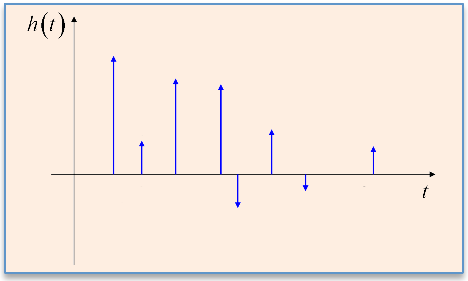
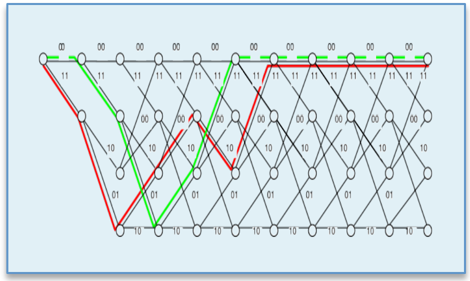
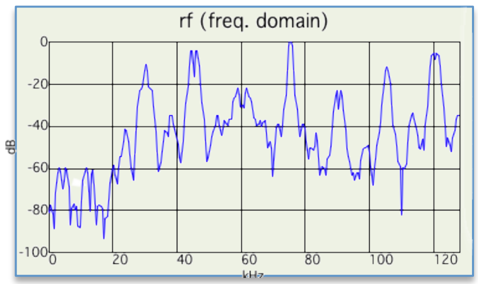
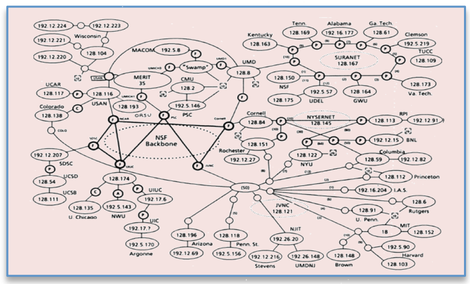








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2011
Lecture #14

- Review+more on frequency response
- Review+more on DT Fourier Series (DTFS)
- Using the DTFS for finite-duration signals

Complex Exponentials through LTI Systems

$x[n]=e^{j\Omega n}$ → h[.] → $y[n]=H(\Omega)e^{j\Omega n}$

Exponentials are special for LTI systems (which comprise scaling and DT time-shifting operations on an input) because time-shifting an *exponential* yields a scaled version of the *same* exponential.

FREQUENCY RESPONSE
(definition)

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$

$$= \left(\sum_m h[m]\cos(\Omega m)\right) - j\left(\sum_m h[m]\sin(\Omega m)\right)$$

$$= C(\Omega) - jS(\Omega) = \left|H(\Omega)\right|.\exp\{j < H(\Omega)\}$$

An infinite sum in general, but well behaved if h[.] is absolutely summable, i.e., if the system is stable --- our standing assumption.

From Complex Exponentials to Sinusoids

$$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n})/2$$

So response to a cosine input is:

$$A\cos(\Omega_0 n + \varnothing_0) \rightarrow \boxed{H(\Omega)} \rightarrow |\mathbf{H}(\Omega_0)|\,A\cos(\Omega_0 n + \varnothing_0 + \angle\mathbf{H}(\Omega_0))$$

Key Properties of Frequency Response

(i) **$H(\Omega)$** repeats periodically on the frequency (Ω) axis, with period 2π (because the input $e^{j\Omega n}$ is the same for Ω that differ by integer multiples of 2π, so the corresponding output is the same).

So only the interval Ω in $[-\pi,\pi]$ is of interest!
$\Omega=0$ is lowest frequency, namely constant 1 or "DC".
$\Omega=\pm\pi$ is highest frequency, namely $(-1)^n$.

(ii) For *real* h[n]:

Real part of $H(\Omega)$ & **magnitude** are EVEN functions of Ω.

Imaginary part & **phase** are ODD functions of Ω.

(iii) For real and *even* $h[n] = h[-n]$, $H(\Omega)$ is purely real.
For real and *odd* $h[n] = -h[-n]$, $H(\Omega)$ is purely imaginary.

Exercise: Frequency response of h[n-D]

Given an LTI system with unit sample response h[n] and associated frequency response $H(\Omega)$,

determine the frequency response $H_D(\Omega)$ of an LTI system whose unit sample response is

