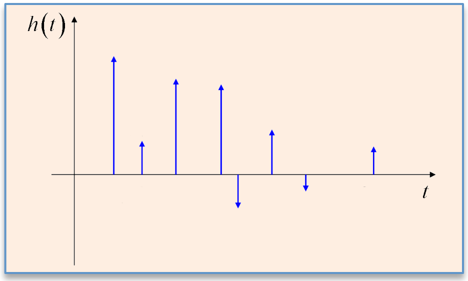
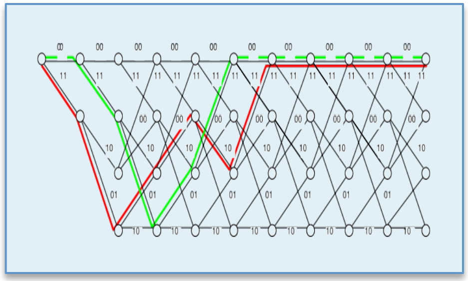
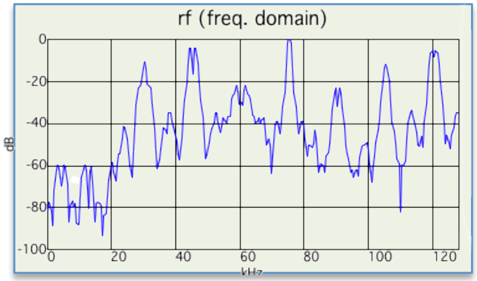
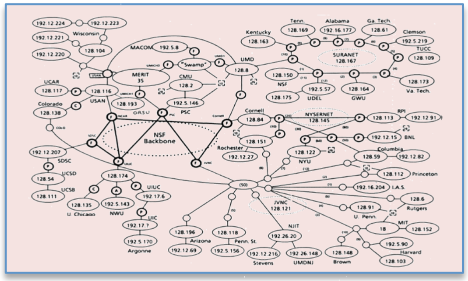




INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2011
Lecture #15

- More on signal spectra
- Modulation & demodulation
- Frequency-division multiplexing (FDM)

Main Messages So Far (in words)

1. A huge class of discrete-time (and continuous-time) signals (periodic, as we've seen, but also absolutely summable/square summable/slow growth)

can be written --- using **Fourier** series and transforms --- as

weighted sums of sinusoids
(ranging from very slow to very fast)
or (equivalently, but more compactly)
complex exponentials.

The sums can be discrete $\sum$ or continuous $\int$ (or both).

2. **LTI** systems act very simply on sums of sinusoids:
superposition of responses to each sinusoid, with the **frequency response** determining the frequency-dependent scaling of magnitude, shifting in phase.

Weighted Sums of Complex Exponentials

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n} = \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n}$$

Synthesis equation for periodic signals of period P, or signals of finite duration P

$$X_k = A_k P = \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

Analysis equation

Sinusoids versus Complex Exponentials (six of one or half-a-dozen of the other)

For a real signal, $A_{-k} = A_k^*$, so we can combine

$$A_{-k}\exp(-j\Omega n) \; + \; A_k\exp(j\Omega n)$$

into the following single term for $1 \leq k \leq P/2$ [for even P, otherwise $(P-1)/2$]:

2 x Real part of $A_k\exp(j\Omega n)$

$$= 2\ |A_k|\cos(\Omega n + \angle A_k)$$

So:

$$x[n] = A_0 + \sum_{k \geq 1} 2\,|\,A_k\,|\cos(\Omega_k n + \angle A_k)$$

(Note that A_0 is real anyway.)

Some examples of periodic signals, period P

- $x[n] = 1$ for all n

$\Rightarrow A_0 = 1$, all other $A_k = 0$, i.e., signal content concentrated at "DC"

- $x[n] = \cos(\Omega_3 n)$, $\Omega_3 = 3(2\pi/P)$

$\Rightarrow A_{-3} = A_3 = 0.5$, all other $A_k = 0$, i.e., content at Ω_3 , all cosine

- $x[n] = \sin(\Omega_3 n)$

$\Rightarrow A_{-3} = 0.5j$, $A_3 = -0.5j$, all other $A_k = 0$, i.e., content at Ω_3 , all sine

- $x[n] = \cos(\Omega_3 n+\phi)$

$\Rightarrow A_{-3} = 0.5 \exp(-j\phi)$, $A_3 = 0.5 \exp(j\phi)$, all other $A_k = 0$, i.e., content at Ω_3 , mixed sine and cosine

- $x[n] = [1, 0, 0, \ldots , 0]$ (P values, repeated periodically, $\delta_P[n]$)

$\Rightarrow$ all $A_k=1/P$, i.e., content uniformly spread over all frequencies!

DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad H(\Omega) = \sum_n h[n] e^{-j\Omega n}$$

Any contiguous interval of length 2π

then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$, not just P specific values.

Forget you saw that!

(More in 6.003, 6.011)

(OK, maybe just two more slides on it here!)

Another way to come at the DTFT

In the DTFS, let P go to infinity, to indicate a non-periodic signal. Then with $2\pi / P \approx d\Omega$ and replacing summation by integration, you can see how the DTFS

$$x[n] = \frac{1}{2\pi} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n} \frac{2\pi}{P} \ ,$$

$$X_k = \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

could become the DTFT

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

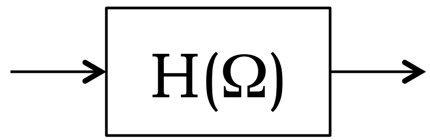
where

$$X(\Omega) = \sum_{n} x[n] e^{-j\Omega n}$$

This is why I like the X_k choice for DTFS ceofficients, rather than the A_k choice.

