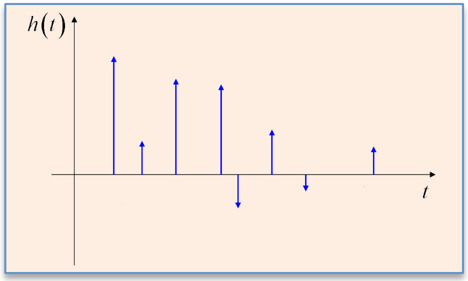
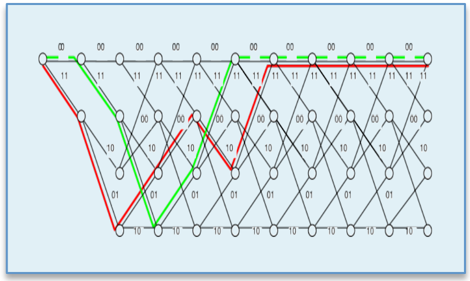
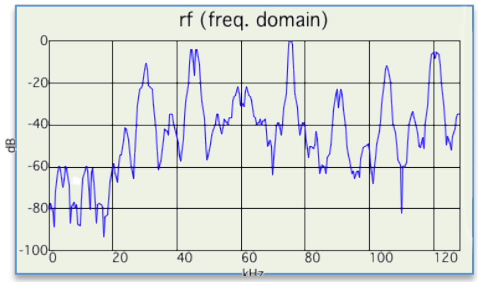
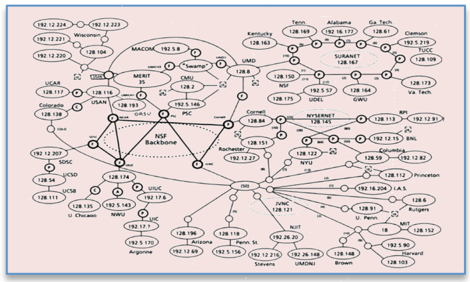




INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2011
Lecture #13

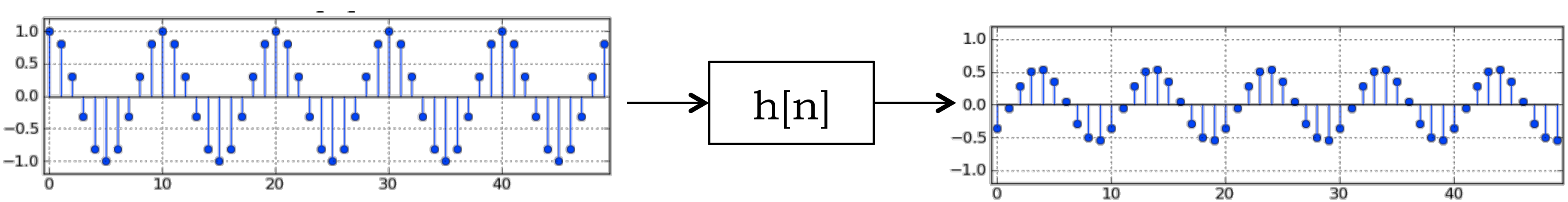
- More on frequency response
- Filters
- Determining spectral content of a periodic signal: DT Fourier Series

Sinusoidal Inputs to LTI Systems

Sinusoidal inputs, i.e.,

$$x[n] = \cos(\Omega n + \theta)$$

yield sinusoidal outputs at the same ‘frequency’ Ω rads.



Complex Exponentials

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$

In the complex plane, $e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$ is a point on the unit circle, at an angle of φ with respect to the positive real axis. Increasing φ by 2π brings you back to the same point! So any function of $e^{j\varphi}$ only needs to be studied for φ in [-π, π] .

Complex Exponentials as "Eigenfunctions" of LTI System

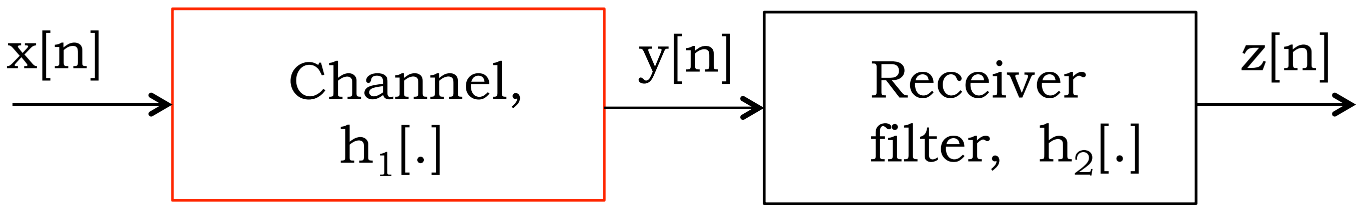
$x[n]=e^{j\Omega n}$ → h[.] → $y[n]=H(\Omega)e^{j\Omega n}$

Eigenfunction: Undergoes only scaling -- by H(Ω) in this case

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$
$$= \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)$$

This is an infinite sum in general, but is well behaved if h[.] is absolutely summable, i.e., if the system is stable.

Example: "Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

$h_1[n]=\delta[n]+0.8\delta[n-1]$

$$H_1(\Omega) = ?? = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

$$= 1+0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$ *EVEN function of* Ω;

$<H_1(\Omega) = \arctan\,[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$ *ODD*.

Sketch these!!

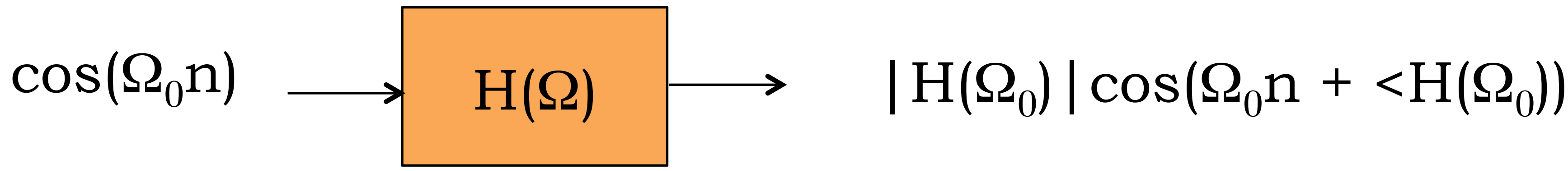
From Complex Exponentials to Sinusoids

$$\cos(\Omega n)=(e^{j\Omega n}+e^{-j\Omega n})/2$$

So response to this cosine input is

$$(H(\Omega)e^{j\Omega n}+H(-\Omega)e^{-j\Omega n})/2 = \text{Real part of } H(\Omega)e^{j\Omega n}$$

$$= \text{Real part of } |H(\Omega)|e^{j(\Omega n+<H(\Omega))}$$



Properties of $H(\Omega)$

Repeats periodically on the frequency (Ω) axis, with period 2π, because the input $e^{j\Omega n}$ is the same for Ω that differ by integer multiples of 2π. So only the interval Ω in $[-\pi,\pi]$ is of interest!

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$\Omega = 0$, i.e., $e^{j\Omega n} = 1$, corresponds to a constant (or "DC", which stands for "direct current", but now just means constant) input, so $H(0)$ is the "DC gain" of the system, i.e., gain for constant inputs.

$H(0) = \sum h[m]$ --- show this from the definition!