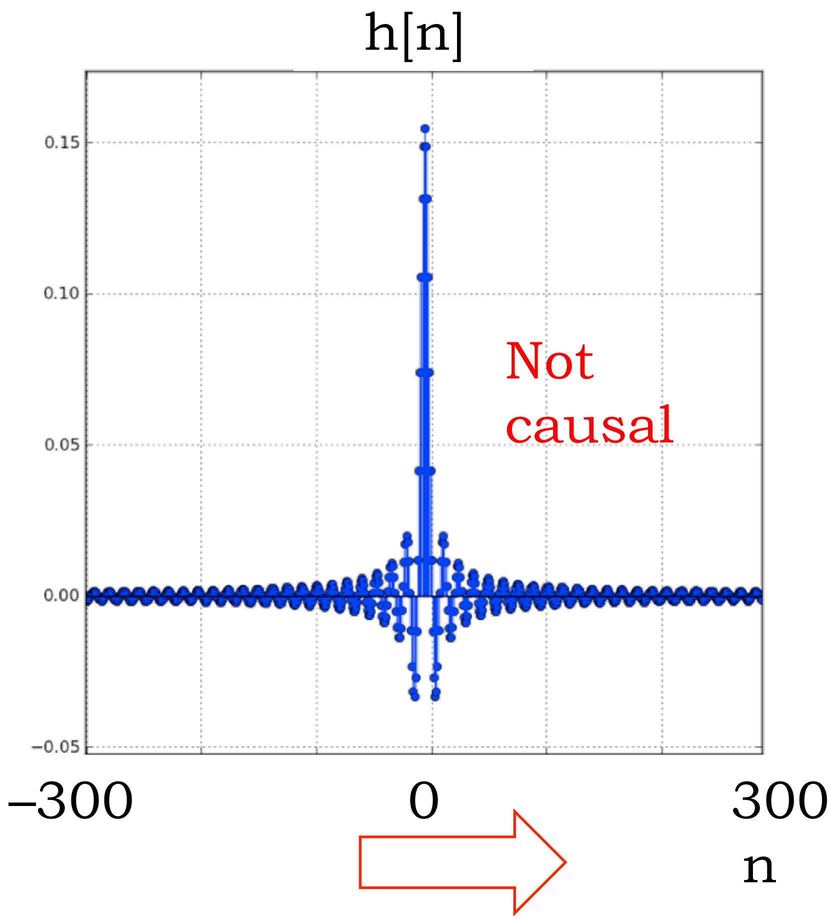
$$h_D[n] = h[n-D].$$

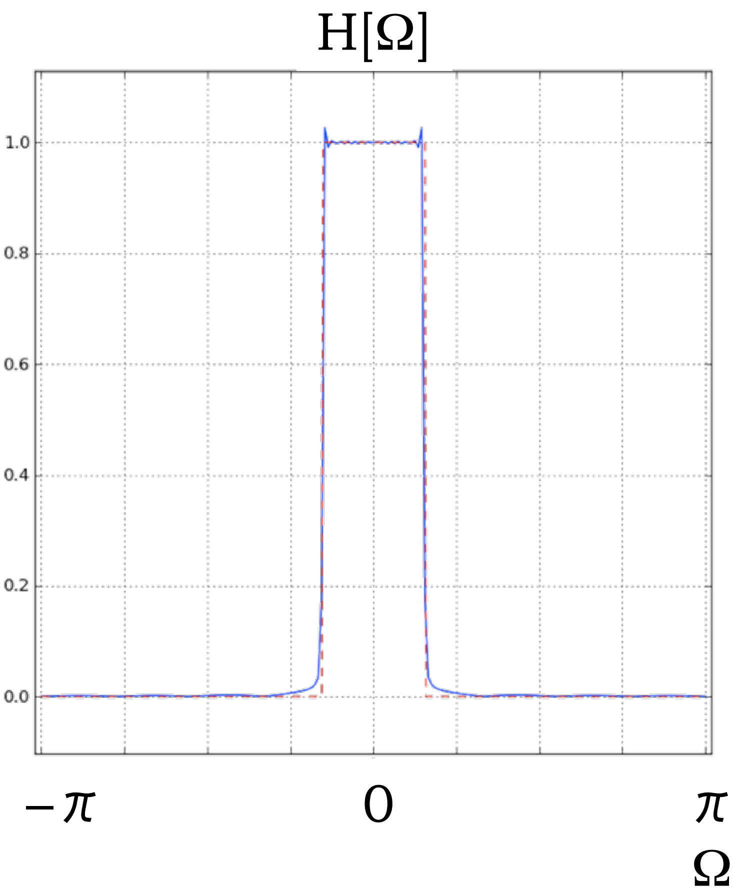
Answer: $H_D(\Omega) = \exp\{-j\Omega D\}.H(\Omega)$

so : $|H_D(\Omega)| = |H(\Omega)|$, i.e., magnitude unchanged

$\angle H_D(\Omega) = -\Omega D + \angle H(\Omega)$, i.e., linear phase term added