Input/Output Behavior of LTI System in Frequency Domain

$$x[n] = \sum_{k=\langle P \rangle} A_k e^{j\Omega_k n}$$



$$y[n] = \sum_{k=\langle P \rangle} H(\Omega_k) A_k e^{j\Omega_k n}$$

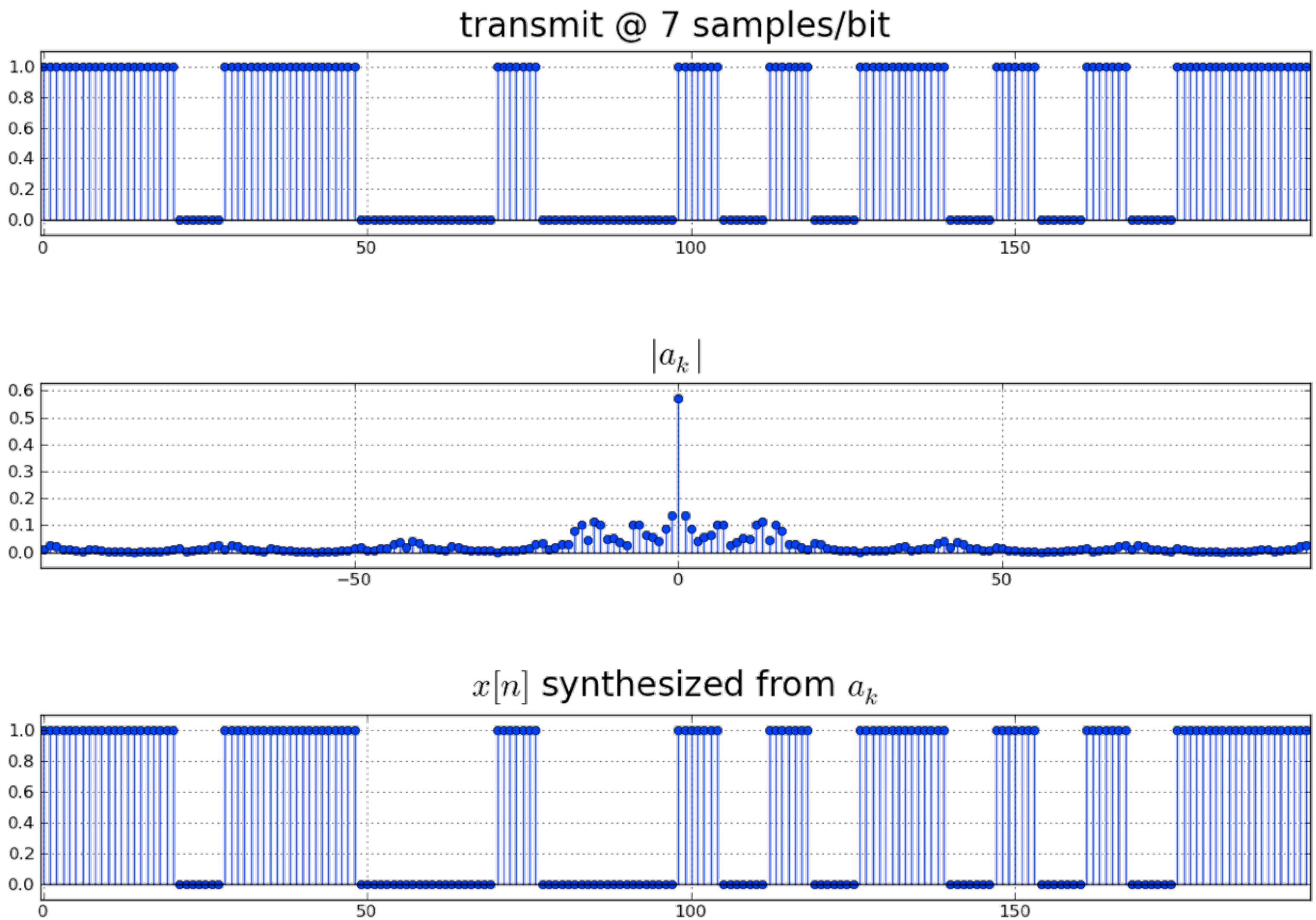
$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

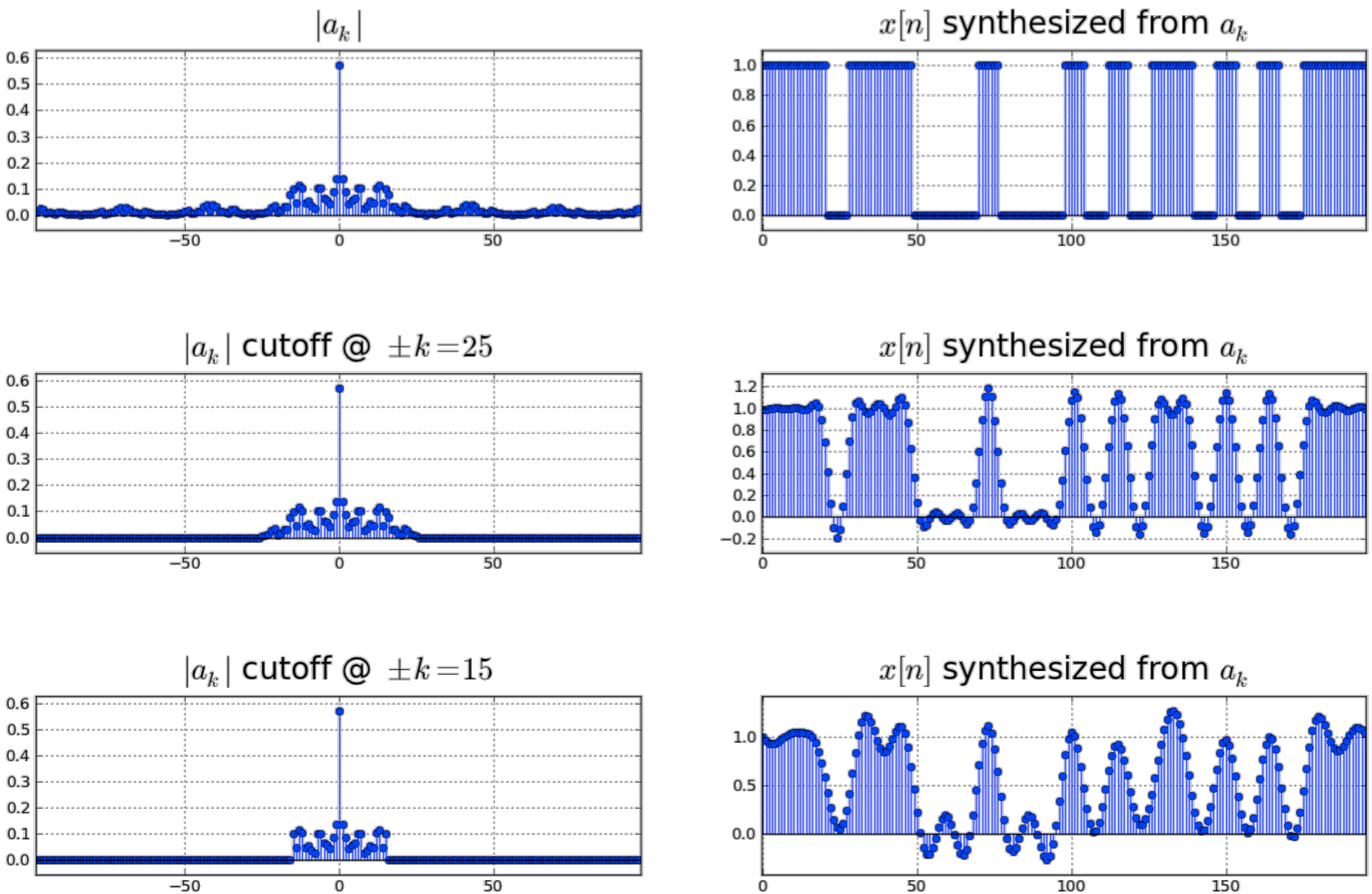
$$Y_k = H(\Omega_k) X_k$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Spectrum of Digital Transmissions