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$\Omega = \pi$ or $-\pi$, i.e., $Ae^{j\Omega n}=(-1)^n A$, corresponds to the highest-frequency variation possible for a discrete-time signal, so $H(\pi)=H(-\pi)$ is the high-frequency gain of the system.

$H(\pi) = \sum (-1)^m h[m]$ --- show from definition!

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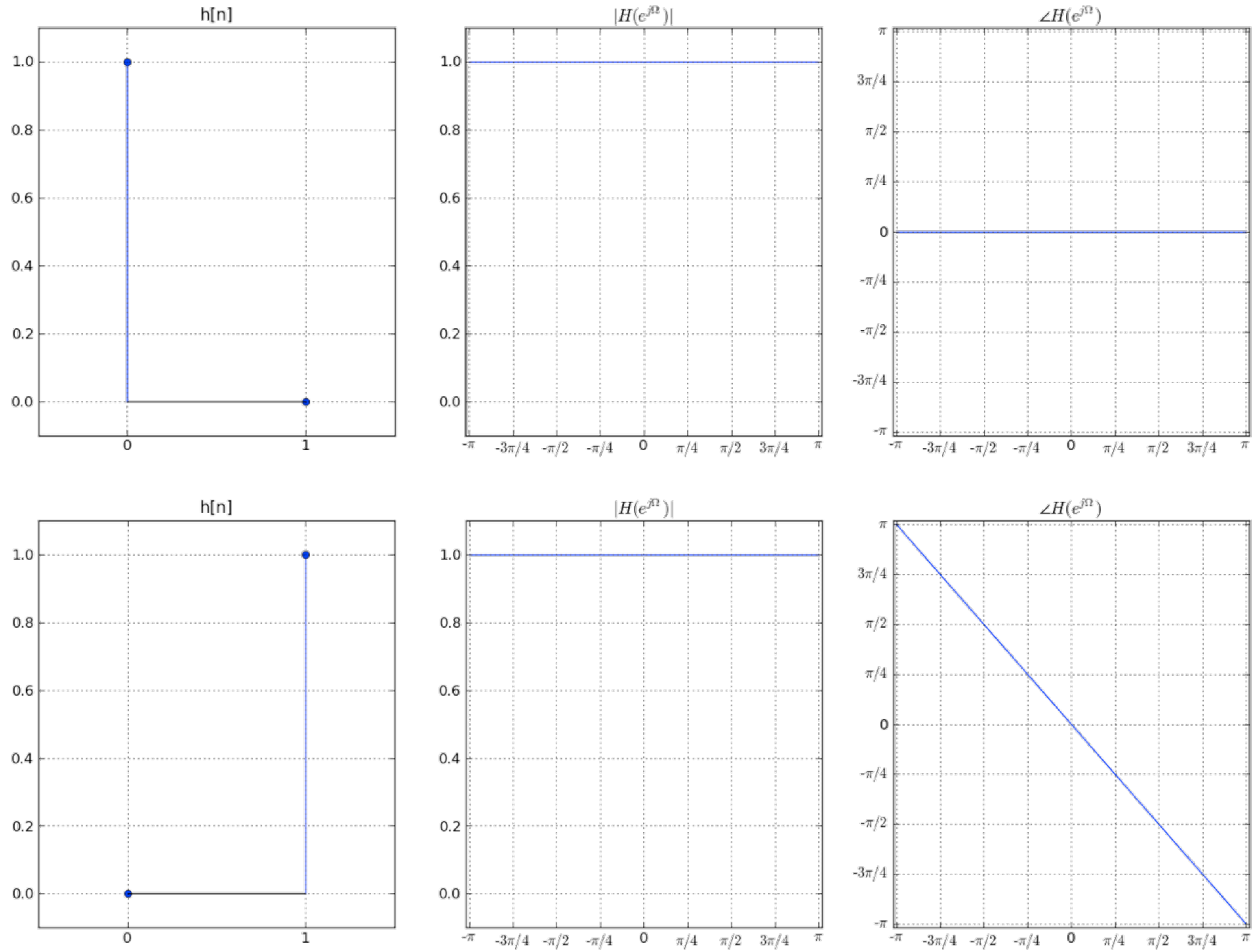
$H(\pi) = \sum (-1)^m h[m]$ --- show from definition!

For real $h[n]$:

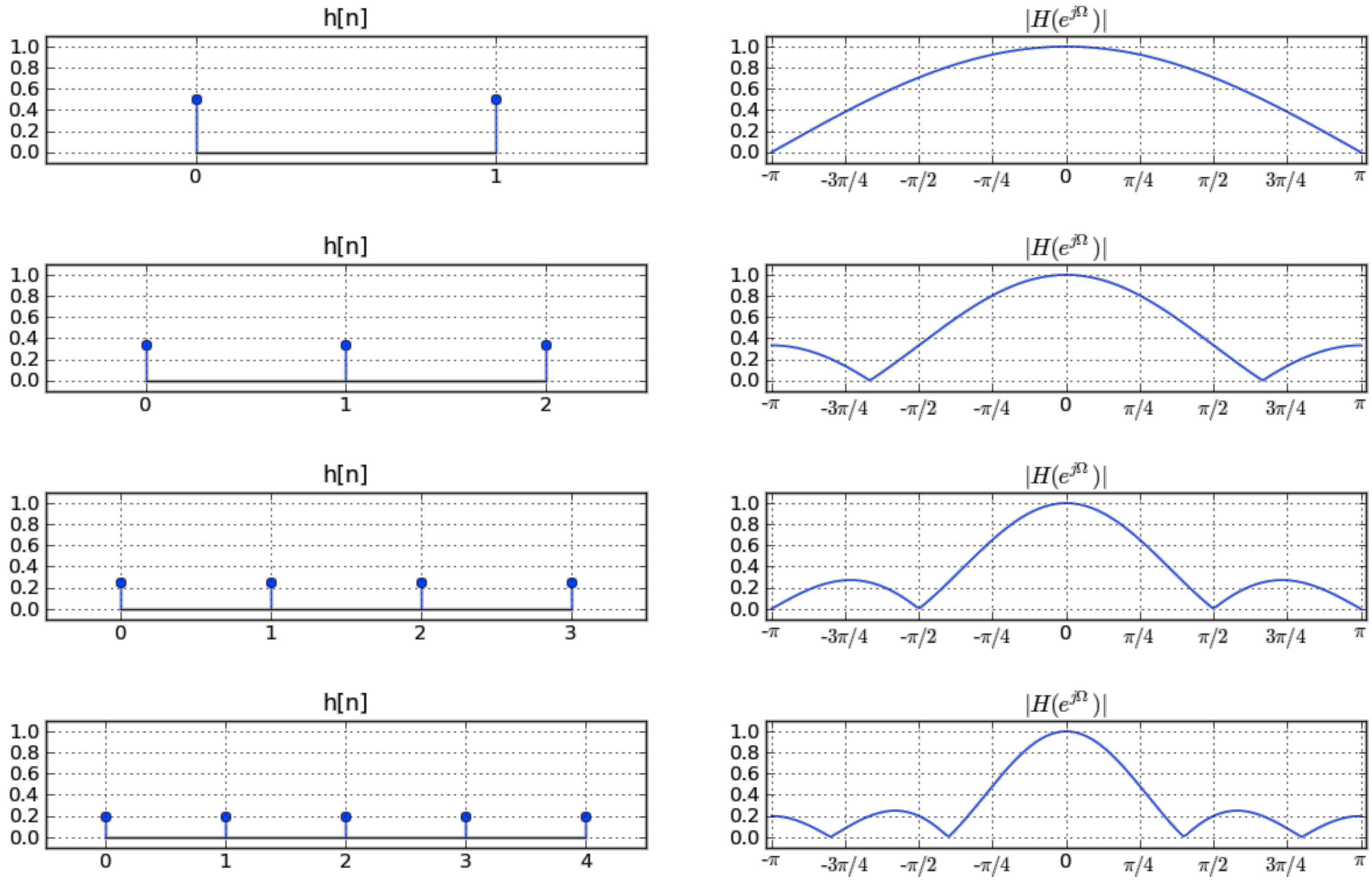
Real part of $H(\Omega)$ & **magnitude** are EVEN functions of Ω.

