Rec. 6

A review (of material relevant to Quiz 4, plus some
excursions), using Prob. 2 & Lab 3 problems as the
vehicle:

**Deconvolution**

\[
x[n] \rightarrow h[n] \rightarrow y[n] = h[0]x[n] + h[1]x[n-1] + \ldots + h[L]x[n-L]
\]

Causal
LTI,
assume unit sample
response (USR) has
finite duration \(0 \to L\)
(i.e., \(h[k] = 0\) for \(k \notin [0, L]\))

Assume for now this model perfectly represents \(x, y\) relation

\[
L \xrightarrow{\text{Suppose for now that}} \frac{1}{h[0]} \xrightarrow{\text{LTI}} \frac{1}{h[0]} y[n] = \frac{1}{h[0]} (y[n] - h[1]x[n-1] - \ldots - h[L]x[n-L])
\]

Rearrange:

\[
x[n] = \frac{1}{h[0]} \left( y[n] - h[1]x[n-1] - \ldots - h[L]x[n-L] \right)
\]

This is a system that takes in the sequence \(y[n]\) values and produces the sequence \(x[n]\) values:

\[
x[n] \xrightarrow{\text{deconvolve}} y[n]
\]

Start from \(n = 0\),
assuming \(x[k] = 0\) for at least \(x[-L] \to x[-1]\).

Is the deconvolver (i.e., is Eq. (**)) time-

invariant? causal? linear?

Yes, only uses present and past values of \(y[k]\) to generate \(x[n]\)

Yes, rule by which input \(y[n]\) is processed to give \(x[n]\) is fixed, no dependence on \(n\).
As for linearity, (***) is a linear constraint among the variables, so superposition holds, i.e., if \( x[n] = x'[n] \), \( y[n] = y'[n] \) is one set of signals satisfying (***) and if \( x^\beta[n], y^\beta[n] \) is another set, then \( c_1 x^\beta[n] + c_2 x'[n], c_1 y^\beta[n] + c_2 y'[n] \) will also satisfy (***) for any scalars \( c_1, c_2 \).

So the deconvolver here is a causal LTI system, whose unit sample response we can denote by \( g[n] : y[n] \xrightarrow{g[n]} x[n] \).

How to find \( g[n] \)?

One way: Use (***): set \( y[n] = s[n] \), then \( x[n] \) will be \( g[n] \). Specifically,

\[
    g[0] = \frac{1}{h[0]},
\]

\[
    g[1] = -\frac{1 - h[1]}{h[0]} h[0],
\]

\[
    g[2] = \frac{1}{h[0]} \left[ \frac{h[2]}{h[0]} \right]^2 - \frac{h[2]}{h[0]},
\]

etc., etc.

May not seem very revealing, but already suggests that powers \( \frac{-h[m]}{h[0]} \) will appear \((m = 1 \text{ to } L)\). Could mean \( g[n] \) will grow in magnitude if \( \left| \frac{h[m]}{h[0]} \right| > 1 \) for some \( m \).
Another way to think of $g[n]$:

Channel model (assumed perfect, for now)

Decovolver, constructed using knowledge of $h[n]$.

Reproduces $x[n]$ at output, no matter what the input $x[n]$ is. So if $x[n] = s[n]$, then $y[n] = h[n]$, and $x[n]$ at output again = $s[n]$. So we learn that:

$$h[n] \rightarrow g[n] \rightarrow s[n] \rightarrow h * g = s \quad (***)$$

Convolving $h[]$ with $g[]$ gives $s[n]$.

This is what a decovolver does. The decovolver "inverts" the convolution (hence also called the inverse system).

A simple example:

Suppose

$$h[n] = s[n] + 2s[n-1]$$

i.e.

$$y[n] = x[n] + 2x[n-1] \rightarrow (*) \text{ for this example}$$

and (rearranging) we get the inverse system:

$$x[n] = y[n] - 2x[n-1] \rightarrow (**) \text{ for this example}$$

We know $g[n]$ is causal:

Find $g[0]$, $g[1]$, $g[2]$, ... using (***)
Since \( h \ast g = s \), and using the ‘flip, slide, inner product’ operation to compute the convolution, we get:

\[
\begin{align*}
h[0]g[0] &= 1 \implies g[0] = 1 \\
h[1]g[0] + h[0]g[1] &= 0 \implies g[1] = -2 \\
\end{align*}
\]

etc.

\( g[n] = \left(\frac{1}{2}\right)^n u[n] \) (Another way to get this: use (**) with \( y[n] = s[n] \), solve forwards.)

Deconvolver has an exponentially growing unit sample response! Doesn’t seem like a good system to work with.

(In the language of 6.003, it’s not “bounded-input bounded-output” stable — a bounded input can produce an unbounded output. The telltale sign is that \( \sum_{n=-\infty}^{\infty} |g[n]| \) is not finite.)

