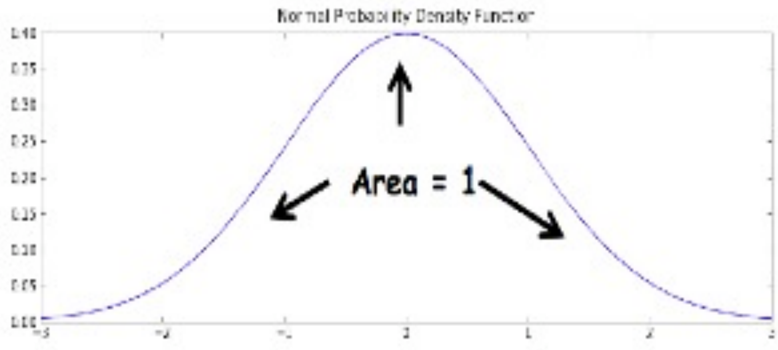
We can now start to think about calculating the bit error rate (BER). Our bit error rate is going to be the sum of the probabilities for each sequence of possible inputs multiplied by the probability of an error in the received current bit. First let's review our notation. We write the probability of transmitting the sequence *pcn* as $P_{(ncp)}$, where n is the next bit, c is the current bit, and p is the previous bit. And $P_{(ncp)\bar{c}}$ is the probability that an error in the current bit occurred. So we can write the BER for our example as:

$$BER = P_{(000)}P_{(000)1} + P_{(001)}P_{(001)1} + P_{(100)}P_{(100)1} + P_{(101)}P_{(101)1} + P_{(010)}P_{(010)0} + P_{(011)}P_{(011)0} + P_{(110)}P_{(110)0} + P_{(111)}P_{(111)0}$$

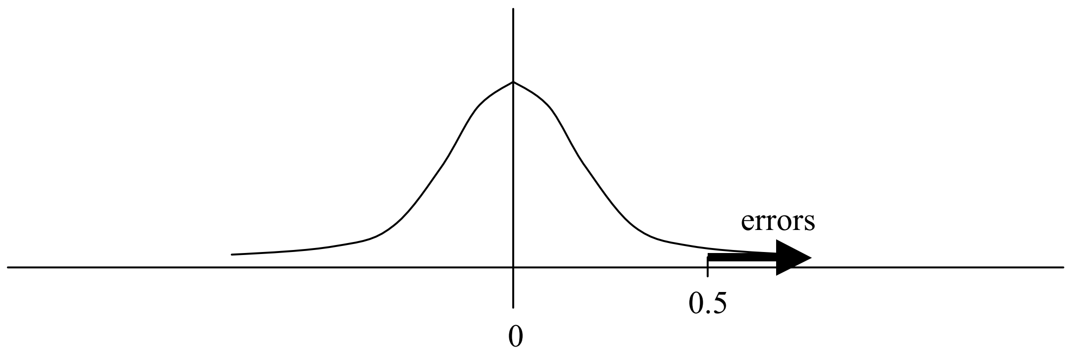
If we assume all input streams are equally likely to be sent, we can simplify this to

$$BER = \frac{1}{8}\left(P_{(000)1} + P_{(001)1} + P_{(100)1} + P_{(101)1} + P_{(010)0} + P_{(011)0} + P_{(110)0} + P_{(111)0}\right).$$

Before we can finish our calculation, we need to know what our noise looks like and how it is distributed. Let us assume our noise has a standard normal distribution so our PDF looks like a Gaussian curve with a $\mu=0$ and a $\sigma=1$:



Let's begin by calculating the first term in our BER, $P_{(000)1}$. We look at the *noise free* + *noise* distribution and find that we have the noise distribution above centered around 0.0 and our error is the part of the Gaussian tail greater than $0.5 = V_{thresh}$.



So, $P_{(000)1} = 1 - CDF(.5)$. Remember that our CDF is the cumulative distribution function and is defined in relation to the probability distribution, PDF:

$$CDF(x_1) = P(X \le x_1) = \int_{-\infty}^{x_1} f_X(x)dx,$$

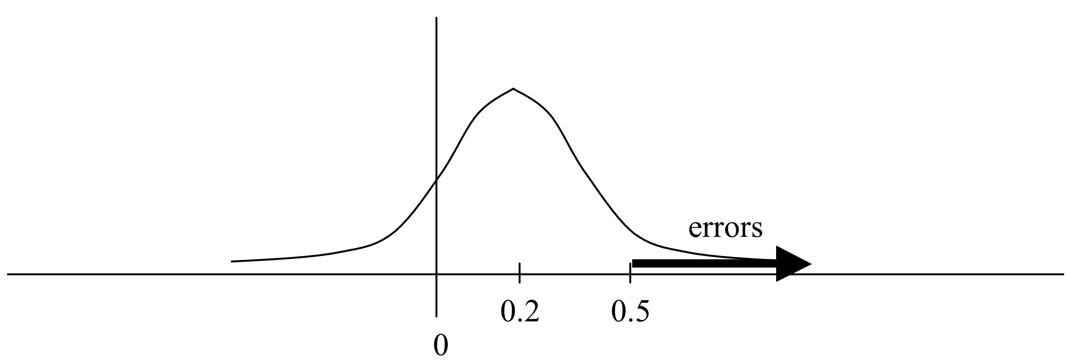
where $f_X(x)$ is the PDF. Also, it is useful to note that

$$1 - CDF(x_1) = P(X \ge x_1) = \int_{x_1}^{\infty} f_X(x)dx,$$

and

$$1 - CDF(x_1) = CDF(-x_1).$$

Continuing for the next term, $P_{(001)1}$, we again draw the *noise free* + *noise* distribution:



This time, even though our current bit is still a '0' our noise distribution is not centered around 0 because our noise free signal gives us the nominal sampled voltage of 0.2. Thus our noise distribution is added to our noise free distribution and looks shifted. Our error still occurs for any voltage greater than 0.5 and $P_{(000)1} = 1 - CDF(0.5 - 0.2) = 1 - CDF(0.3)$.

Continuing in this method will result in:

$$BER = \frac{1}{8}\left(P_{(000)1} + P_{(001)1} + P_{(100)1} + P_{(101)1} + P_{(010)0} + P_{(011)0} + P_{(110)0} + P_{(111)0}\right)$$
$$= \frac{1}{8}\begin{pmatrix} [1 - CDF(.5)] + [1 - CDF(.3)] + [1 - CDF(.3)] + [1 - CDF(.1)] + [CDF(-.1)] \\ +[CDF(-.3)] + [CDF(-.3)] + [CDF(-.5)] \end{pmatrix}$$

Each term in the bottom equation corresponds to the terms in the top equation. Can you verify this result?

We can simplify further to obtain the following result:

$$BER = \frac{1}{4}\left(\left[1 - CDF(.5)\right] + 2 \cdot [1 - CDF(.3)] + [1 - CDF(.1)]\right)$$
$$\approx \frac{1}{4}\left([1 - .6915] + 2 \cdot [1 - .6179] + [1 - .5398]\right)$$
$$\approx .383225$$