Imaginary part & **phase** are ODD functions of Ω.

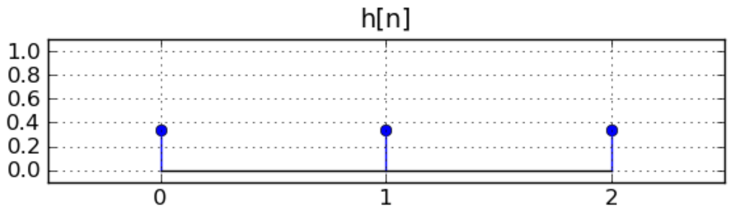
Examples: h[n], |H(Ω)| and <H(Ω)

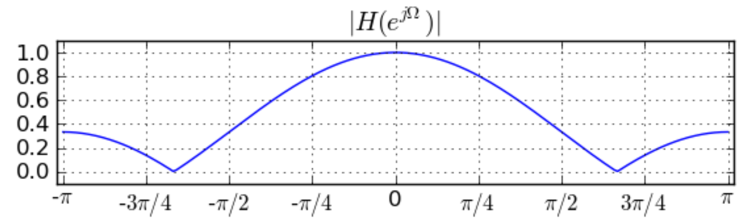


Frequency Response of "Moving Average" Filters



H(Ω) with Zeros





$$H(\Omega) = \sum_m h[m]e^{-j\Omega m} = h[0]e^{-j\Omega 0} + h[1]e^{-j\Omega 1} + h[2]e^{-j\Omega 2}$$

$$= h[0] + h[1](e^{-j\Omega}) + h[2](e^{-j\Omega})^2$$

Hmm. A quadratic equation with two roots at $\Omega=\pm\varphi$:

$$(e^{-j\Omega} - e^{-j\varphi})(e^{-j\Omega} - e^{j\varphi})$$

$$= (e^{-j\Omega})^2 - (e^{j\varphi} + e^{-j\varphi})(e^{-j\Omega}) + e^{j\varphi}e^{-j\varphi}$$

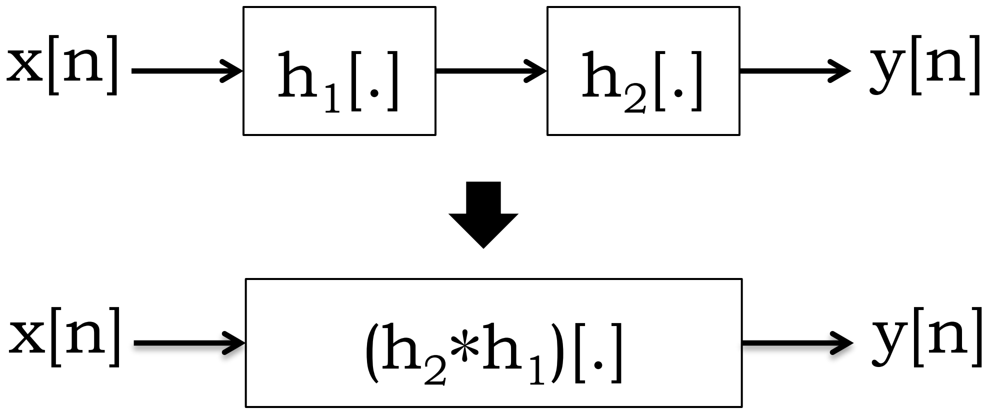
$$= 1 - 2\cos(\varphi)(e^{-j\Omega}) + (e^{-j\Omega})^2$$

Matching terms in the two equations, we see that this LTI system would have a frequency response that went to zero at $\pm\varphi$ if

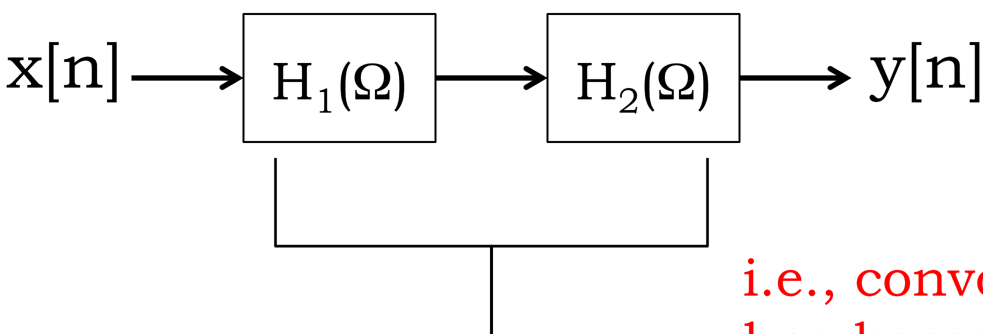
h[0]=1, h[1]=–2cos(φ) and h[2] = 1.

Series Interconnection of LTI Systems

From Lecture 11:



In the frequency domain (i.e., thinking about input-to-output frequency response):