Here's the problem with the deconvolver:

By superposition, output

\[
w[n] = x[n] + \text{the component contributed by }
\]

inevitable nonidealty

\[
\text{For unstable inverse system } g[-n], \quad \implies g \ast \text{noise}
\]

This blows up!!
Now consider the specific $h[n]$ from Prob. 2.

In the noise-free case,

$$
y[n] = 0.1 x[n] + 0.2 x[n-1] + 0.1 \sum_{m=2}^{n} x[n-m] \tag{1}
$$

$$
\Rightarrow x[n] = 10 y[n] - 2 x[n-1] - \frac{1}{8} \sum_{m=2}^{n} x[n-m] \tag{**}
$$

This is the $h[n]$ equation in Prob. 2.

Is the $g[n]$ that goes with this deconvolver well-behaved (stable)? Using $h * g = x$, or alternatively choosing $y[n] = 8[n]$ in (**), and seeing what the corresponding $x[n] = g[n]$ is, we find $g[n]$, i.e. steadily growing in magnitude, unstable.

So in the presence of noise, i.e. for $w[n]$, this will do poorly.

The essence of the problem in (***) is that

$$
\left| \frac{h[1]}{h[0]} \right| = 2 > 1,
$$

$g[n]$ tends to grow in magnitude by around this factor at each step. So consider approximating $h[n]$ by the following:

$$
0.2 h_2[n]
$$

i.e., $h_2[0] = 0$ instead of 0.1, and $h_2[n] = h[n]$ for $n \neq 0$.

What is the deconvolver $g_2[n]$ (or inverse system) for $h_2[n]$?
Note first that there is no causal \( g_2[n] \) such that \( h_2 * g_2 = \delta \), because that would require \( h_2[0] \cdot g_2[0] = 1 \), impossible.

But a slightly noncausal \( g_2[n] \) will work, i.e.,

\[
\begin{align*}
\text{For this we need:} & \quad h_2[1] \cdot g_2[1] = 1 \\
h_2[2] \cdot g_2[1] + h_2[1] \cdot g_2[0] = 0 \\
h_2[3] \cdot g_2[1] + h_2[2] \cdot g_2[0] + h_2[1] \cdot g_2[-1] = 0 \\
\text{etc.}
\end{align*}
\]

\[ g_2[-1] = 5, \quad g_2[0] = -2.5, \quad g_2[1] = -1.25 \]

etc. — seems better behaved as a deconvolution.

To recover \( x[n] \) from \( y[n] \) if we had

\[ x[n] \rightarrow \begin{array}{c}
\mathbf{h}_2[n] \\
\end{array} \rightarrow y[n] \]

we would start with


and solve for \( x[n-1] \). This is exactly where the \( \mathbf{W}_2[n] \) equation comes from for the second deconvolver.

In Prob 2:

\[ \mathbf{W}_2[n-1] = \frac{1}{h_2[1]} \left( y[n] - h_2[2] \mathbf{W}_2[n-2] - \ldots - h_2[8] \mathbf{W}_2[n-8] \right) \]

or

\[ \mathbf{W}_2[n-1] = 5y[n] - \frac{1}{2} \sum_{m=0}^{8} h_2[n-m] \mathbf{W}_2[n-m] \]  \( \ast \ast \)

Note non-causality — we need \( y[n] \) to determine \( \mathbf{W}_2[n-1] \), i.e., we generate our estimate \( \mathbf{g} \cdot x[n] \) with a one-sample delay.
Looking at the response of \( W_2[n] \), we notice that it's a little more 'unsettled' than the response of the next deconvolver, \( W_3[n] \), and also does not quite settle at 1 when \( y[n] \) is 1.

What is the steady-state value of \( W_2[n] \) if \( y[n] \) is kept at 1? Set \( W_2[n] = \overline{W_2} \), constant, and \( y[n] = 1 \) in (**) on previous page:

\[
\overline{W_2} = 5 - \frac{1}{2} \cdot 7\overline{W_2} \implies \overline{W_2} = \frac{10}{9}, \text{ i.e.,}
\]

the values settle slightly higher than 1. The reason the steady-state is \( \frac{10}{9} \) is because \( \sum h_2[m] = 0.9 \neq \sum h[m] = 1.0 \), i.e., our approximation to \( h[n] \) has a unit step response that settles to \( \frac{9}{10} \) instead of 1. To fix this, try a different approximation to \( h[n] \):

Now \( \sum h_3[m] = \sum h[m] = 1 \),

so unit step response settles to 1.

Proceeding as before, the associated deconvolver is (as in Prob. 2):

\[
W_3[n-1] = \frac{10}{3} y[n] - \frac{1}{3} \sum_{m=2}^{\infty} W_3[n-m] \quad (**)
\]

Comparing with (**) on p. 6, we see this is more stable, because of the factor \( \frac{1}{3} \) that weights past values of \( W_3[n] \), versus \( \frac{1}{2} \) for \( W_2[n] \).

--- I leave you to find \( g_3[n] \).