Recitation 5

9-22-2011

Today

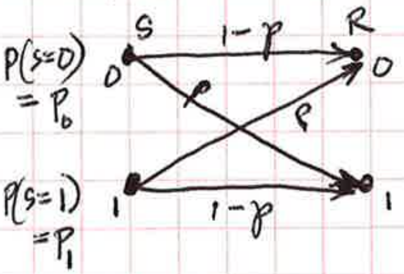
1. BSC review
2. Overview of PDF, CDF – Random Processes
   - eg. Binomial Distr.
   - eg. Gaussian Distr.
3. Optimum Detection
   - Digitizing thresholds & Decision Rule
   - BER calculations and Packet Error Rate

Note: Independent vs. Disjoint

(*) Events E_1 and E_2 are disjoint (mutually exclusive) if
$P(E_1 \cup E_2) = P(E_1) + P(E_2)$
ie $P(E_1 \cap E_2) = \phi$

(*) Events E_1 and E_2 are independent if $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

(1) BSC Review



Model:

(*) Consider source (sender) S producing symbols 0, 1 with marginal probabilities (a priori) P_0 and P_1 respectively.

over a channel with error probability (crossover) of p.

ie. $P(R=0|S=1) = P(R=1|S=0) = p$.

Find:

(A) $P(R=1) = ?$

Soln: Total Probability Thm:

$$\boxed{P(A) = \sum_{i=1}^{n} \underbrace{P(A|E_i)}_{i^{th}\text{ cond. prob.}} \cdot \underbrace{P(E_i)}_{\text{marginal prob.}}}$$

for collectively exclusive and disjoint events.

So, $P(R=1) = \underbrace{P(R=1|S=0)}_{p} \cdot \underbrace{P(S=0)}_{P_0} + \underbrace{P(R=1|S=1)}_{(1-p)} \cdot \underbrace{P(S=1)}_{P_1}$ ~ ie. sum of all probs into "1"

$= pP_0 + (1-p)P_1$ ✓

(B) $P(S=0|R=1) = ?$ Soln: Bayes' Rule:

$$\boxed{P(E_i|A) = \frac{P(A|E_i) \cdot P(E_i)}{P(A)} = \frac{P(A|E_i) \cdot P(E_i)}{\sum_{i=1} [P(A|E_i) \cdot P(E_i)]}}$$

So that $P(S=0|R=1) = \dfrac{P(R=1|S=0) \cdot P(S=0)}{P(R=1)}$ ~ from above, using total prob. thm.

$= \dfrac{p \cdot P_0}{pP_0 + (1-p) \cdot P_1}$ ✓

(2) $\boxed{\text{Random Processes}}$

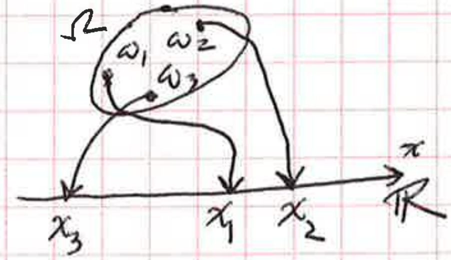
— Good model for describing noise and communication generally.

— Information is inherently random, ie. uncertainty!
~ cannot predict with certainty.

(*) RP is RV with time as parameter.

What is RV? — RV is a mapping (function) of outcomes in Ω to $\mathbb{R}$ (set of real numbers)

ie. $\omega \xrightarrow[\text{maps}]{X} x$ $\qquad X(\omega) = x$



— RVs represented by capital X, Y, etc.
~ individual values of X denoted $X(\omega) = x$.

(*) RV's characterized by the Probability Density Functions (pdf) $f_X(x)$ defined as

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

where $F_X(x)$ is the cumulative distribution function (cdf)

with following properties:

① The cdf is a non-decreasing probability distribution that gives

$$F_X(x) = P(X \leq x) = P\{\omega \in \Omega \mid X(\omega) = x\}$$

② $P(a < X \leq b) = F_X(b) - F_X(a)$

③ $0 \leq F_X(x) \leq 1$ ~ a probability fnc.