e.g.: Approximating an ideal lowpass filter

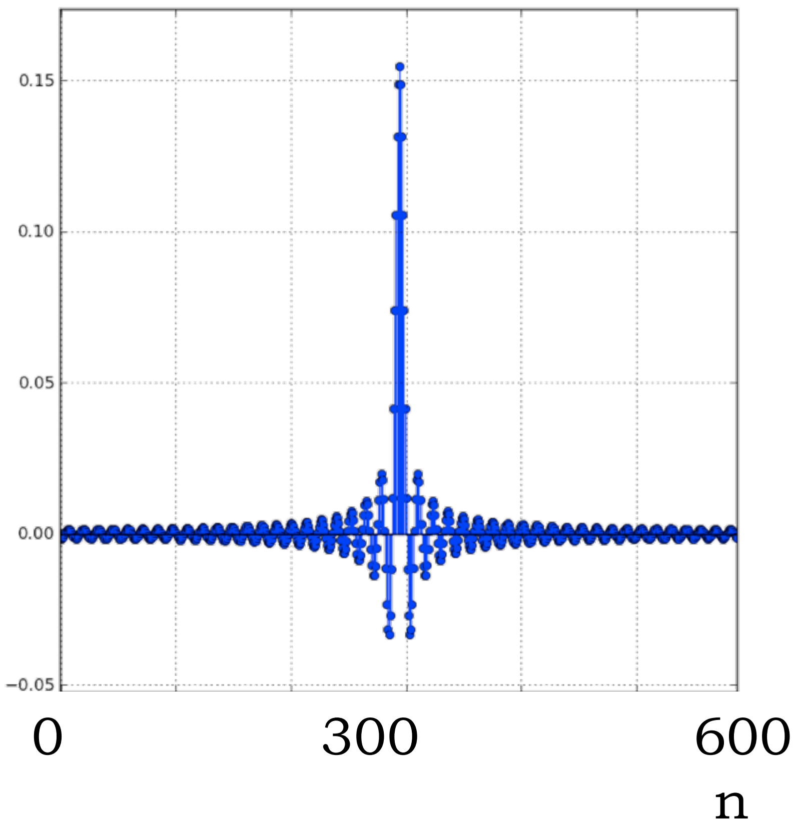




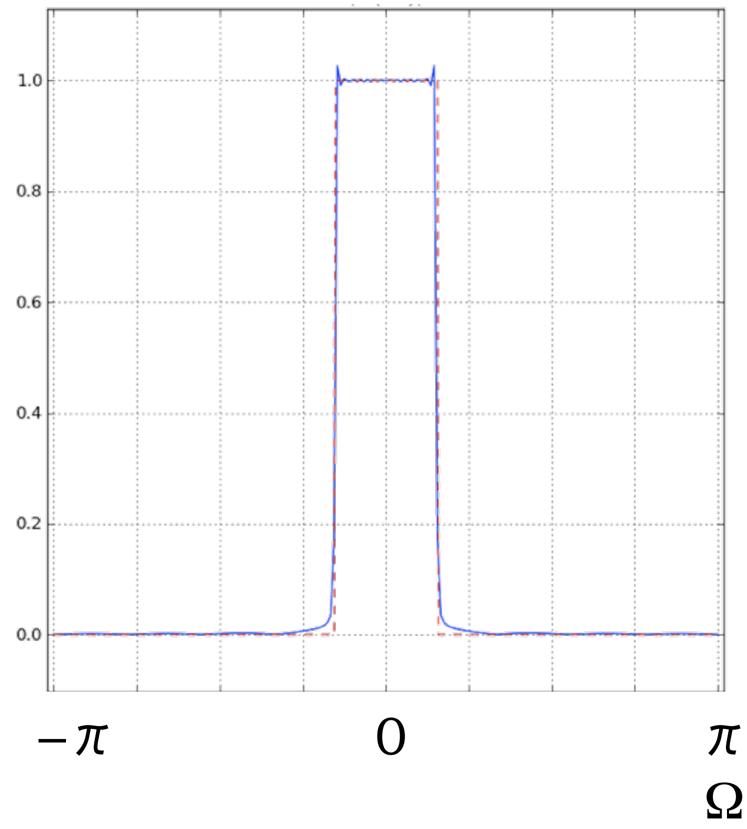
Idea: shift h[n] right to get causal LTI system.
Will the result still be a lowpass filter?

Causal approximation to ideal lowpass filter

$h_C[n] = h[n-300]$



$|H_C[\Omega]|$



Determine $\angle H_C(\Omega)$

Determining h[n] from H(Ω)

$$H(\Omega) = \sum_m h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) 2π interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega = \int_{<2\pi>} \sum_m h[m] e^{-j\Omega(m-n)} d\Omega$$

$$= 2\pi \cdot h[n]$$

$$\Rightarrow \quad h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

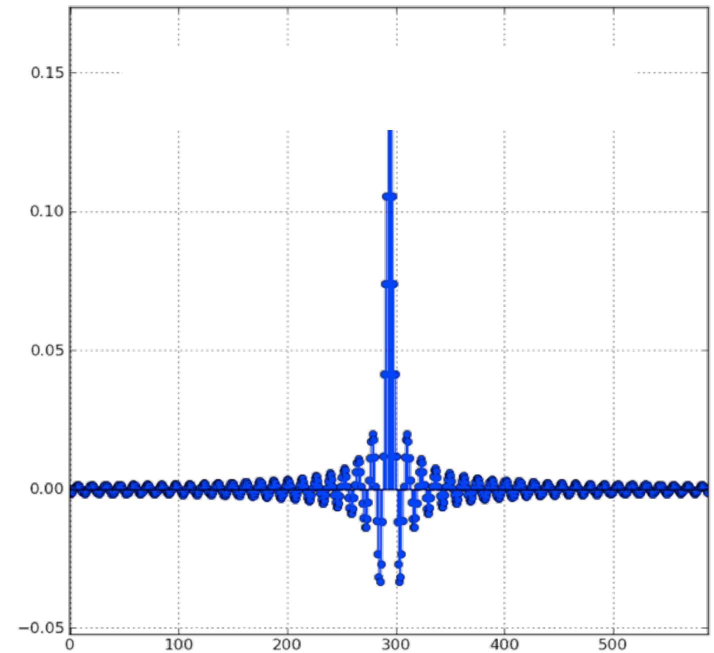
Design ideal lowpass filter with cutoff frequency Ω_C and $H(\Omega)=1$ in passband

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_C n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_C / \pi \quad , \quad n = 0$$



DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n))

What about other non-sinusoidal periodic inputs?

Any *strictly* periodic DT signal of period P

$x[n+P]=x[n]$ for all n

e.g., $6.\sin((2\pi n/P)+0.17) + 4.\cos(3(2\pi n/P)+0.82)$

can be written as
a weighted combination (generally with complex weights)
of P complex exponentials

whose frequencies are
consecutive integer multiples of the *fundamental frequency* $2\pi/P=\Omega_1$
(so each exponential term has period P)

This is the
Discrete-Time Fourier Series (DTFS)
or discrete spectral representation.