Effect of Band-limiting a Transmission



The Need for Speed:
Fast Fourier Transform (FFT)

$$x[n] = \frac{1}{P} \sum_{k=\langle P \rangle} X_k e^{j\Omega_k n} \quad , \quad X_k = \sum_{n=\langle P \rangle} x[n]\, e^{-j\Omega_k n}$$

Computing these series involves ~P^2 operations, so when P gets large, the computations get very s l o w ….

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform and its inverse (aka the **FFT**, IFFT), exploiting the nice structure of complex exponentials, rediscovering a technique known to Gauss (the "Prince of Mathematicians" was there first!). This method takes ~$P\log_2 P$ operations.

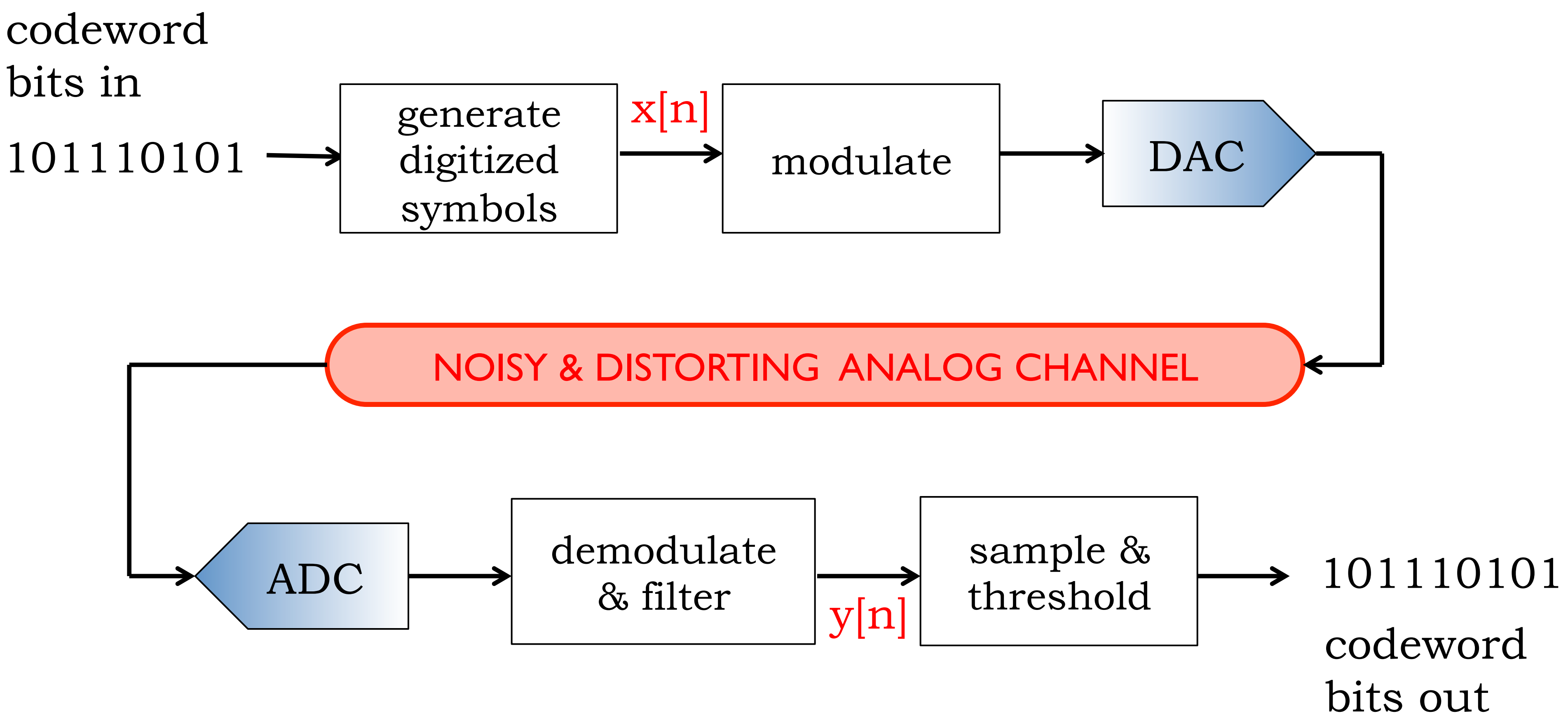
$P = 1{,}024$; $P^2 = 1{,}048{,}576$; $P\log_2 P \approx 10{,}240$

OK, back to digital communications

better equipped with language and tools to understand and analyze

a key part of the system

From Baseband to Modulated Signal, and Back



From Brant Rock tower, radio age was sparked
By Carolyn Y. Johnson, Globe Staff | July 30, 2006

MARSHFIELD, MA -- A century ago*, radio pioneer **Reginald A. Fessenden** used a massive 420-foot radio tower that dwarfed Brant Rock to send voice and music to ships along the Atlantic coast, in what has become known as the world's first voice radio broadcast.

This week, Marshfield will lay claim to its little-known radio heritage with a three-day extravaganza to celebrate the feat -- including pilgrimages to the base of the long-dismantled tower, a cocktail to be named the Fessenden Fizz, and a dramatic reenactment of the historic moment, called ``Miracle at Brant Rock."

Audio Signals Carried on Electromagnetic Waves Propagating through the Atmosphere

*Christmas Eve, 1906

Some ballpark numbers

"420 ft antenna," say 150 m. For this to be comparable with a quarter-wavelength of sinusoidal EM wave traveling at 3×10^8 m/s, we need a frequency of $3 \times 10^8/(4 \times 150) = 0.5 \times 10^6$ Hz = 500 kHz.

Fessenden, using a special rotating electrical generator built by GE, managed 50 kHz! --- and not with much power. But he invented and demonstrated amplitude modulation (AM): amplitude variations of the sinusoidal "carrier" signal carry the signal of interest.