Generally,
$$\int_A f_X(x)\,dx = P(X \in A)$$

In particular,
$$\int_{a^+}^{b^+} f_X(x)\,dx = P(a < X \leq b)$$

(*) Let's look more closely @ ~~two~~ one of the most useful pdfs in communication and statistics

① Binomial Distribution ??? (Not today)

✓ ② Gaussian (Normal) Distribution

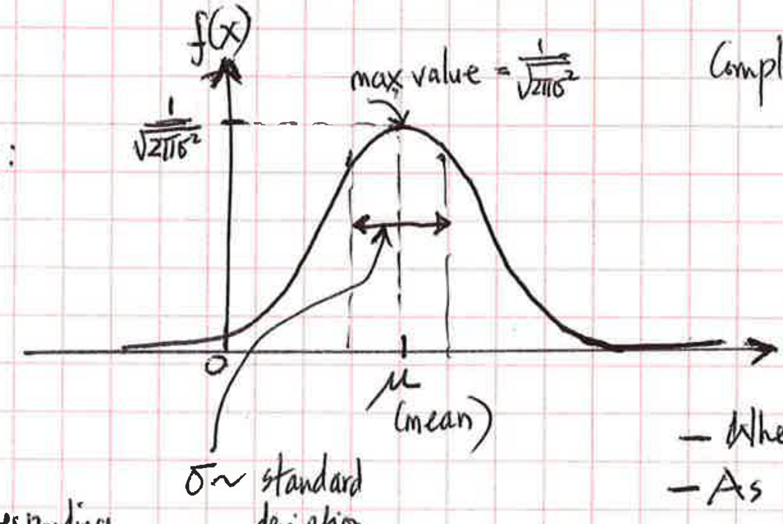
Gaussian (Normal) Distribution – perhaps the most important distr. fnc. in communications!

PDF: $$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$$

where $\mu \sim$ mean
$\sigma^2 \sim$ variance, $(\text{standard deviation})^2$
$\sigma \sim$ standard deviation.

Completely specified by μ and σ^2.

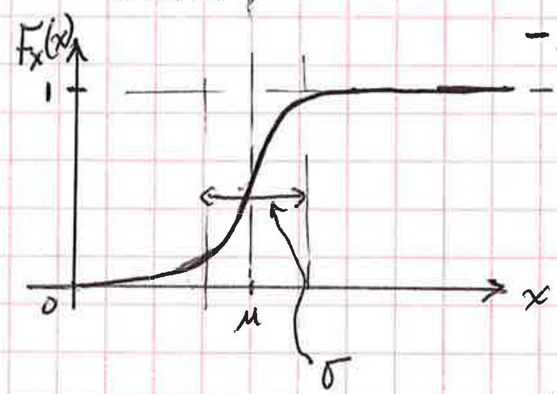
Plot:



Some observations:

– When $x=\mu$, $f(x)$ peaks at $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}$
– As $\sigma \to 0$, $f(x) \to$ delta fnc at $x=\mu$.
– As $\sigma \to$ large, peak drops and $f(x)$ spreads out.

Its corresponding cdf:



$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f_X(x)\,dx$$
$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Let $t = -\left(\frac{x-\mu}{\sigma}\right)$
So that $dt = -\frac{1}{\sigma}dx$
Also $x \to -\infty \Rightarrow t \to +\infty$
$x = a \Rightarrow t = -\left(\frac{a-\mu}{\sigma}\right)$

$$= -\int_{+\infty}^{\frac{\mu-a}{\sigma}} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-t^2/2} \cdot \sigma\,dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\mu-a}{\sigma}}^{\infty} e^{-t^2/2}\,dt = Q\left(\frac{\mu-a}{\sigma}\right)$$

i.e. $$\boxed{F_X(a) = Q\left(\frac{\mu-a}{\sigma}\right)}$$

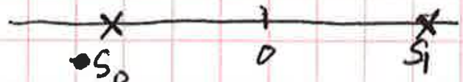
where $$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\,dt$$

(3) Optimum Detection

Consider binary PAM signals given by

$$s_1 = -s_0 = \sqrt{E_b}\,,$$ where E_b ~ energy per bit.

Let $P(s_1) = P_1$
$P(s_0) = P_0$



Problem: We wish to transmit these signals over an AWGN channel.

Goal: Find optimum decision rule when transmitted signal corrupted by AWGN noise! (for receiver)

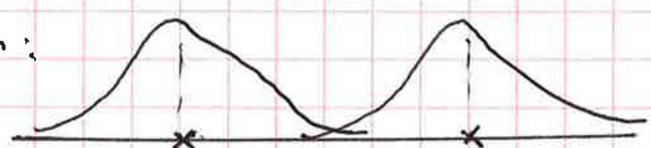
Soln: Received signal $r = \pm\sqrt{E_b} + n$, where n is a zero-mean Gaussian R.V. with variance $\sigma^2 = \frac{N_0}{2}$

where N_0 ~ power spectral density

i.e. r will be a Gaussian distribution with mean $\pm\sqrt{E_b}$ — Cf. Central limit Thm.

Central Limit Thm: Sum of independent RVs yield PDF ≈ Gaussian generally!

s_0: Received r:



Conditional pdfs:
likelihood ~

$$f(r|s_1) = \frac{1}{\sqrt{\pi N_0}}\, e^{-(r-\sqrt{E_b})^2/N_0}$$

$$f(r|s_0) = \frac{1}{\sqrt{\pi N_0}}\, e^{-(r+\sqrt{E_b})^2/N_0}$$

$2\sigma^2 = N_0$ power spectral density.