Discrete-Time Fourier Series (DTFS)

If x[n] is periodic with period P (convenient to assume P is even, so P/2 is integer, but odd P can be handled too), it can be expressed as the sum of P "spectral components" --- scaled complex exponentials of period P:

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{jk\Omega_1 n}$$

Complex exponentials with *fundamental frequency* $2\pi/P = \Omega_1$. Frequency of term k is $\Omega_k = k\Omega_1$.

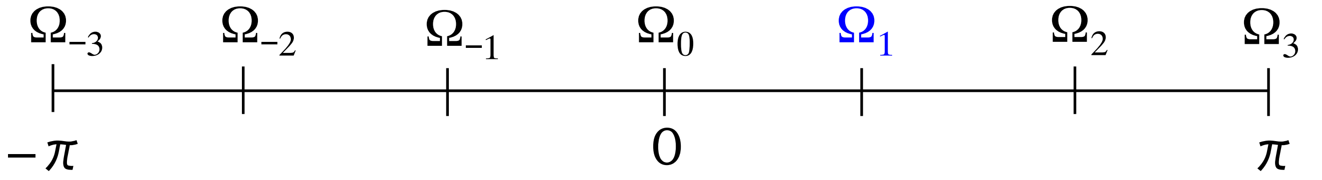
k ranges over any P consecutive integers. Common choices:

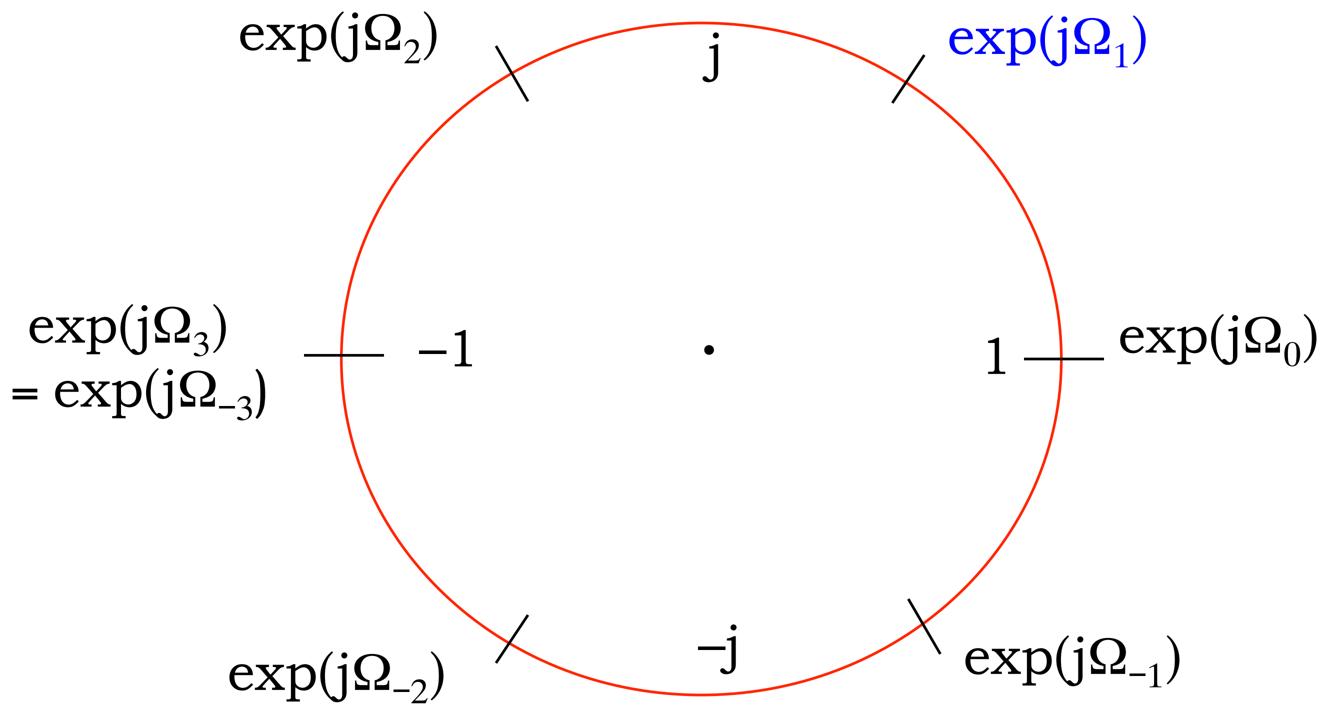
- k for 0 to P–1 ; $0 \le k\Omega_1 \le 2\pi - \Omega_1$
- k for –(P/2) to (P/2)–1 for even P ; $-\pi \le k\Omega_1 \le \pi - \Omega_1$
- k symmetrically out from 0 for odd P ; $-\pi + (\Omega_1/2) \le k\Omega_1 \le \pi - (\Omega_1/2)$