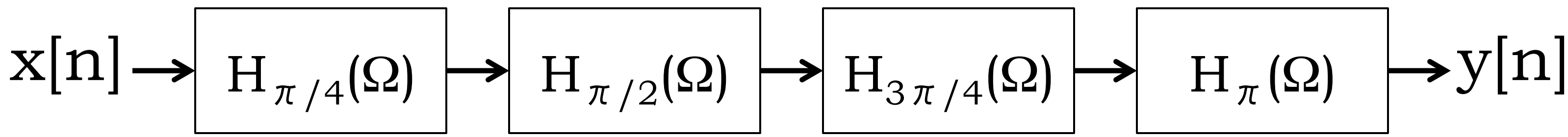


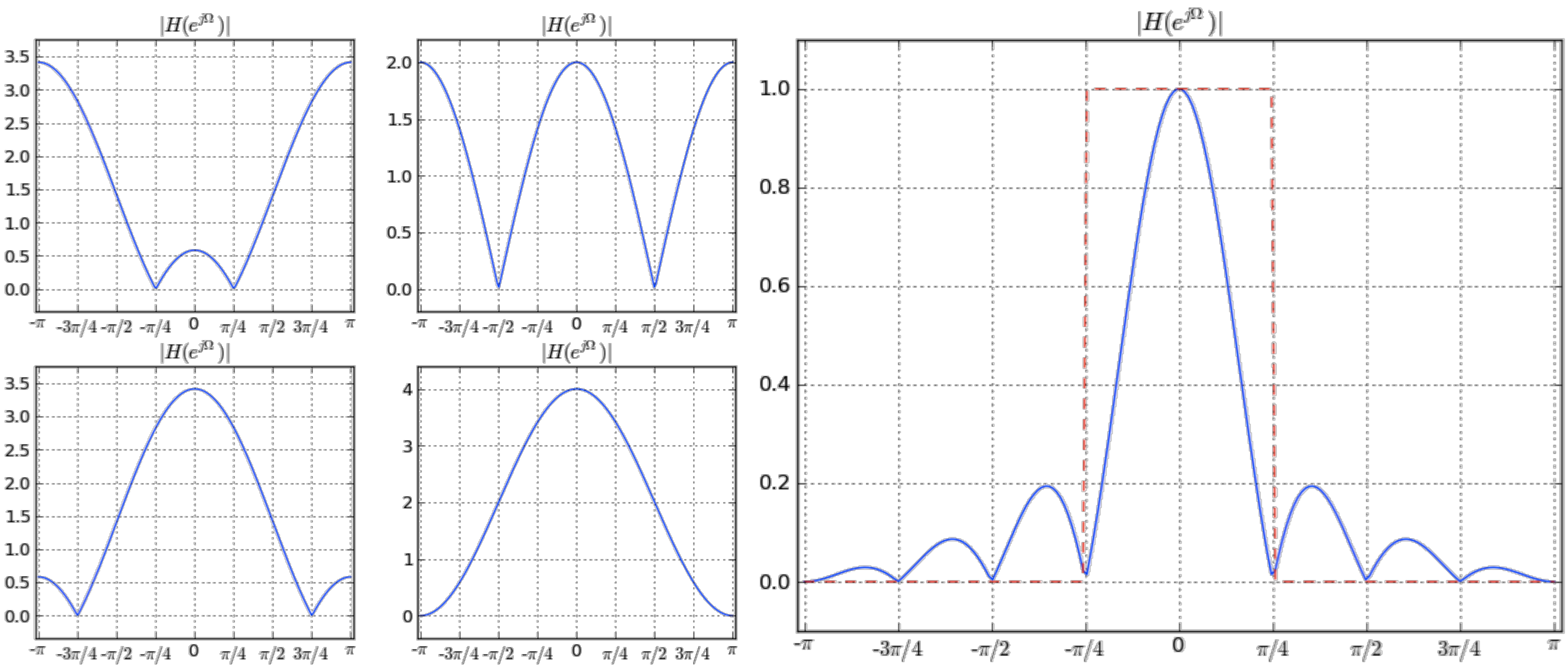
$$H(\Omega)=H_2(\Omega)H_1(\Omega)$$

i.e., convolution in time has become multiplication in frequency!

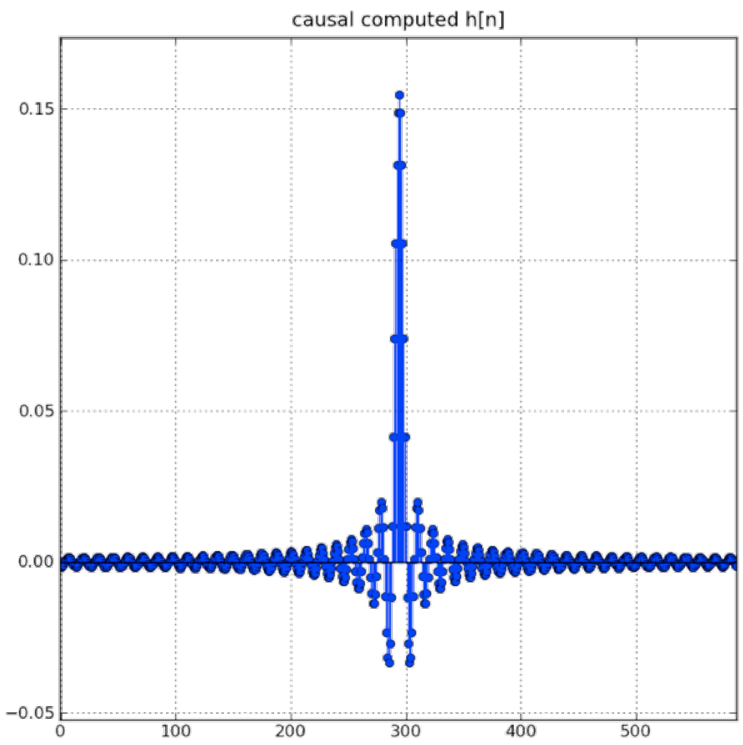
A 10-cent Low-pass Filter

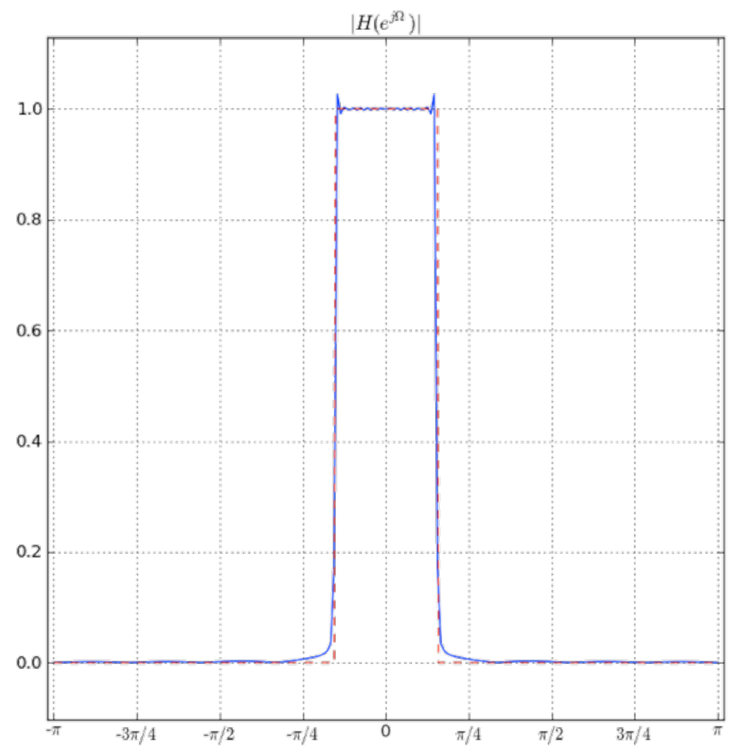
Suppose we wanted a low-pass filter with a cutoff frequency of π/4



