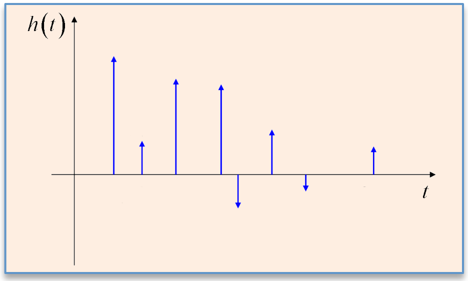
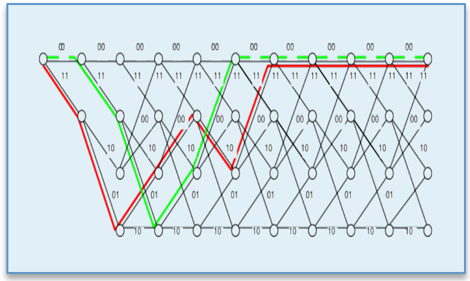
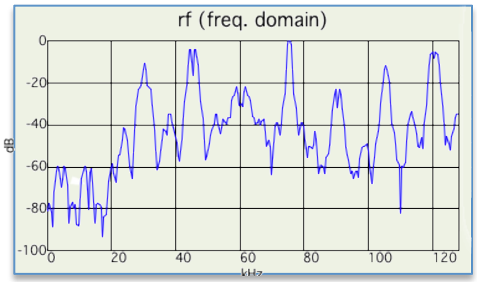
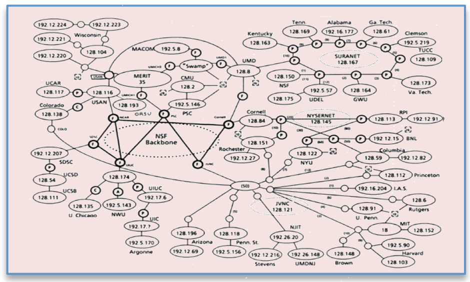








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2011
Lecture #12

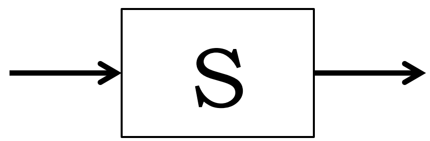
- Alternative ways to look at convolution
- Frequency response
- Filters

Convolution

From last lecture: If system S is both linear and time-invariant (LTI), then we can use the unit sample response h[n] to predict the response to *any* input waveform x[n]:

Sum of shifted, scaled unit sample functions

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

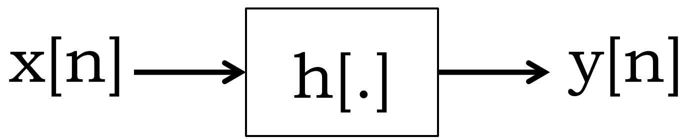


Sum of shifted, scaled unit sample *responses*, with the same scale factors

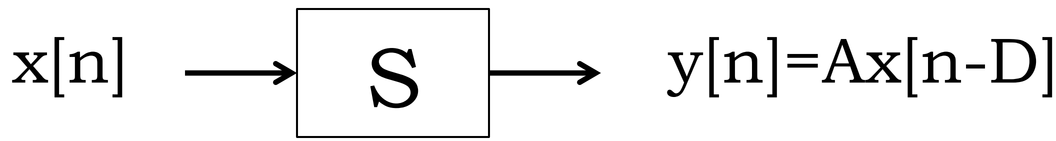
$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CONVOLUTION SUM

Indeed, the unit sample response h[n] completely characterizes the LTI system S, so you often see



Unit Sample Response of a Scale-&-Delay System



If S is a system that scales the input by A and delays it by D time steps (negative 'delay' D = advance), is the system

time-invariant? Yes!

linear? Yes!

Unit sample response is $h[n]=A\delta[n-D]$

General unit sample response

$$h[n]=\ldots + h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1]+\ldots$$

for an LTI system can be thought of as resulting from many scale-&-delays in parallel

A Complementary View of Convolution

So instead of the picture:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{h[.]}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

we can consider the picture:

$$\text{x[n]} \longrightarrow \boxed{\text{h[.]}=\ldots+\text{h[-1]}\delta\text{[n+1]}+\text{h[0]}\delta\text{[n]}+\text{h[1]}\delta\text{[n-1]}+\ldots} \longrightarrow \text{y[n]}$$

from which we get $$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

(To those who have an eye for these things, my apologies for the varied math font --- too hard to keep uniform!)

