More on frequency response
Filters
Determining spectral content of a periodic signal: DT Fourier Series
Sinusoidal Inputs to LTI Systems

Sinusoidal inputs, i.e.,

\[ x[n] = \cos(\Omega n + \theta) \]

yield sinusoidal outputs at the same ‘frequency’ \( \Omega \) rads.
Complex Exponentials

\[ e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \]

\[
\begin{align*}
\cos(\varphi) &= \frac{1}{2} e^{j\varphi} + \frac{1}{2} e^{-j\varphi} \\
\sin(\varphi) &= \frac{1}{2j} e^{j\varphi} - \frac{1}{2j} e^{-j\varphi}
\end{align*}
\]

In the complex plane, \( e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \) is a point on the unit circle, at an angle of \( \varphi \) with respect to the positive real axis. Increasing \( \varphi \) by \( 2\pi \) brings you back to the same point! So any function of \( e^{j\varphi} \) only needs to be studied for \( \varphi \) in \([-\pi, \pi]\).
Complex Exponentials as “Eigenfunctions” of LTI System

\[ x[n] = e^{j\Omega n} \quad \xrightarrow{h[.]} \quad h[.] \quad \xrightarrow{y[n]} H(\Omega)e^{j\Omega n} \]

Eigenfunction: Undergoes only scaling -- by \( H(\Omega) \) in this case

\[
H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m} = \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)
\]

This is an infinite sum in general, but is well behaved if \( h[.]. \) is absolutely summable, i.e., if the system is stable.
Example: “Deconvolving” Output of Channel with Echo

Suppose channel is LTI with

\[ h_1[n] = \delta[n] + 0.8\delta[n-1] \]

\[ H_1(\Omega) = ??? = \sum_{m} h_1[m]e^{-j\Omega m} \]

\[ = 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega) \]

So:

\[ |H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2} \quad \text{EVEN function of } \Omega; \]

\[ <H_1(\Omega) = \arctan \left[-(0.8\sin(\Omega))/[1 + 0.8\cos(\Omega)]\right] \quad \text{ODD}. \]

Sketch these!!
From Complex Exponentials to Sinusoids

$$\cos(\Omega n) = \frac{(e^{j\Omega n} + e^{-j\Omega n})}{2}$$

So response to this cosine input is

$$\left( H(\Omega) e^{j\Omega n} + H(-\Omega) e^{-j\Omega n} \right) / 2 = \text{Real part of } H(\Omega) e^{j\Omega n}$$

$$= \text{Real part of } |H(\Omega)| e^{j(\Omega n + <H(\Omega))}$$
Properties of $H(\Omega)$

Repeats periodically on the frequency ($\Omega$) axis, with period $2\pi$, because the input $e^{j\Omega n}$ is the same for $\Omega$ that differ by integer multiples of $2\pi$. So only the interval $\Omega$ in $[-\pi, \pi]$ is of interest!
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$\Omega = 0$, i.e., $e^{i\Omega n} = 1$, corresponds to a constant (or “DC”, which stands for “direct current”, but now just means constant) input, so $H(0)$ is the “DC gain” of the system, i.e., gain for constant inputs.

$$H(0) = \sum h[m]$$

--- show this from the definition!
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$H(\pi) = \sum (-1)^m h[m]$  --- show from definition!
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\[
H(\pi) = \sum (-1)^m h[m] \quad --- \text{show from definition!}
\]

For real $h[n]$:

**Real part** of $H(\Omega)$ & **magnitude** are **EVEN** functions of $\Omega$.

**Imaginary part** & **phase** are **ODD** functions of $\Omega$.
Examples: $h[n]$, $|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$
Frequency Response of “Moving Average” Filters

- Graph 1: $h[n]$, $|H(e^{j\omega})|$
- Graph 2: $h[n]$, $|H(e^{j\omega})|$
- Graph 3: $h[n]$, $|H(e^{j\omega})|$
- Graph 4: $h[n]$, $|H(e^{j\omega})|$
**H(Ω) with Zeros**

\[ H(Ω) = \sum_{m} h[m]e^{-jΩm} = h[0]e^{-jΩ0} + h[1]e^{-jΩ1} + h[2]e^{-jΩ2} \]

\[ = h[0] + h[1](e^{-jΩ}) + h[2](e^{-jΩ})^2 \]