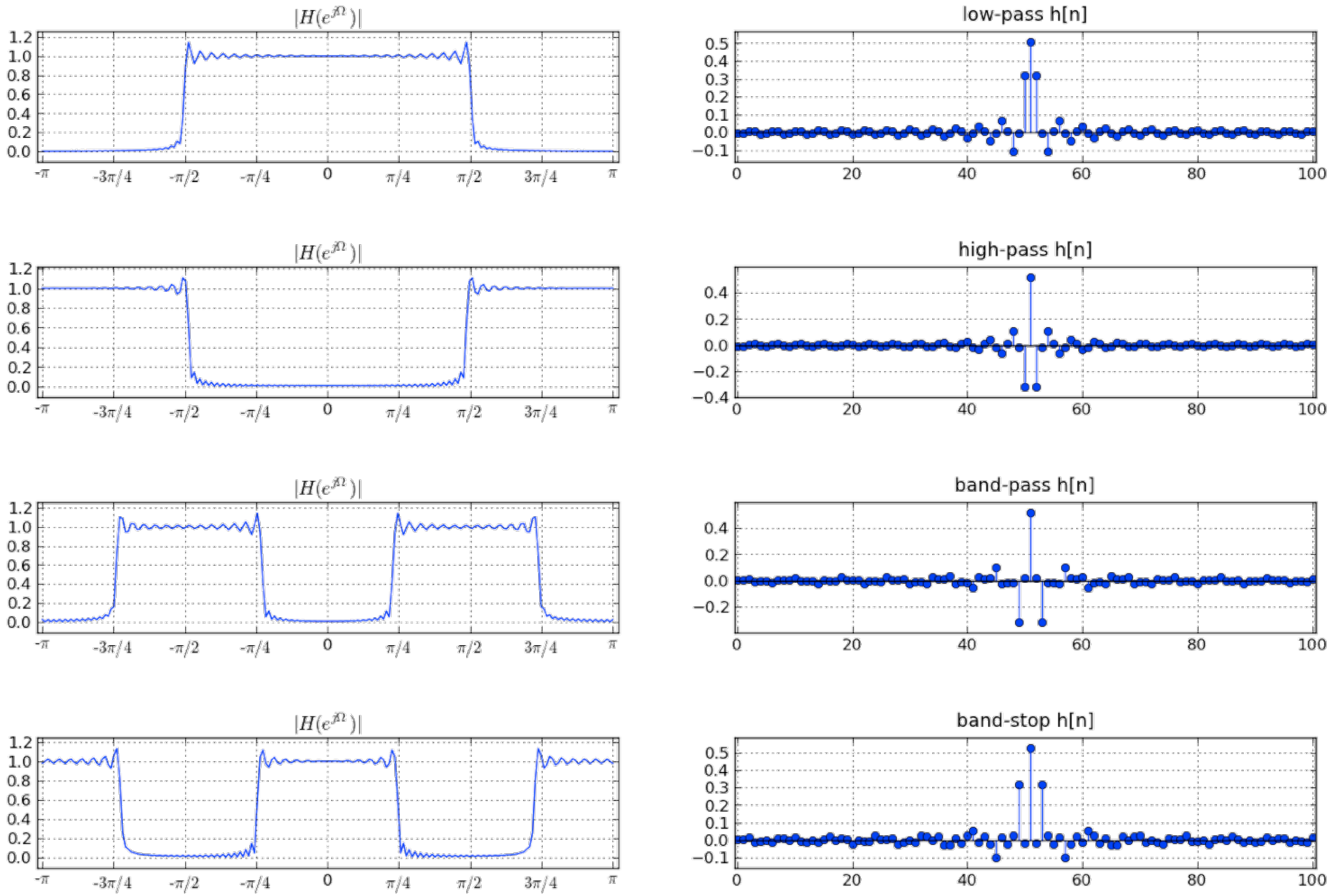


The $4.99 version, h[n] and H(Ω)

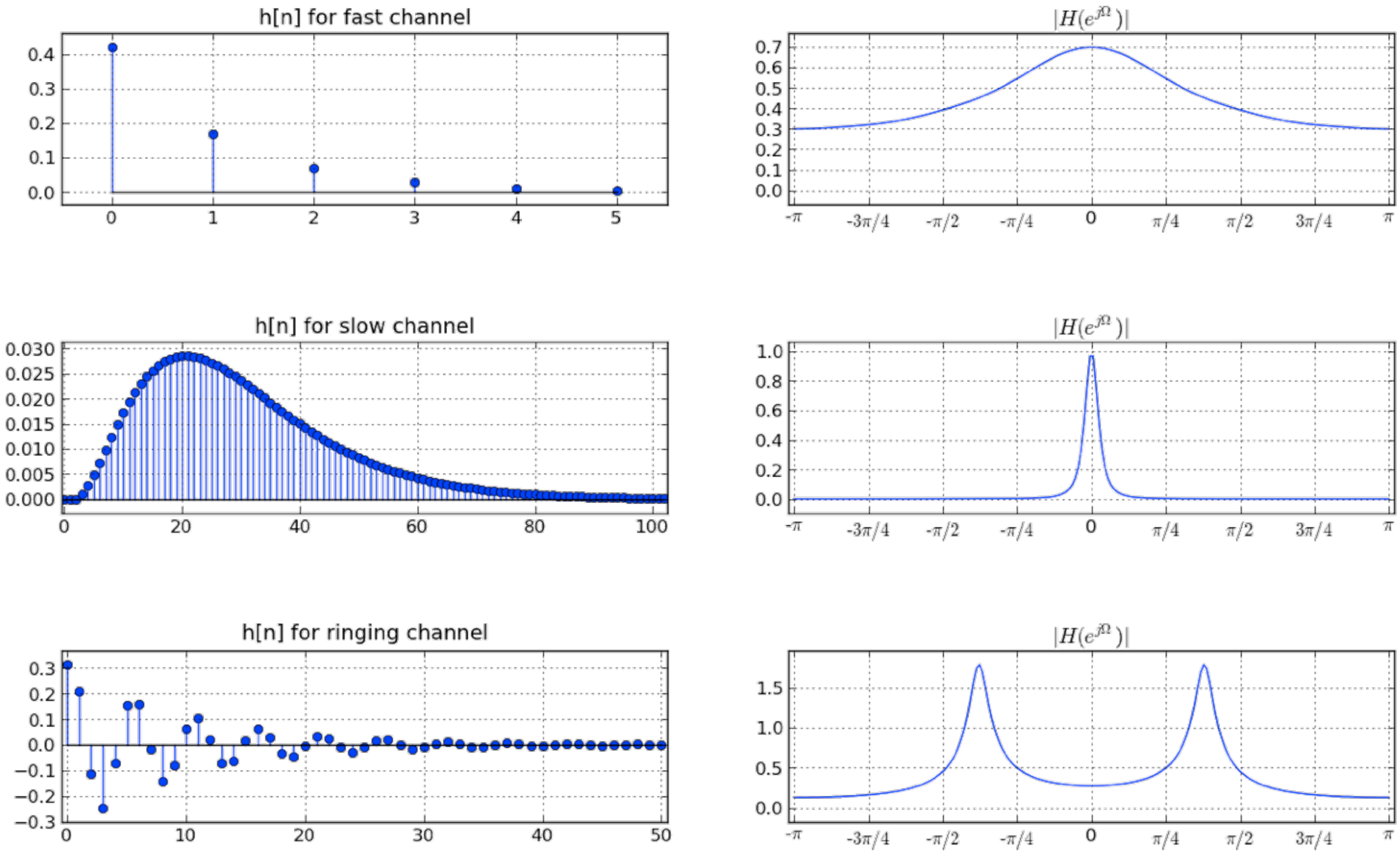




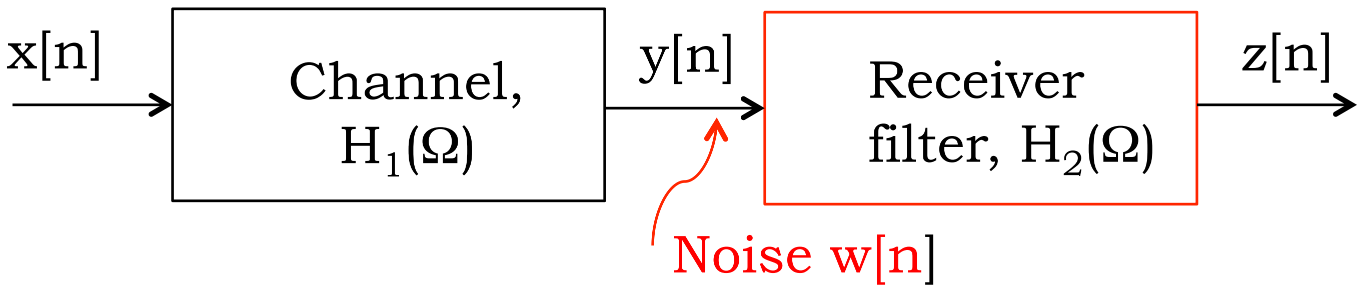
H(Ω) and h[n] for some Useful Filters



h[n] and H(Ω) for some Idealized Channels



A Frequency-Domain view of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

$$\Rightarrow \quad H_2(\Omega) = 1/H_1(\Omega) \qquad \text{“Inverse filter”}$$

$$= (1/|H_1(\Omega)|).\ \exp\{-j\angle H_1(\Omega)\}$$

Inverse filter at receiver does very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

Enough of sinusoidal inputs already!