Fessenden started his scientific work with Edison in NJ.
His application to Edison said "Do not know anything about electricity, but can learn pretty quick."
Edison wrote back to say "Have enough men now that do not know about electricity."

Fessenden was awarded around 500 patents in his lifetime!
Lived in Chestnut Hill, Newton, in a house that's still there, and on the National Register of Historic Places (because he lived there!).

AM Radio today

Broadcast power 250 W to 50 kW. No rotating machines! --- just electronic oscillators.

AM radio stations are on 520 – 1610 kHz (‘medium wave”) in the US, with carrier frequencies of different stations spaced 10 kHz apart.

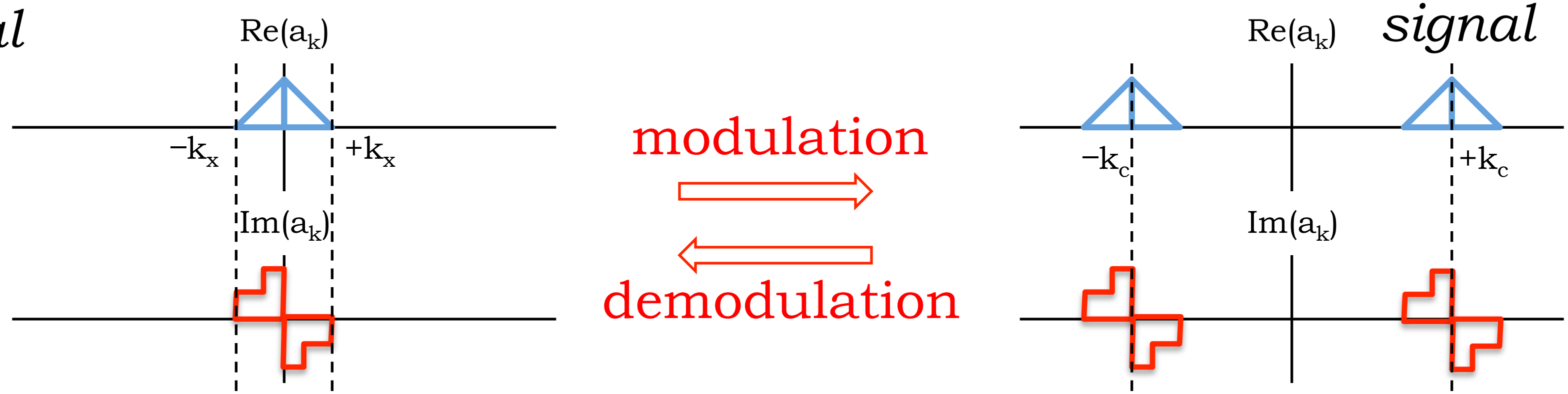
The principles of AM, with various extensions and modifications, find their way into many other settings (not just AM on your radio dial!).

Using Some Piece of the Spectrum

- You have: a band-limited signal x[n] at *baseband* (i.e., centered around 0 frequency).
- You want: the same signal, but centered around some specific frequency $k_c\Omega_1$.
- Modulation: convert from baseband up to $k_c\Omega_1$.
- Demodulation: convert from $k_c\Omega_1$ down to baseband

Spectrum of ***baseband*** *signal*

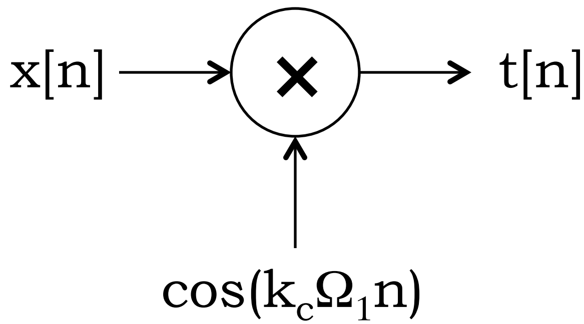
Spectrum of ***transmitted*** *signal*



Signal centered at 0

Signal centered at k_c

Modulation

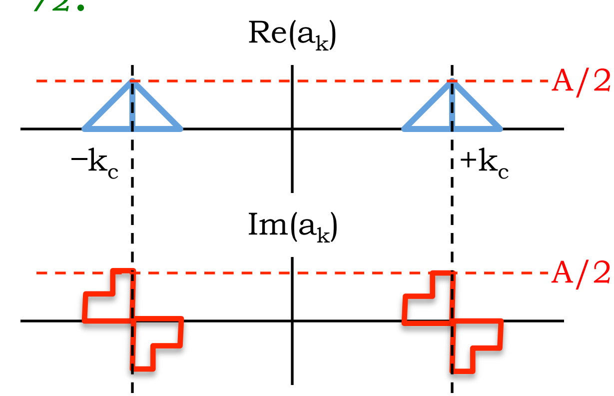


For band-limited signal A_k are nonzero only for small range of $\pm k$

i.e., just replicate baseband signal at $\pm k_c$, and scale by ½.

$$t[n] = \left[\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n}\right]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}$$



“Heterodyne principle” (Fessenden)

$$\cos(\Omega_{\text{mod}} n).\cos(\Omega_{car} n)$$

$$=$$

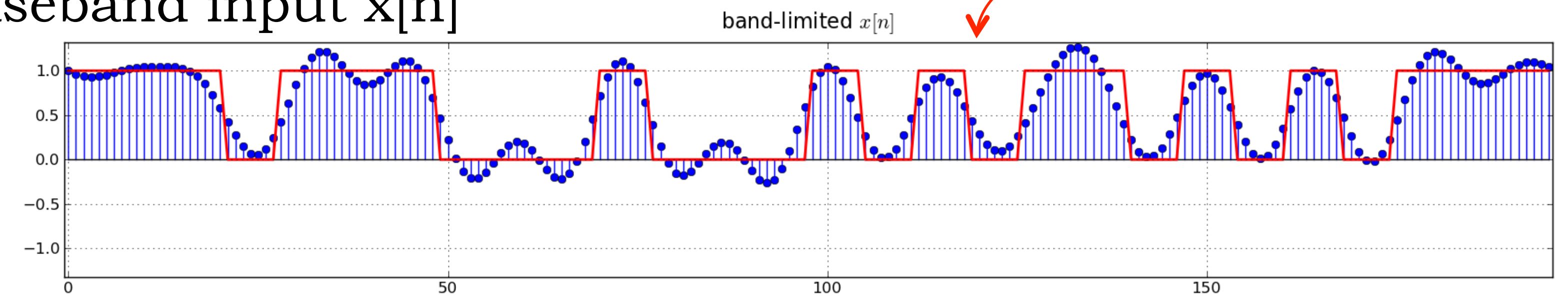
$$0.5\{\cos[(\Omega_{car}+\Omega_{\text{mod}})n]+\cos[(\Omega_{car}-\Omega_{\text{mod}})n]\}$$

Multiplying two sinusoids causes the
sum and difference frequencies to appear.

