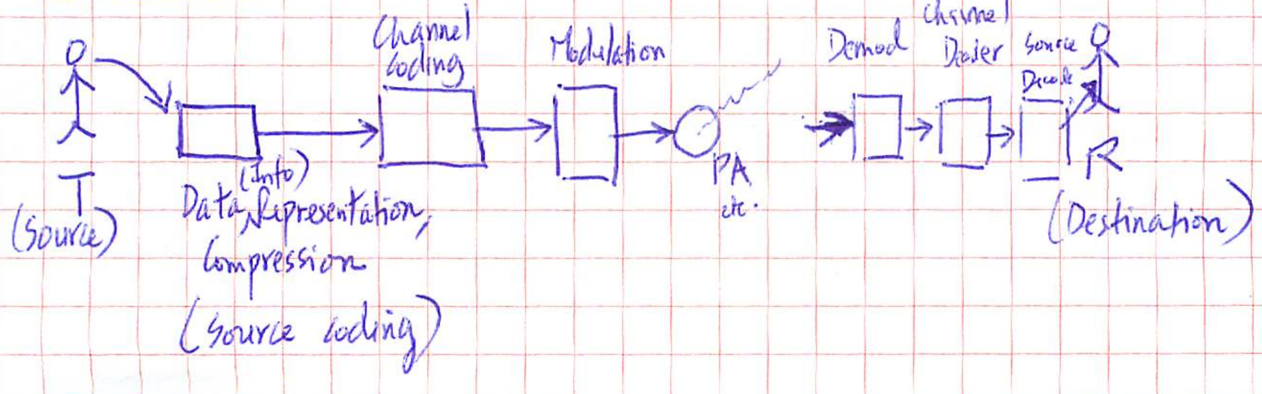
Recitation 1

1. Overview of Comm. System:



2. Information Representation and Measure:

More likely event → less info
Less likely event → more info

ie

$$I \propto \frac{1}{p}$$

✓ Better more quantitative measure ~ $\boxed{I = \log_2\left(\frac{1}{p}\right)}$ bits. – How much info?

(*) If 2 events are equally likely, ie $p_i = \frac{1}{2}$, then $I_i = 1$ bit.

eg. represent events as "0" and "1".

(*) If 2 independent events both happen, $p = p_i \cdot p_j$

so that $I = I_i + I_j$ information 'additive'

(*) For N equiprobable events, if told that M has occured, how much info? $I = \log_2\left(\frac{1}{\left(\frac{M}{N}\right)}\right) = \log_2\left(\frac{N}{M}\right)$ bits.

(eg) If told drawn card is 'spade' (13 possible cards), info you get

$$I = \log_2\left(\frac{1}{\left(\frac{13}{52}\right)}\right) = \log_2\left(\frac{52}{13}\right) = 2 \text{ bits}$$

Calculator:

$$\log_2 X = \frac{\log_{10} X}{\log_{10} 2}$$

2-2: Entropy ~ expected value of information.

$$H(X) = E\left[I(X)\right] = \sum p(x) \cdot \log_2\left(\frac{1}{p(x)}\right)$$

~ average amt. of information needed to resolve all uncertainty!

~ lower bound on number of bits needed to encode message.

Some Examples:

(*) Recall: fair die.

$p(H) = p(T) = \frac{1}{2}$

one-flip of a fair die: $p_i = \frac{1}{2}$

✓ Amount of information $I = \log_2\left(\frac{1}{(1/2)}\right) = \underline{1 \cdot \text{bit}}$.

✓ Expected (ideal) length of representation $= \sum p_i \cdot \log_2\left(\frac{1}{p_i}\right) = \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(1) = \underline{1\text{-bit}}$

Now: unfair die.

$p(H) = 0.98 \quad p(T) = 0.02$

✓ Expected length of representation $= (0.98)\log_2\left(\frac{1}{0.98}\right) + (0.02)\log_2\left(\frac{1}{0.02}\right)$

$\approx \underline{0.14 \text{ bits} << 1\text{-bit}}$.

What does this mean? "0.14" bits ???

* Coin almost always comes up "Heads" → average (information) uncertainty small!

(*) Thus, for 1000 throws (flips), need only 140-bits to represent sequence!

Why — Most likely is 'H', so only need to indicate where 'T' occurs — need only 140-bits to do that!

Another Example:

1. Alice, Bob, Charlie, Deb trying to guess a 3-bit binary number.

Alice is told "number is odd"

Bob " " "not a multiple of 3" (ie, not 0, 3, 6)

Charlie " " "contains exactly two - 1's"

Deb given all the above clues.

Q) How much information did each player receive about the number?

Soln:

A : {001, 011, 101, 111} ~ 4 possibilities out of 8.

$$P_A = \frac{4}{8} =$$

$$I_A = \log_2\left(\frac{1}{(4/8)}\right) = \underline{1\text{-bit}}$$

B : could be {1, 2, 4, 5, 7} ~ 5 possibilities out of 8.

$$P_B = \frac{5}{8}$$

$$I_B = \log_2\left(\frac{1}{(5/8)}\right) = \underline{0.678\text{-bits}}.$$

C : {011, 101, 110} or ${}^3C_2$ = 3 possibilities out of 8

$$P_C = \frac{3}{8}$$

$$I_B = \log_2\left(\frac{1}{(3/8)}\right) = \underline{1.42\text{-bits}}.$$

~~D : {000, 001, 010, 011, 100, 101, 110, 111}~~
B A A A B A

B
D = {000, 001, 010, 011, 100, 101, 110, 111}
A A A A
B(1) B(2) B(4) B(5) B(7)
C C C

$p = 1/8$

$$I = \log_2\left(\frac{1}{(1/8)}\right) = \underline{3.5}$$

$$I = \log_2\left(\frac{1}{p}\right) = \frac{\log_{10}\left(\frac{1}{p}\right)}{\log_{10} 2}$$

where $\log_{10} 2 \approx 0.301$

$$\boxed{\log_a b = \frac{\log_x b}{\log_x a}}$$

Chapter 2 (All the questions come from exercises of chapter 2 note)

Q1: Alice: Possibility $= \frac{4}{8}$ $\quad I = \log 2\left(\frac{1}{\frac{4}{8}}\right) = 1$

Bob: Possibility $= \frac{5}{8}$ $\quad I = \log\left(\frac{1}{\frac{5}{8}}\right) = 0.678$

Charlie: Possibility $= \frac{3}{8}$ $\quad I = \log 2\left(\frac{1}{\frac{3}{8}}\right) = 1.415$

? Deb: Possibility $= \frac{1}{8}$ $\quad I = \log 2\left(\frac{1}{\frac{1}{8}}\right) = 3$

Q2: (a) $I = \log 2\left(\frac{1}{p}\right) = \log 2\left(\frac{1}{0.001}\right) = \log 2\, 1000$

(b) $\sum P_i \log 2\left(\frac{1}{P_i}\right) = 0.25 \log 2\left(\frac{1}{0.25}\right)$

$+ 0.75 \log 2\left(\frac{1}{0.75}\right)$

Q3: $C_2^4 = \frac{4*3}{2*1} = 6$

$I = \log 2\left(\frac{1}{p}\right) = \log 2\left(\frac{1}{\frac{6}{16}}\right) = \log 2\ 2.667$

$= 1.415$

Q4: $P = \frac{49}{52}$

$I = \log_2 \frac{1}{P} = \log_2 \frac{52}{49} = 0.0857$

000
001
010
011
100
101
110
111