What about other periodic inputs?

We'll start with **strictly periodic** inputs:

$$x[n+P] = x[n] \text{ for all } n$$

and some $P>0$.

(Caution: N is usually used instead of P – looks better, but gets confused with n when spoken! You will find N in the labs, rather than P; not a big deal, but stay alert.)

Key claim in (mostly!) words:

Any periodic DT signal of period P
can be written as
a weighted combination* of P complex exponentials
whose frequencies are
consecutive multiples of the *fundamental frequency* $2\pi/P$.

This is called the
Discrete-time (DT) or Discrete Fourier Series
or discrete spectral representation.

(We'll explore the form and implications now, and defer the proof of the claim.)

* generally with complex weights

Discrete-time Fourier Series

If x[n] is periodic with period P (convenient to assume P is even, so P/2 is integer, but odd P can be handled too), then x[n] can be expressed as the sum of scaled periodic complex exponentials:

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{jk\left(\frac{2\pi}{P}\right)n}$$

Complex exponential with period P and *fundamental frequency* $2\pi/P = \Omega_1$.

$$= \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{jk\Omega_1 n}$$

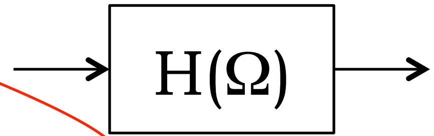
With the notation $X_k = PA_k$ we get an alternate (and often used) normalization.

k ranges over any P consecutive integers. Common choices:

- k for 0 to P–1 ; $0 \le k\Omega_1 \le 2\pi - \Omega_1$
- k for –(P/2) to (P/2)–1 for even P ; $-\pi \le k\Omega_1 \le \pi - \Omega_1$
- k symmetrically out from 0 for odd P ; $-\pi + (\Omega_1/2) \le k\Omega_1 \le \pi - (\Omega_1/2)$

Immediate Consequence:

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$



$$y[n] = \sum_{k=\langle P \rangle} H(\Omega_k) A_k e^{j\Omega_k n}$$

We write $\Omega_k = k\Omega_1 = k(2\pi/P)$, to further simplify the notation

i.e., the frequency response tells us how the system will affect the spectral components in the periodic input. We know the output is periodic, and must have its own Fourier series, with coefficients B_k. So evidently

$$B_k = H(\Omega_k) A_k$$

are the spectral coefficients for y[n]. If we use the alternate normalization, $X_k = A_k P$ and $Y_k = B_k P$, then similarly

$$Y_k = H(\Omega_k) X_k$$

How do we get the Fourier Coefficients?

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$

Synthesis equation

$$A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

Analysis equation

More on this next lecture

- x[n] and A_k are both periodic with period P
- $2\pi/P$ radians/sample is the fundamental frequency. Complex exponentials in Fourier series equations have frequencies which are some harmonic of $2\pi/P$
- If x[n] is real, $A_{-k} = A_k^*$ (i.e., they are complex conjugates)
- A_0 is the average of the x[n] over one period
- $A_{P/2}$ (when P is even) is the average of $(-1)^n x[n]$ over one period

$$x[n] = \sin(r\frac{2\pi}{P}n)$$

Let's do it "by inspection". First rewrite x[n]:

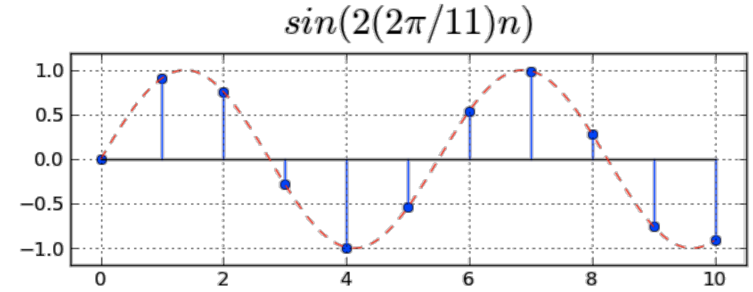
$$x[n] = \frac{1}{2j}e^{jr\frac{2\pi}{P}n} - \frac{1}{2j}e^{j(-r)\frac{2\pi}{P}n}$$

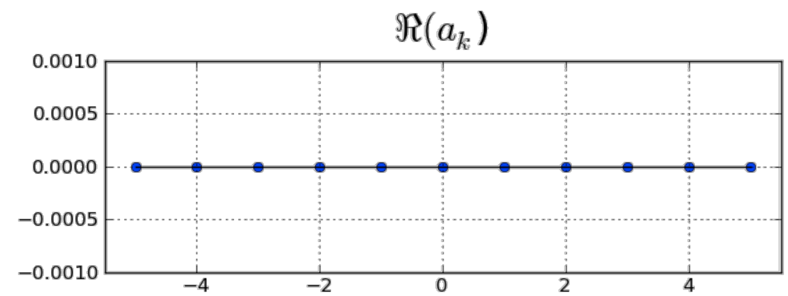
Now x[n] is a sum of complex exponentials and we can determine the A_k directly from the equation:

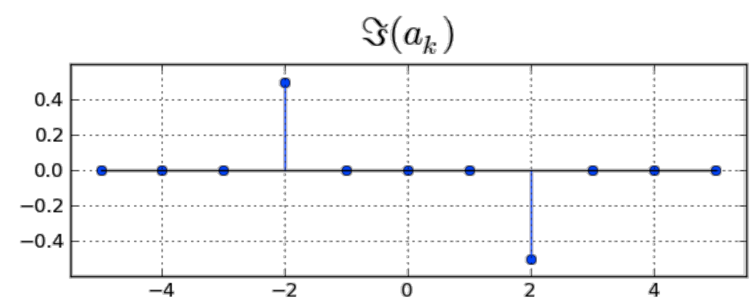
$$A_r = \frac{1}{2j} = -\frac{j}{2}$$

$$A_{-r} = -\frac{1}{2j} = \frac{j}{2}$$

$$A_k = 0 \quad \text{otherwise}$$



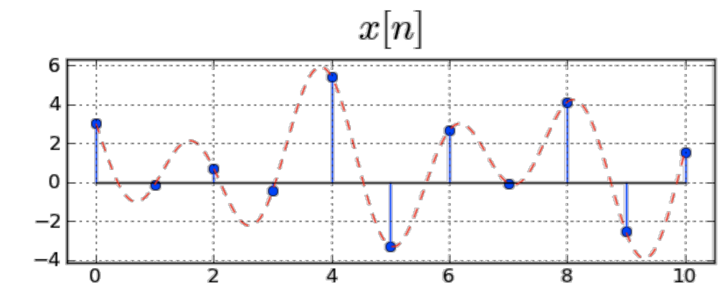




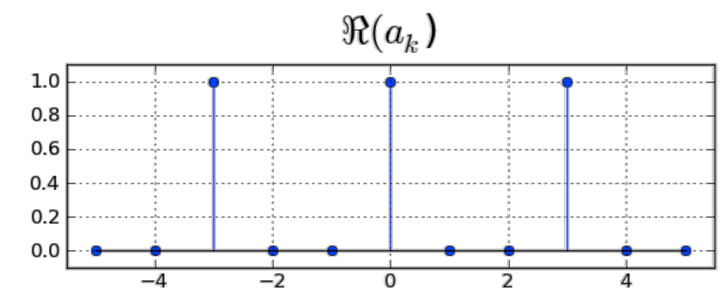
P is *odd* here, so the end points of the frequency scale are at $\pm(\pi - (\pi / P))$, not $\pm\pi$.

$$x[n] = 1 + 2\cos(3\frac{2\pi}{11}n) - 3\sin(5\frac{2\pi}{11}n)$$

Again, by inspection: since the cos and sin are at different frequencies, we can analyze them separately.