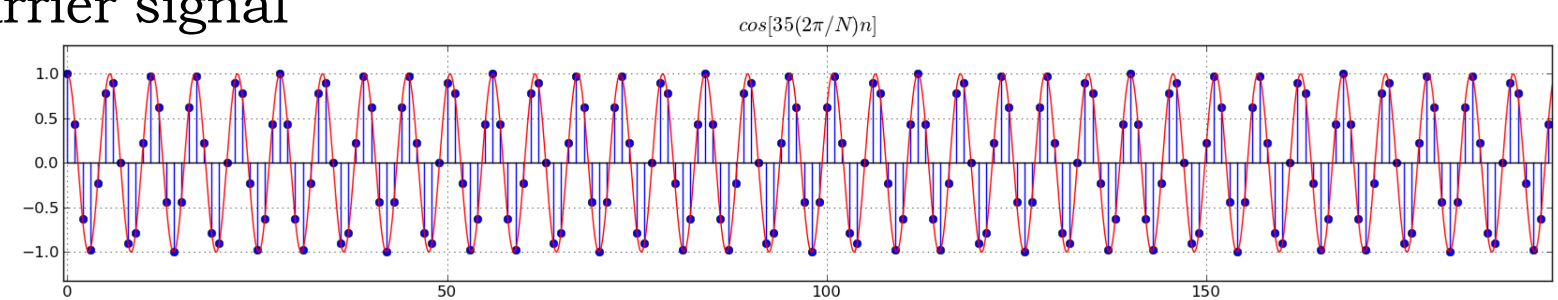
Example: Modulation (time)

Shaped pulses! Chosen because we know the channel is bandlimited

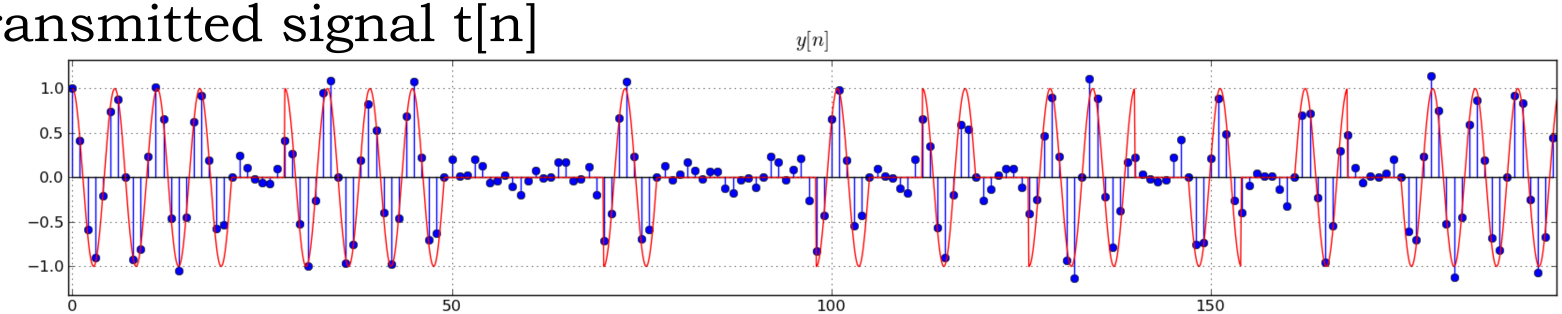
Baseband input x[n]