(side by side)

$$y[n] =$$

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad = \quad \sum_{m=-\infty}^{\infty} h[m]x[n-m] = (h * x)[n]$$

Input term x[0] at time 0 launches scaled unit sample response x[0]h[n] at output

Unit sample response term h[0] at time 0 contributes scaled input h[0]x[n] to output

Input term x[k] at time k launches scaled shifted unit sample response x[k]h[n-k] at output

Unit sample response term h[m] at time m contributes scaled shifted input h[m]x[n-m] to output

To Convolve (but not to "Convolute"!)

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

A simple graphical implementation:

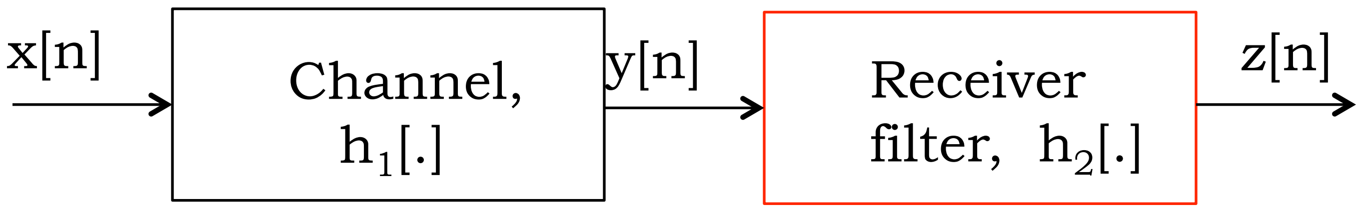
Plot x[.] and h[.] as a function of the dummy index (k or m above)

Flip (i.e., reverse) one signal in time, **slide** it right by n (slide left if n is –ve), take the **dot.product** with the other.

This yields the value of the convolution at the single time n.

'flip one & slide by n …. dot.product with the other'

"Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

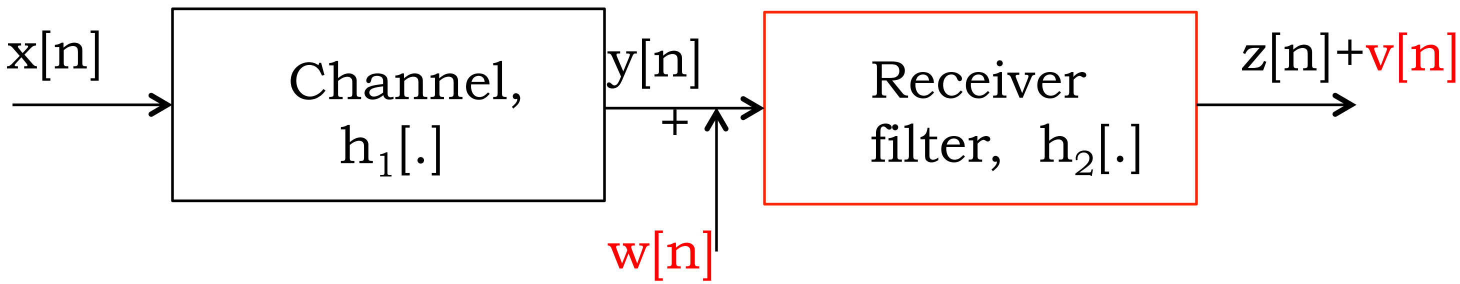
$$h_1[n]=\delta[n]+0.8\delta[n-1]$$

Find $h_2[n]$ such that $z[n]=x[n]$

$$\Rightarrow \quad h_2 * h_1[n]=\delta[n]$$

Good exercise in applying Flip/Slide/Dot.Product

"Deconvolving" Output of Channel with Echo



Even if channel was well modeled as LTI and $h_1[n]$ was known, noise on the channel can greatly degrade the result, so this is usually not practical.

Time now for a Frequency-Domain Story

in which
convolution
is transformed to
multiplication,
and other
good things
happen

A First Step

Do periodic inputs to an LTI system, i.e., x[n] such that

$$x[n+P] = x[n] \text{ for all } n, \text{ some fixed } P$$

(with P usually picked to be the smallest positive integer for which this is true) yield periodic outputs? If so, of period P?

Yes! --- use Flip/Slide/Dot.Product to see this easily: sliding by P gives the same picture back again, hence the same output value.

Alternate argument: Since the system is TI, using input x delayed by P should yield y delayed by P. But x delayed by P is x again, so y delayed by P must be y.

But much more is true for Sinusoidal Inputs to LTI Systems

Sinusoidal inputs, i.e.,

$$x[n] = \cos(\Omega n + \theta)$$

yield sinusoidal outputs at the same 'frequency' Ω rads.

And observe that such inputs are not even periodic in general!

Periodic if and only if $2\pi/\Omega$ is rational, $=P/Q$ for some integers $P(>0)$, Q. The smallest such P is the period.