A_0 = average value = 1

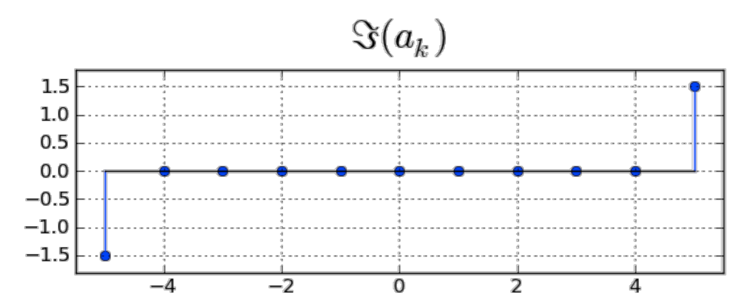


$A_{\pm 3} = 2(1/2) = 1$ [from cos term]

$A_{-5} = -3(j/2) = -1.5j$ [from sin term]

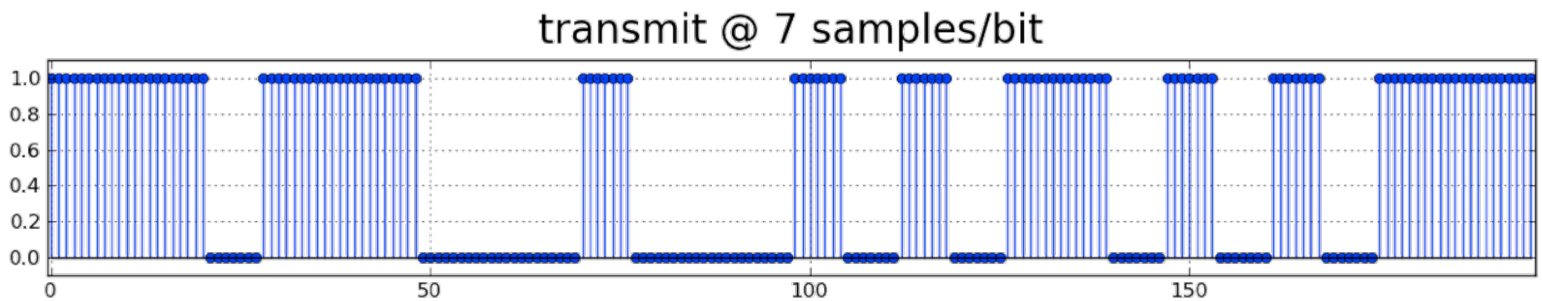
$A_5 = -3(-j/2) = 1.5j$

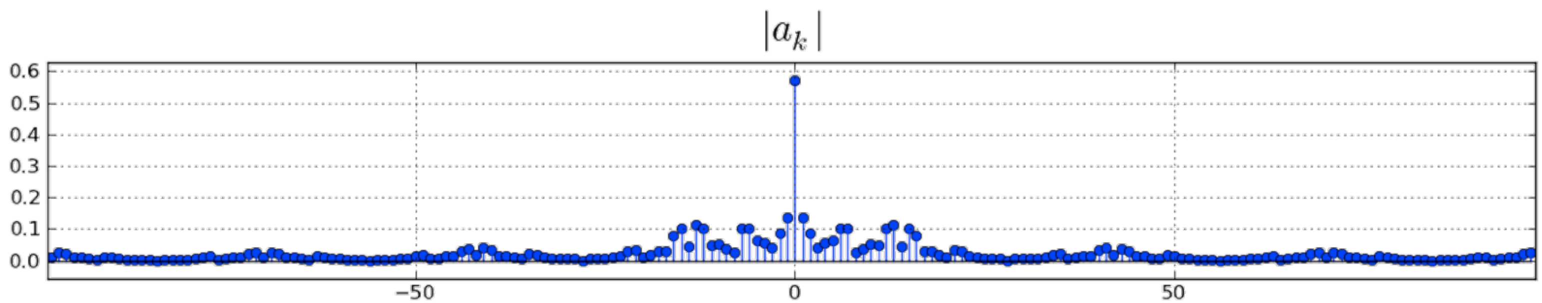
$A_k = 0$ otherwise

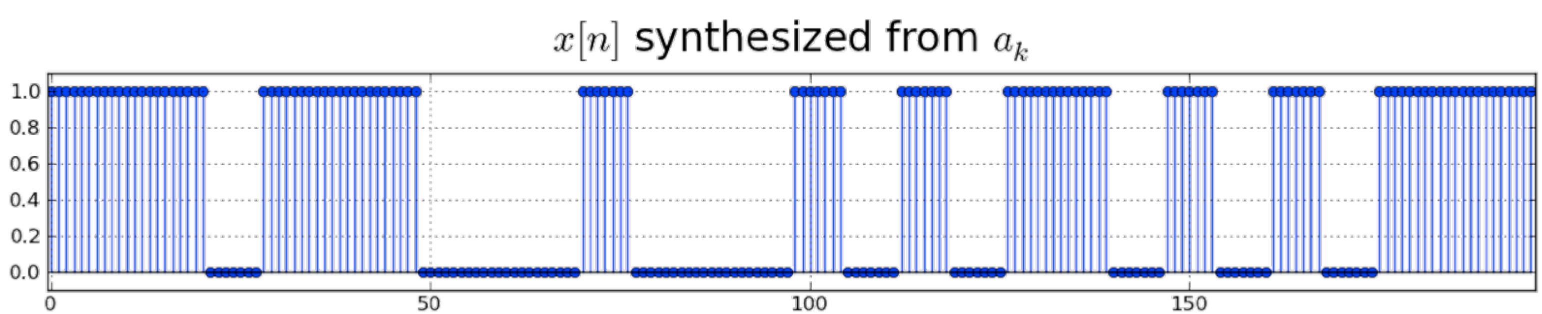


Again, P is *odd* here, so the end points of the frequency scale are at $\pm(\pi - (\pi/P))$, not $\pm\pi$.