Carrier signal

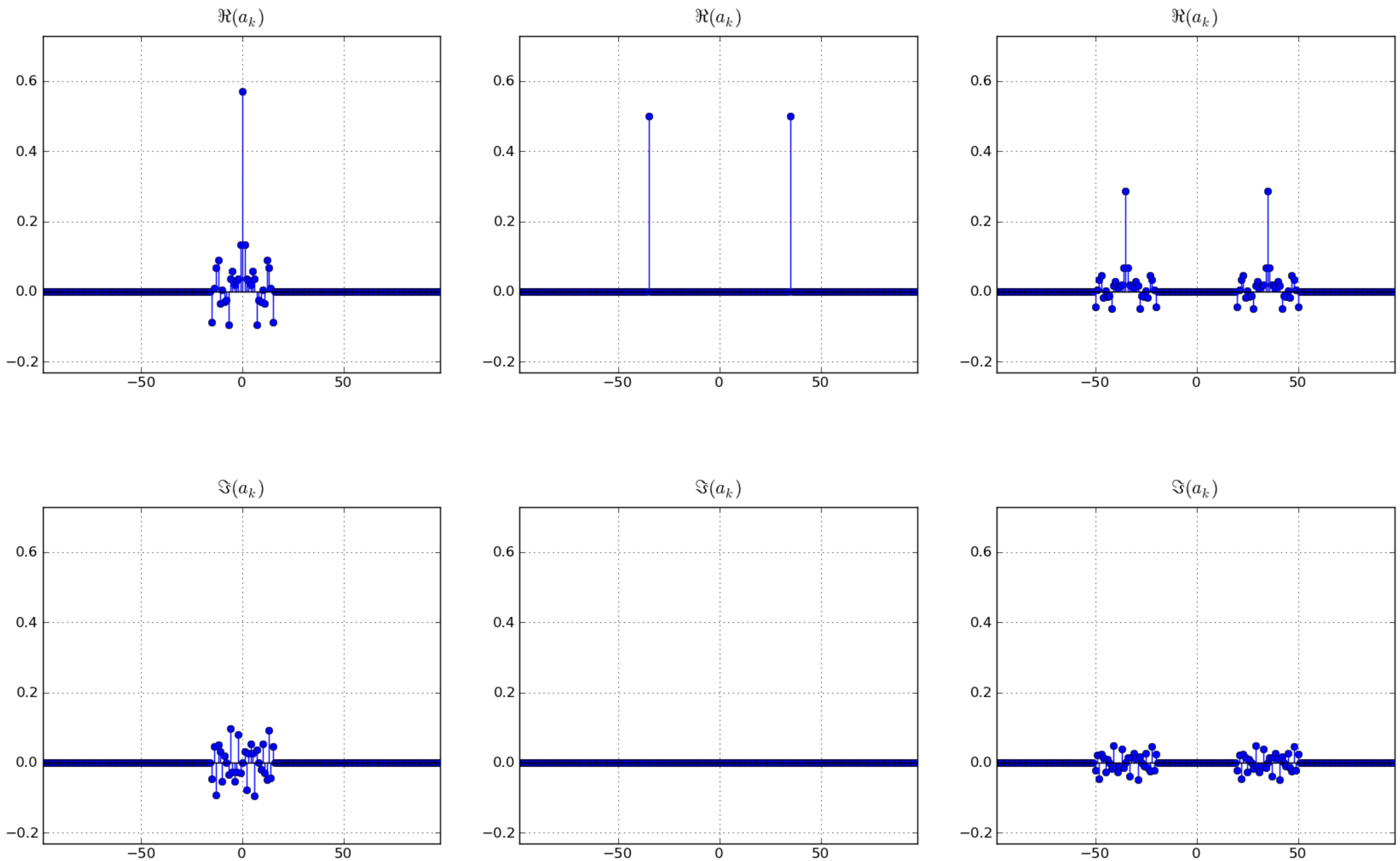


Transmitted signal t[n]



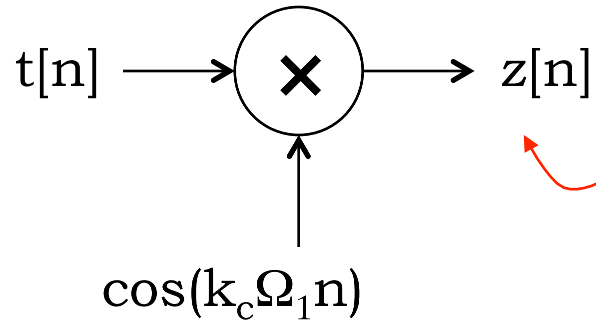
Example: Modulation (freq domain picture)

Band-limited x[n] $\cos(35\Omega_1 n)$ t[n]



Demodulation

Assuming no distortion or noise on channel, so what was transmitted is received



Hmm. So z[n] has what we want at baseband, but has signal we don't want at $\pm 2k_c\Omega_1$

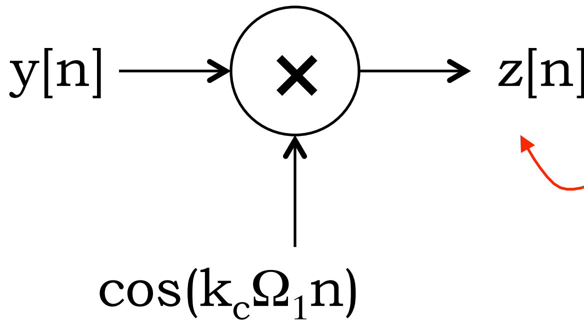
$$z[n] = t[n]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \left[\frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}\right]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k+2k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n} + \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k-2k_c)\Omega_1 n}$$

What we want

Demodulation



Hmm. So z[n] has what we want at baseband, but has signal we don't want at $\pm 2k_c\Omega_1$

$$z[n] = y[n]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \left[\frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}\right]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k+2k_c)\Omega_1 n} + \underbrace{\frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n}}_{\text{What we want}} + \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k-2k_c)\Omega_1 n}$$

What we want

That is just Fessenden's heterodyne principle at work again:

Taking the sum and difference frequencies of

$$\Omega_c$$

with the sum and difference frequencies of

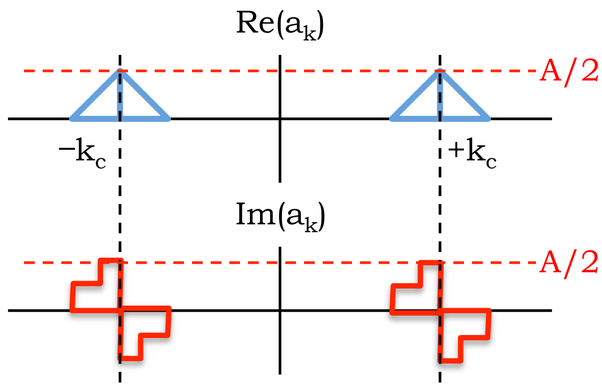
$$\Omega_c \text{ and } \Omega_m ,$$

i.e., $\pm\Omega_c \pm \Omega_m$

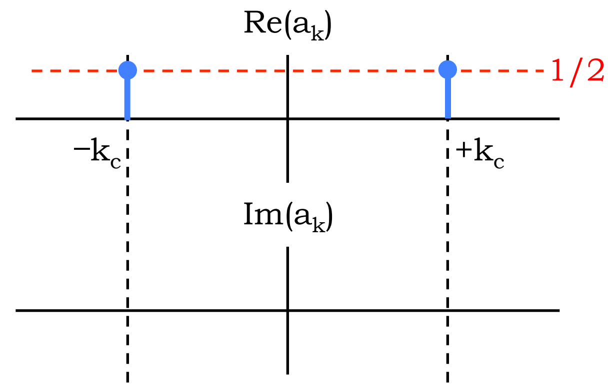
results in components at frequencies that are

$$\pm\Omega_m \text{ away from } 0, -2\Omega_c , +2\Omega_c$$

Demodulation Frequency Diagram

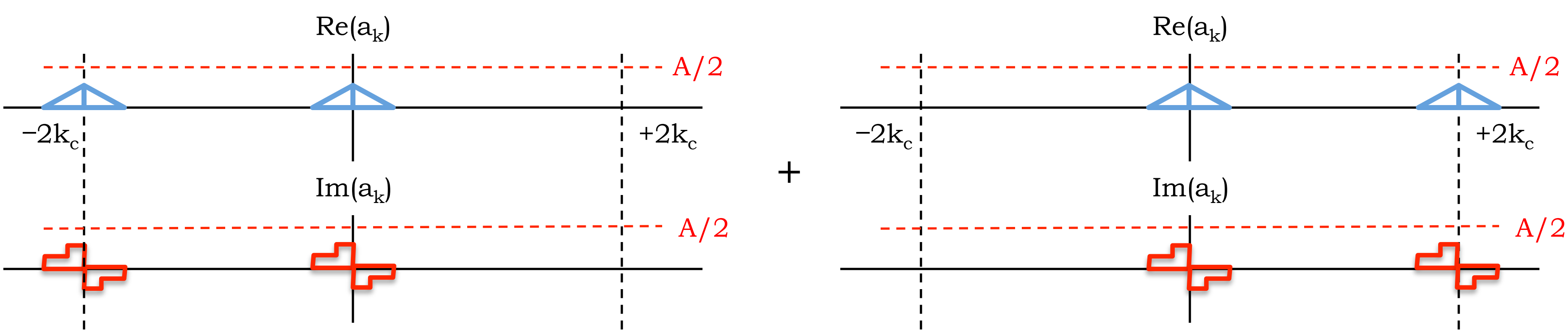


t[n]