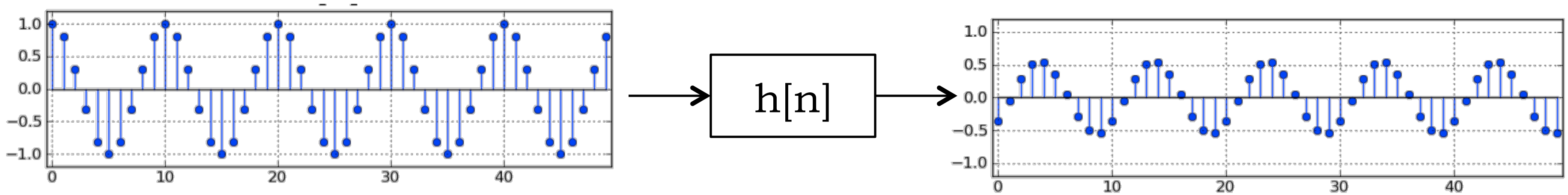
Nevertheless, we often refer to $2\pi/\Omega$ as the 'period' of this sinusoid, whether or not it is a periodic discrete-time sequence. This is the period of an underlying continuous-time signal.

Examples

$\cos(3\pi n/4)$ has frequency $3\pi/4$ rad, and period 8; shifting by integer multiples of 8 yields the same sequence back again, and no integer smaller than 8 accomplishes this.

$\cos(3n/4)$ has frequency ¾ rad, and is not periodic as a DT sequence because $8\pi/3$ is irrational, but we could still refer to $8\pi/3$ as its 'period', because we can think of the sequence as arising from sampling the periodic continuous-time signal $\cos(3t/4)$ at integer t.

Complex Exponentials and LTI Systems



A very important property of LTI systems or channels:

> If the input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

Let's prove this to be true … but use complex exponentials instead, for clean derivations that take care of sines and cosines (or sinusoids of arbitrary phase) simultaneously.

Complex Exponentials

A complex exponential is a complex-valued function of a single argument – an angle measured in radians. Euler's formula shows the relation between complex exponentials and our usual trig functions:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$

In the complex plane, $e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$ is a point on the unit circle, at an angle of φ with respect to the positive real axis. Increasing φ by 2π brings you back to the same point! So any function of $e^{j\varphi}$ only needs to be studied for φ in [-π, π] .

Useful Properties of $e^{j\varphi}$

When $\varphi = 0$:

$$e^{j0} = 1$$

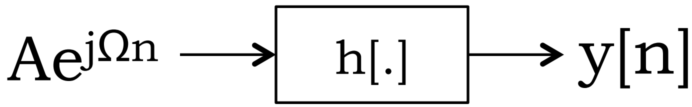
When $\varphi = \pm\pi$:

$$e^{j\pi} = e^{-j\pi} = -1$$

$$e^{j\pi n} = e^{-j\pi n} = (-1)^n$$

(More properties later)

Frequency Response



Using the convolution sum we can compute the system's response to a complex exponential (of frequency Ω) as input:

$$y[n] = \sum_m h[m]x[n-m]$$

$$= \sum_m h[m]Ae^{j\Omega(n-m)}$$

$$= \left(\sum_m h[m]e^{-j\Omega m}\right)Ae^{j\Omega n}$$

$$= H(\Omega)\cdot x[n]$$

where we've defined the *frequency response* of the system as

$$\boxed{H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}}$$

More on $H(\Omega) \equiv \sum_m h[m] e^{-j\Omega m}$

This complex function of Ω repeats periodically on the frequency (Ω) axis, with period 2π, because the input $Ae^{j\Omega n}$ is the same for Ω that differ by integer multiples of 2π.

So only the interval Ω in [−π,π] is of interest!

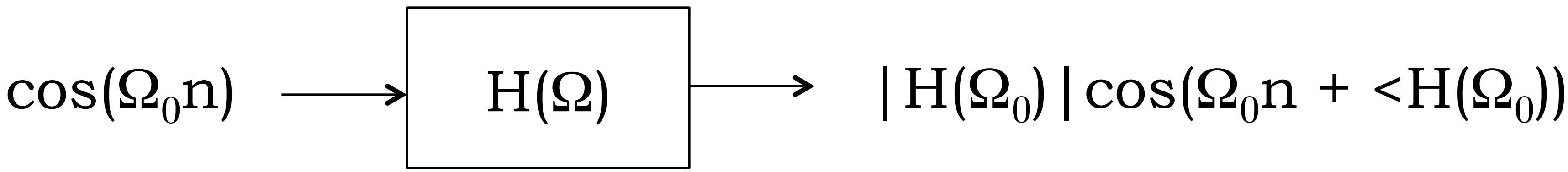
Ω = 0, i.e., $Ae^{j\Omega n} = A$, corresponds to a constant (or "DC", which stands for "direct current", but now just means constant) input, so H(0) is the "DC gain" of the system, i.e., gain for constant inputs.