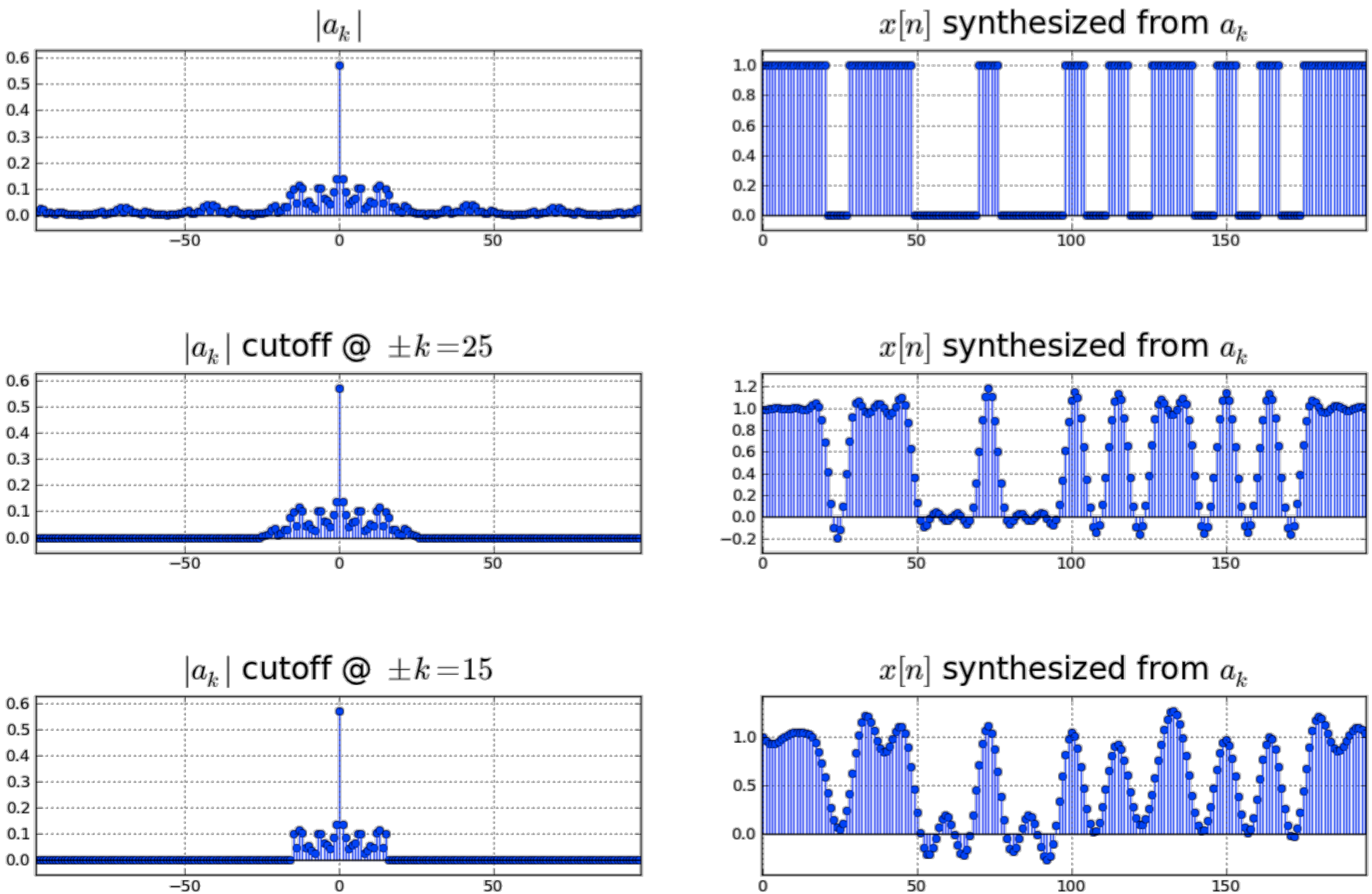
Spectrum of Digital Transmissions



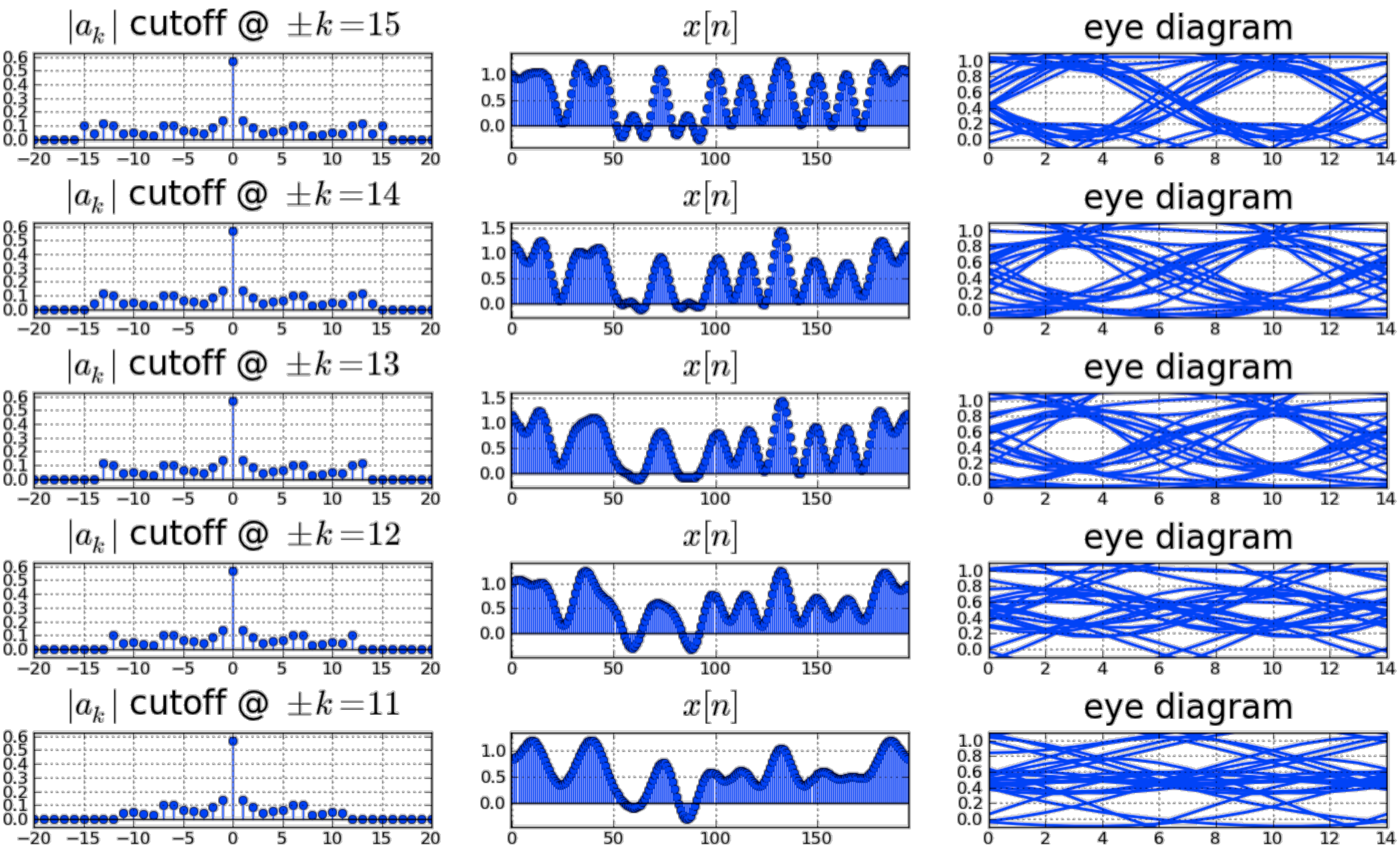




Effect of Band-limiting a Transmission



How Low Can We Go?



7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14