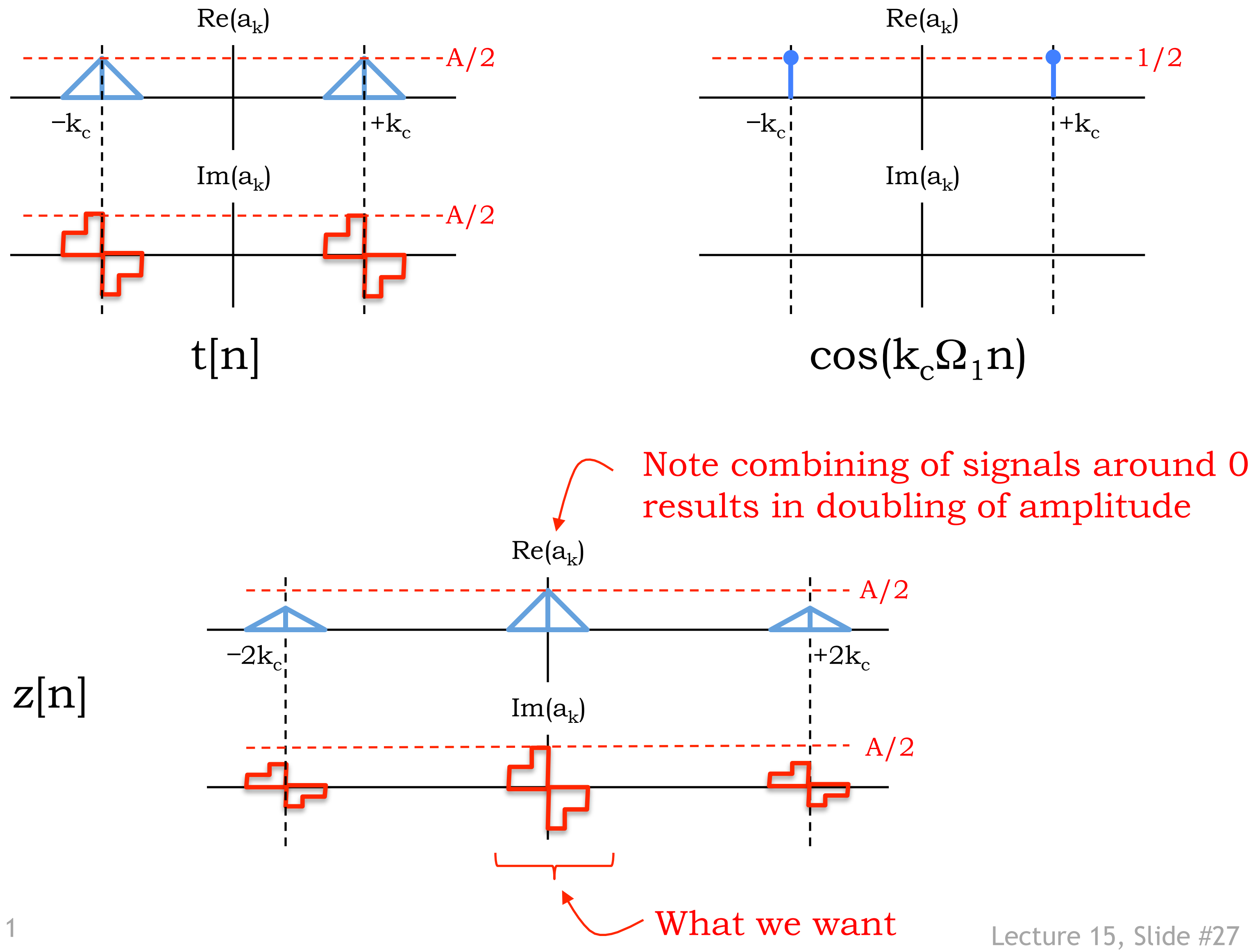


$\cos(k_c\Omega_1 n)$

z[n]



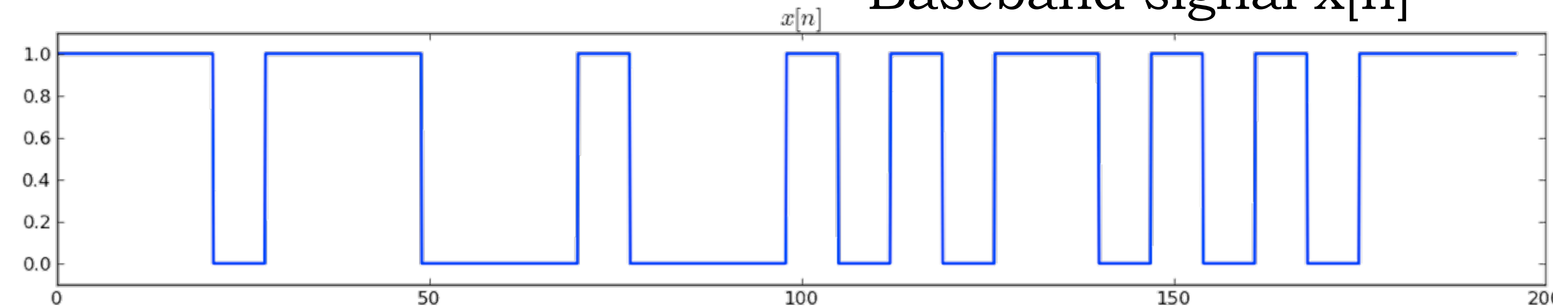
Demodulation Frequency Diagram



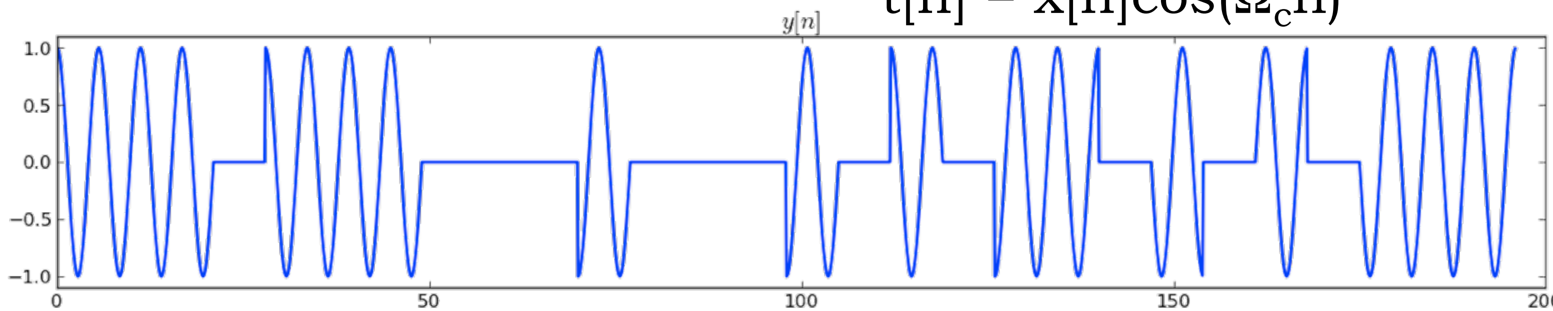
Example: Demodulation (time)