$$= \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{jk\Omega_1 n}$$

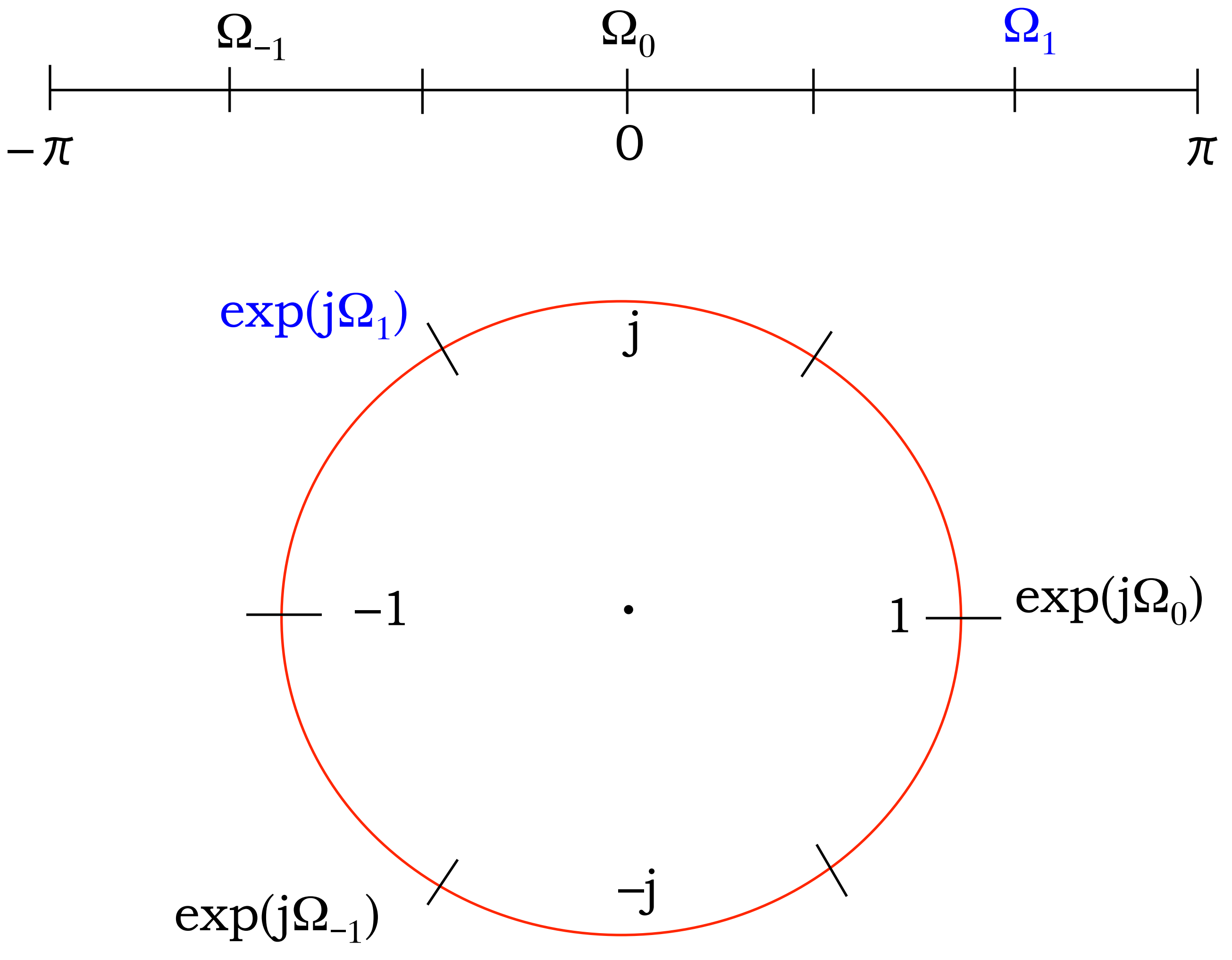
With the notation $A_k = X_k/P$, we get an alternate (and often used) normalization.

Where do the Ω_k live?
e.g., for P=6 (even)



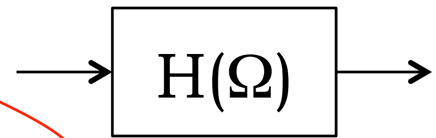


Where do the Ω_k live? e.g., for P=3 (odd)



Consequence for Periodic Input to LTI System

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$



$$y[n] = \sum_{k=\langle P \rangle} H(\Omega_k) A_k e^{j\Omega_k n}$$

We write $\Omega_k = k\Omega_1 = k(2\pi/P)$, to further simplify the notation; so $\Omega_{-k} = -\Omega_k$.

i.e., the frequency response tells us how the system will affect the spectral components in the periodic input. We know the output is periodic, and must have its own Fourier series, with coefficients B_k. So evidently

$$B_k = H(\Omega_k) A_k$$

are the spectral coefficients for y[n]. If we use the alternate normalization, $X_k = A_k P$ and $Y_k = B_k P$, then similarly

$$Y_k = H(\Omega_k) X_k$$

Determining the Fourier Series Coefficients

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$

Synthesis equation

$$X_k = A_k P = \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

Analysis equation

Parenthetical remark: compare with

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

$$H(\Omega) = \sum_n h[n] e^{-j\Omega n}$$

Derivation of equation for A_k

Start with:

$$x[n] = \sum_{m=\langle P \rangle} A_m e^{j\Omega_m n}$$

Multiply both sides by $e^{-j\Omega_k n}$ and sum over P terms:

$$\sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n} = \sum_{n=\langle P \rangle} \sum_{m=\langle P \rangle} A_m e^{j\Omega_m n} e^{-j\Omega_k n}$$

$$= \sum_{m=\langle P \rangle} A_m \sum_{n=\langle P \rangle} e^{j(m-k)\Omega_1 n}$$

$$= A_k P$$

= 0 if m-k ≠ 0, and
= P otherwise

$$A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$

DTFS Properties

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$

$$A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] e^{-j\Omega_k n}$$

- $x[n]$ and A_k are both periodic with period P
- If $x[n]$ is real, $A_{-k} = A_k^*$ (i.e., they are complex conjugates)
- A_0 is the average of the $x[n]$ over one period
- $A_{P/2}$ (for even P) is the average of $(-1)^n x[n]$ over one period
- It takes P numbers to specify this periodic $x[n]$, and it takes P numbers to specify its Fourier series coefficients

$$x[n] = \sin(r\frac{2\pi}{P}n)$$

Let's do it "by inspection". First rewrite x[n]:

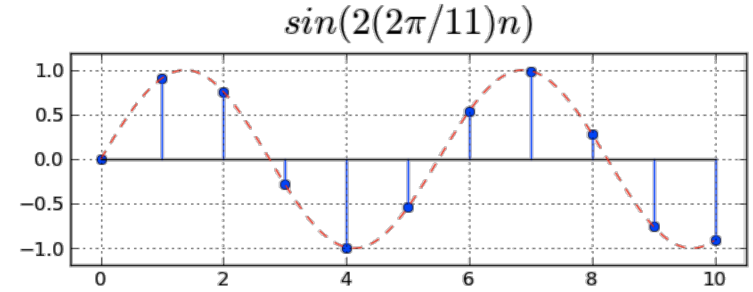
$$x[n] = \frac{1}{2j}e^{jr\frac{2\pi}{P}n} - \frac{1}{2j}e^{j(-r)\frac{2\pi}{P}n}$$

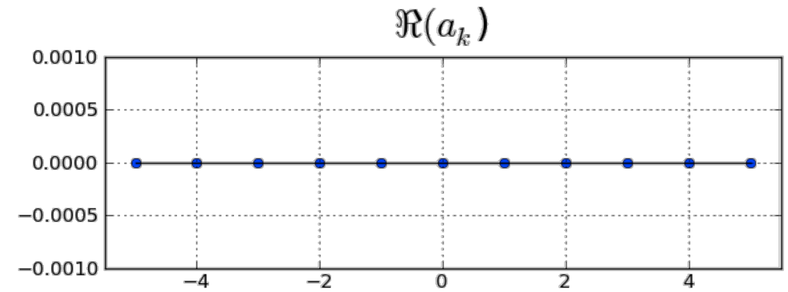
Now x[n] is a sum of complex exponentials and we can determine the A_k directly from the equation:

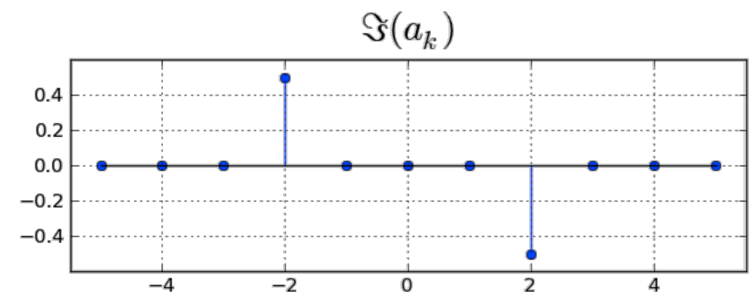
$$A_r = \frac{1}{2j} = -\frac{j}{2}$$

$$A_{-r} = -\frac{1}{2j} = \frac{j}{2}$$

$$A_k = 0 \quad \text{otherwise}$$







P is *odd* here (=11), so the end points of the frequency scale are at $\pm(\pi - (\pi/P))$, not $\pm\pi$.

$$x[n] = 1 + 2\cos(3\frac{2\pi}{11}n) - 3\sin(5\frac{2\pi}{11}n)$$

Again, by inspection: since the cos and sin are at different frequencies, we can analyze them separately.

A_0 = average value = 1

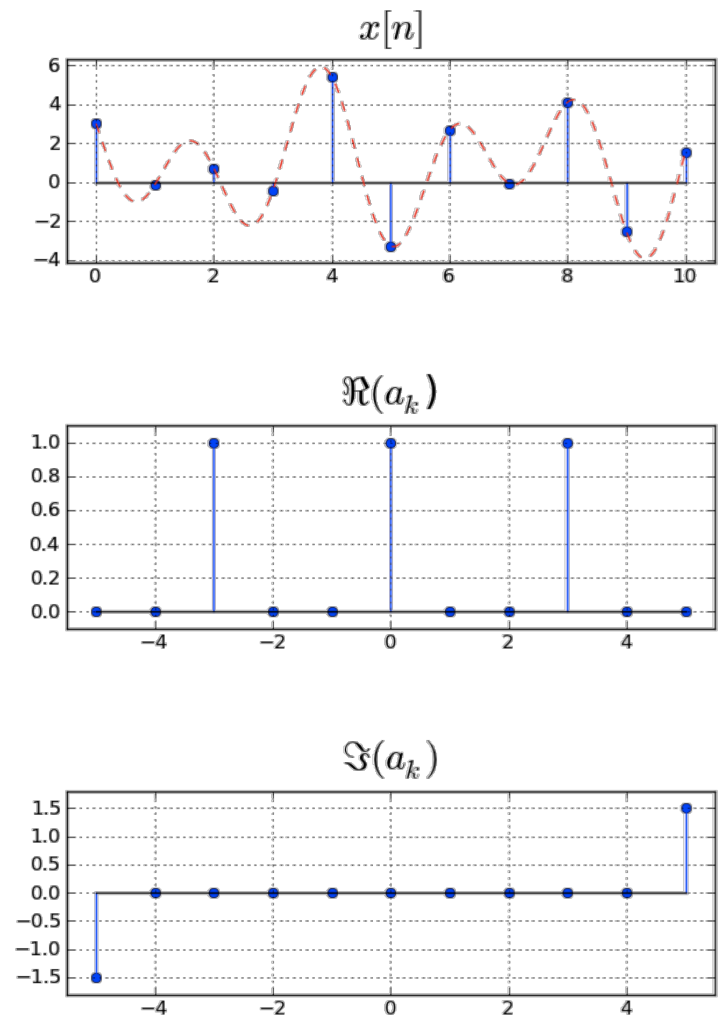
$A_{\pm 3} = 2(1/2) = 1$ [from cos term]