Hmm. A quadratic equation with two roots at \( Ω=±φ \):

\[ (e^{-jΩ} - e^{-jφ})(e^{-jΩ} - e^{jφ}) \]

\[ = (e^{-jΩ})^2 - (e^{jφ} + e^{-jφ})(e^{-jΩ}) + e^{jφ}e^{-jφ} \]

\[ = 1 - 2\cos(φ)(e^{-jΩ}) + (e^{-jΩ})^2 \]

Matching terms in the two equations, we see that this LTI system would have a frequency response that went to zero at \( ±φ \) if

\[ h[0]=1, \quad h[1]=-2\cos(φ) \quad \text{and} \quad h[2]=1. \]
Series Interconnection of LTI Systems

From Lecture 11:

\[ x[n] \rightarrow h_1[.] \rightarrow h_2[.] \rightarrow y[n] \]

\[ x[n] \rightarrow (h_2*h_1)[.] \rightarrow y[n] \]

In the frequency domain (i.e., thinking about input-to-output frequency response):

\[ x[n] \rightarrow H_1(\Omega) \rightarrow H_2(\Omega) \rightarrow y[n] \]

\[ H(\Omega) = H_2(\Omega)H_1(\Omega) \]

i.e., convolution in time has become multiplication in frequency!
A 10-cent Low-pass Filter

Suppose we wanted a low-pass filter with a cutoff frequency of $\pi/4$

$$x[n] \rightarrow H_{\pi/4}(\Omega) \rightarrow H_{\pi/2}(\Omega) \rightarrow H_{3\pi/4}(\Omega) \rightarrow H_{\pi}(\Omega) \rightarrow y[n]$$
The $4.99 version, $h[n]$ and $H(\Omega)$
$H(\Omega)$ and $h[n]$ for some Useful Filters

- **Low-pass** $h[n]$
- **High-pass** $h[n]$
- **Band-pass** $h[n]$
- **Band-stop** $h[n]$
\( h[n] \) and \( H(\Omega) \) for some Idealized Channels
A Frequency-Domain view of Deconvolution

Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get $z[n]=x[n]$?

$H_2(\Omega) = \frac{1}{H_1(\Omega)}$  “Inverse filter”

$= \left(\frac{1}{|H_1(\Omega)|}\right) \cdot \exp\{-j<\Omega \cdot H_1(\Omega)\}$

Inverse filter at receiver does very badly in the presence of noise that adds to $y[n]$:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!
Enough of sinusoidal inputs already!

What about other **periodic inputs**?

We’ll start with **strictly periodic** inputs:

\[ x[n+P] = x[n] \text{ for all } n \]

and some \( P > 0 \).

(Caution: \( N \) is usually used instead of \( P \) – looks better, but gets confused with \( n \) when spoken! You will find \( N \) in the labs, rather than \( P \); not a big deal, but stay alert.)
Key claim in (mostly!) words:

Any periodic DT signal of period P can be written as a weighted combination* of P complex exponentials whose frequencies are consecutive multiples of the fundamental frequency $2\pi / P$.

This is called the Discrete-time (DT) or Discrete Fourier Series or discrete spectral representation.

(We’ll explore the form and implications now, and defer the proof of the claim.)

* generally with complex weights
If \( x[n] \) is periodic with period \( P \) (convenient to assume \( P \) is even, so \( P/2 \) is integer, but odd \( P \) can be handled too), then \( x[n] \) can be expressed as the sum of scaled periodic complex exponentials:

\[
x[n] = \sum_{k=\langle P \rangle} A_k e^{jk \left( \frac{2\pi}{P} \right) n}
\]

Complex exponential with period \( P \) and fundamental frequency \( 2\pi/P = \Omega_1 \).

\[
= \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j k \Omega_1 n}
\]

With the notation \( X_k = PA_k \) we get an alternate (and often used) normalization.