Showing idealized signals ---
no bandwidth limit on channel

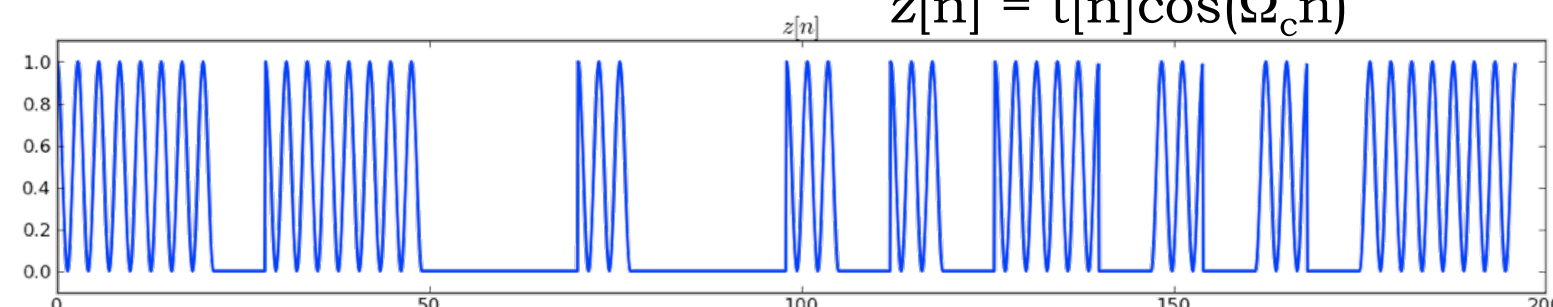
Baseband signal x[n]



$t[n] = x[n]\cos(\Omega_c n)$

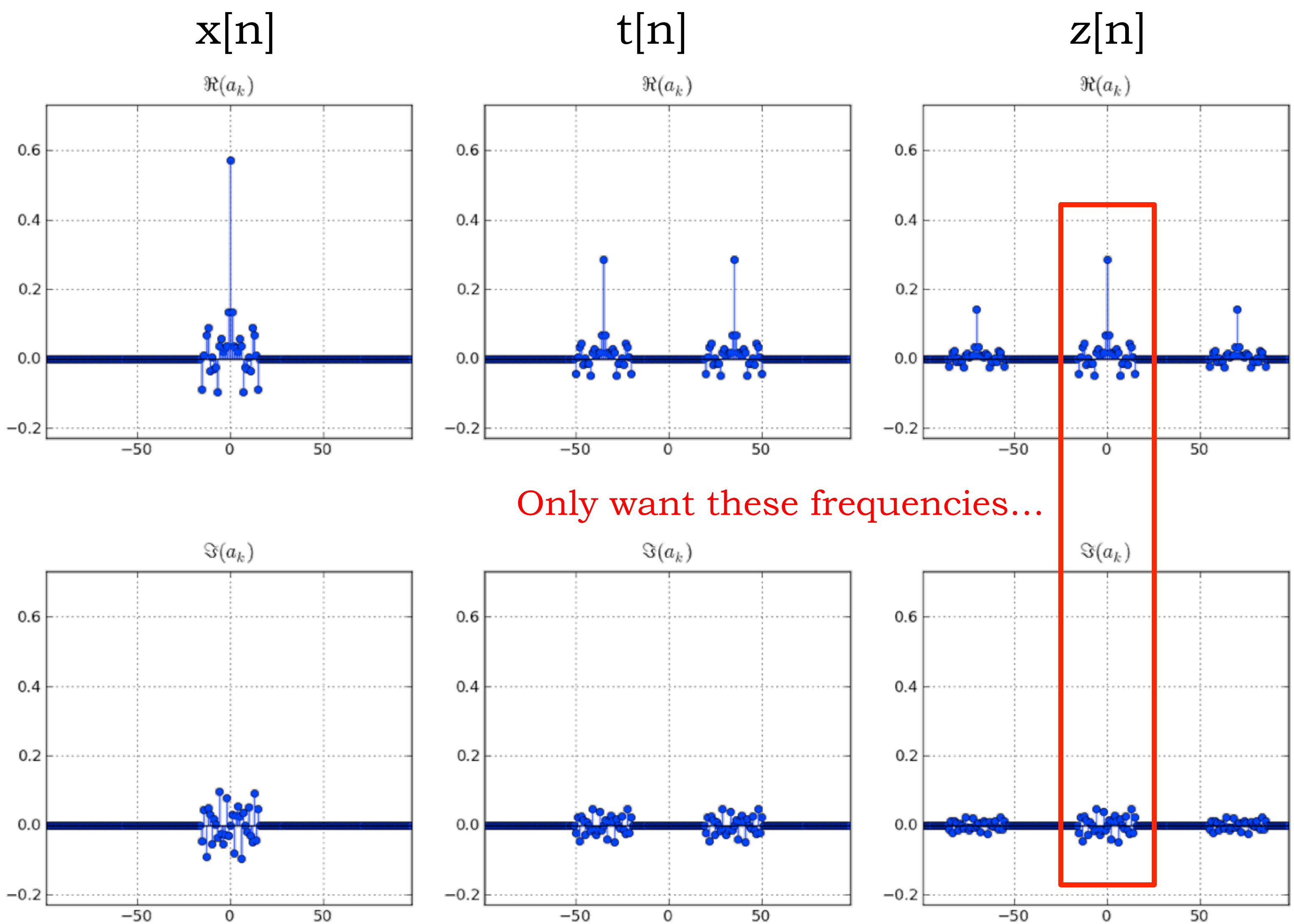


$z[n] = t[n]\cos(\Omega_c n)$



Note: lowpass filtering of this signal will yield x[n]/2 !

Example: Demodulation (freq)



Demodulation + LPF