$A_{-5} = -3(j/2) = -1.5j$ [from sin term]

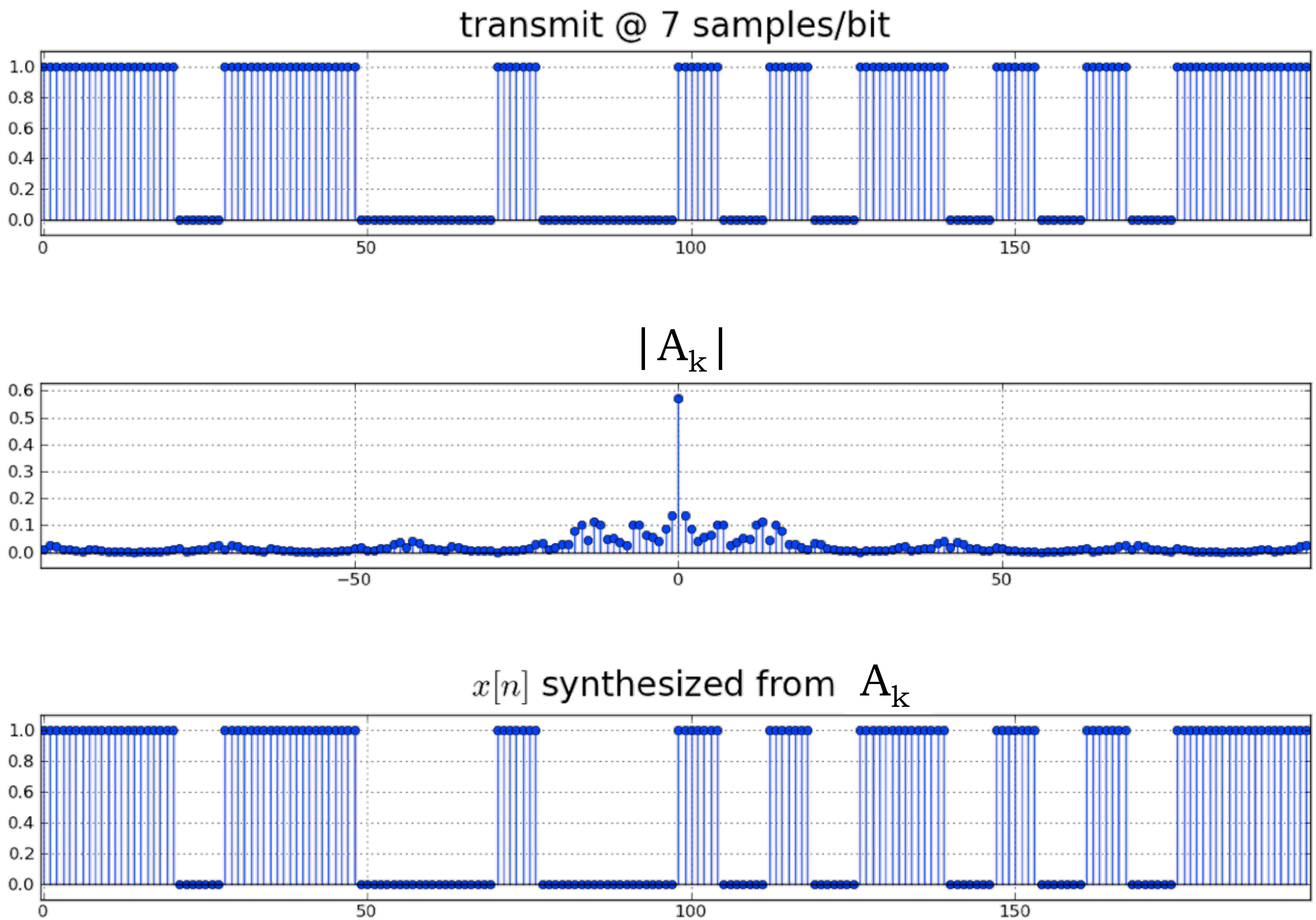
$A_5 = -3(-j/2) = 1.5j$

$A_k = 0$ otherwise

Again, P is *odd* here (=11), so the end points of the frequency scale are at $\pm(\pi - (\pi/P))$, not $\pm\pi$.