Ω = π or −π, i.e., $Ae^{j\Omega n}=(-1)^n A$, corresponds to the highest-frequency variation possible for a discrete-time signal, so H(π)=H(–π) is the high-frequency gain of the system.

The notation $H(e^{j\Omega})$ is also used for the frequency response, and has the virtue of making the 2π periodicity evident, plus other advantages when being compared to transforms we will not be using in 6.02. So let's try and stick to simpler notation.

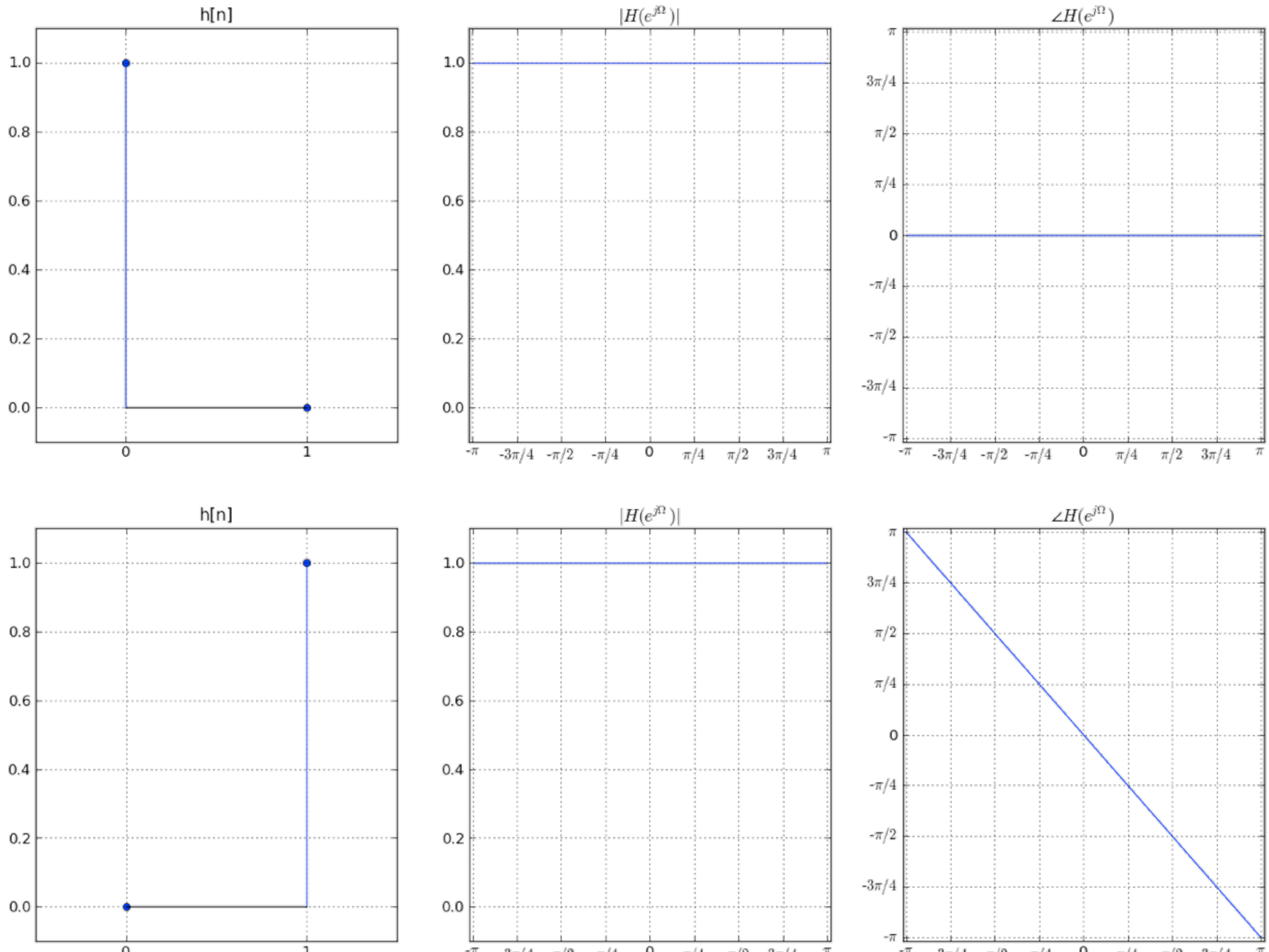
Back to Sinusoidal Inputs

Invoking the result for complex exponential inputs, it is easy to deduce what an LTI system does to sinusoidal inputs:



This is IMPORTANT

Example h[n] and H(Ω)



Frequency Response of "Moving Average" Filters