\( k \) ranges over any \( P \) consecutive integers. Common choices:
- \( k \) for 0 to \( P-1 \); \( 0 \leq k\Omega_1 \leq 2\pi-\Omega_1 \)
- \( k \) for \(-(P/2)\) to \((P/2)-1\) for even \( P \); \(-\pi \leq k\Omega_1 \leq \pi-\Omega_1 \)
- \( k \) symmetrically out from 0 for odd \( P \); \(-\pi+(\Omega_1/2)\leq k\Omega_1 \leq \pi-(\Omega_1/2) \)
Immediate Consequence:

\[ x[n] = \sum_{k=P} A_k e^{j\Omega_k n} \rightarrow H(\Omega) \rightarrow y[n] = \sum_{k=P} H(\Omega_k) A_k e^{j\Omega_k n} \]

We write \( \Omega_k = k\Omega_1 = k(2\pi/P) \), to further simplify the notation.

i.e., the frequency response tells us how the system will affect the spectral components in the periodic input. We know the output is periodic, and must have its own Fourier series, with coefficients \( B_k \). So evidently

\[ B_k = H(\Omega_k)A_k \]

are the spectral coefficients for \( y[n] \). If we use the alternate normalization, \( X_k = A_k P \) and \( Y_k = B_k P \), then similarly

\[ Y_k = H(\Omega_k)X_k \]
How do we get the Fourier Coefficients?

\[ x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n} \]

**Synthesis equation**

\[ A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n} \]

**Analysis equation**

- \( x[n] \) and \( A_k \) are both periodic with period \( P \)
- \( 2\pi/P \) radians/sample is the fundamental frequency. Complex exponentials in Fourier series equations have frequencies which are some harmonic of \( 2\pi/P \)
- If \( x[n] \) is real, \( A_{-k} = A_k^* \) (i.e., they are complex conjugates)
- \( A_0 \) is the average of the \( x[n] \) over one period
- \( A_{P/2} \) (when \( P \) is even) is the average of \( (-1)^n x[n] \) over one period

More on this next lecture
\[ x[n] = \sin(r \frac{2\pi}{P} n) \]

Let’s do it “by inspection”. First rewrite \( x[n] \):

\[ x[n] = \frac{1}{2j} e^{jr \frac{2\pi}{P} n} - \frac{1}{2j} e^{j(-r) \frac{2\pi}{P} n} \]

Now \( x[n] \) is a sum of complex exponentials and we can determine the \( A_k \) directly from the equation:

\[ A_r = \frac{1}{2j} = -\frac{j}{2} \]

\[ A_{-r} = -\frac{1}{2j} = \frac{j}{2} \]

\[ A_k = 0 \quad \text{otherwise} \]

\( P \) is odd here, so the end points of the frequency scale are at \( \pm(\pi - (\pi / P)) \), not \( \pm \pi \).
\[ x[n] = 1 + 2 \cos(3 \frac{2\pi}{11} n) - 3 \sin(5 \frac{2\pi}{11} n) \]

Again, by inspection: since the \( \cos \) and \( \sin \) are at different frequencies, we can analyze them separately.

\( A_0 = \) average value = 1

\( A_{\pm3} = 2(1/2) = 1 \) [from \( \cos \) term]

\( A_{-5} = -3(j/2) = -1.5j \) [from \( \sin \) term]

\( A_5 = -3(-j/2) = 1.5j \)

\( A_k = 0 \) otherwise

Again, \( P \) is odd here, so the end points of the frequency scale are at \( \pm(\pi - (\pi /P)) \), not \( \pm \pi \).
Spectrum of Digital Transmissions

transmit @ 7 samples/bit

\[ |a_k| \]

\[ x[n] \text{ synthesized from } a_k \]
Effect of Band-limiting a Transmission

$|a_k|$ cutoff @ $\pm k = 25$

$x[n]$ synthesized from $a_k$

$|a_k|$ cutoff @ $\pm k = 15$

$x[n]$ synthesized from $a_k$
How Low Can We Go?

7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14