Spectrum of Digital Transmissions



Observations on previous figure

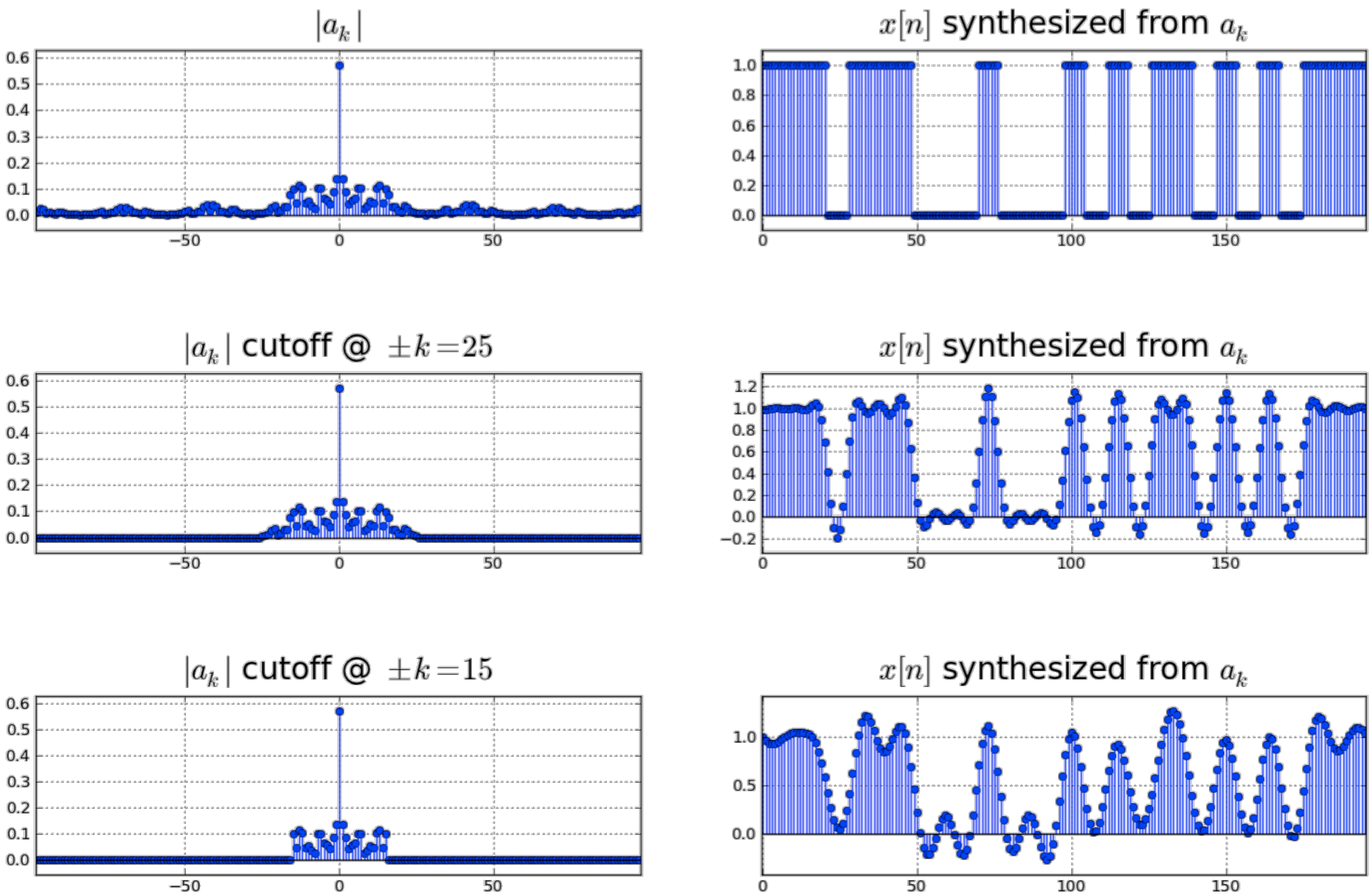
- The waveform x[n] cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (The is rough and qualitative, as x[n] is not sinusoidal.)

- A period of 14 corresponds to a frequency of $2\pi/14 = \pi/7$, which is 1/7 of the way from 0 to the positive end of the frequency axis at π (so k approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

- There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

- And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.

Effect of Band-limiting a Transmission



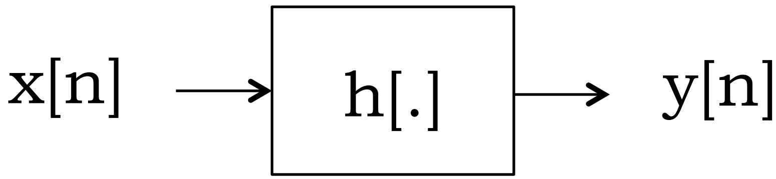
The DTFS is also good for finite-duration signals!

Claim: Over any contiguous interval of length P that we may be interested in --- say n=0,1,…,P–1 for concreteness --- an arbitrary DT signal x[n] can be written in the form

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$

What's going on here? If we know we will only be interested in the interval [0,P–1], then it doesn't matter that our representation above will create periodically repeating extensions outside the interval of interest.

Application



Suppose x[n] is nonzero only over the time interval $[0, n_x]$, and h[n] is nonzero only over the time interval $[0, n_h]$.

In what time interval can the non-zero values of y[n] be guaranteed to lie? The interval $[0, n_x + n_h]$.

Since all the action we are interested in is confined to this interval, choose $P - 1 \geq n_x + n_h$, then use the DTFS to represent x[n] and y[n] over this interval.

This is actually the much more common use of the DTFS!

The Need for Speed: Fast Fourier Transform (FFT)

$$x[n] = \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n} \quad , \quad X_k = \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

Computing these series involves $O(P^2)$ operations – when P gets large, the computations get very s l o w….

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

$P = 1000, \; P^2 = 1000000, \; P \log P \approx 10000$