The optimum detector seeks to maximize $\boxed{PM(r,s) = f(r|s)\cdot p(s)}$

i.e. $PM(r,s_1) = f(r|s_1)\cdot P_1$
$PM(r,s_0) = f(r|s_0)\cdot P_0$

If $PM(r,s_1) > PM(r,s_0)$ we pick s_1 else pick s_2.

ie. $$\frac{PM(r, s_1)}{PM(r, s_0)} \underset{s_0}{\overset{s_1}{\gtrless}} 1$$

Substituting expressions for $PM(r,s_1)$ and $PM(r,s_0)$,

$$\frac{PM(r, s_1)}{PM(r,s_0)} = \frac{P_1}{P_0} e^{\left[(r+\sqrt{E_b})^2-(r-\sqrt{E_b})^2\right]/N_0} \underset{s_0}{\overset{s_1}{\gtrless}} 1$$

Taking logs of both sides,

$$\ln \frac{P_1}{P_0} + \frac{(r+\sqrt{E_b})^2-(r-\sqrt{E_b})^2}{N_0} \underset{s_0}{\overset{s_1}{\gtrless}} 0$$

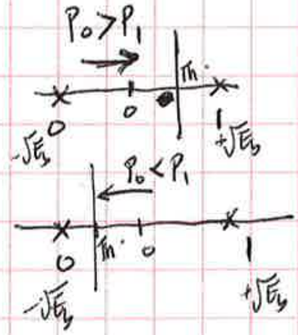
ie. $$\frac{4r\sqrt{E_b}}{N_0} \underset{s_0}{\overset{s_1}{\gtrless}} \ln\left(\frac{P_0}{P_1}\right)$$

OR $$\boxed{\underbrace{r\sqrt{E_b}}_{\text{Correlation metric}} \underset{s_0}{\overset{s_1}{\gtrless}} \underbrace{\frac{N_0}{4} \ln \frac{P_0}{P_1}}_{\text{threshold}}}$$

The optimum detector calculates the correlation metric and compares with a threshold.

Special case: If $P_0 = P_1 = \frac{1}{2}$, $\ln\left(\frac{P_0}{P_1}\right) = 0$

and $$\underline{r \underset{s_0}{\overset{s_1}{\gtrless}} 0}$$ — which is the case we examined in class for equiprobable a priori probs!



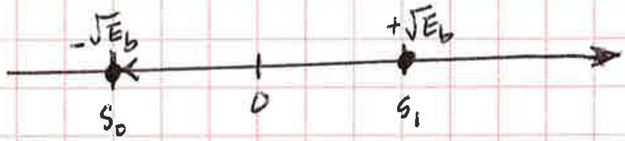
Q
If $P_0 > P_1$ where does threshold move to?

If $P_0 < P_1$ where does threshold move to?

(4) P(Error) for Binary Signalling

Consider binary antipodal signals s_0 and s_1 with energy E_b each, whose geometric representations are the one-dimensional vector (respectively)

$$s_0 = -\sqrt{E_b} \qquad s_1 = +\sqrt{E_b}$$



signal pts for binary antipodal signals.

The received signal r, assuming s_1 was transmitted, is given by

$$r = s_1 + n = \sqrt{E_b} + n,$$ where n is a zero-mean AWGN with variance $\sigma^2 = \frac{N_0}{2}$.

The conditional pdf's of r are respectively

$$f(r|s_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(r-\sqrt{E_b})^2/2\sigma^2} \quad \text{and} \quad f(r|s_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(r+\sqrt{E_b})^2/2\sigma^2}$$

Given that s_1 was transmitted, the prob. of error is the prob. that $r < 0$ (assuming equally likely signals)

i.e. $$p(e|s_1) = \int_{-\infty}^{0} f(r|s_1)\,dr = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{0} e^{-(r-\sqrt{E_b})^2/2\sigma^2}\,dr$$

By substituting $x = -\frac{(r-\sqrt{E_b})}{\sqrt{2}\sigma}$, $dr = -\sqrt{2}\,\sigma\,dx$; $x \to +\infty$, when $r \to -\infty$
and $x = \sqrt{E_b}/(\sqrt{2}\cdot\sigma)$ when $r = 0$

The above expression becomes

$$P(e|s_1) = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{\sqrt{E_b}/\sqrt{2}\sigma} e^{-x^2}\,dx = \frac{1}{\sqrt{\pi}} \int_{\frac{\sqrt{E_b}}{\sqrt{2}\sigma}}^{\infty} e^{-x^2}\,dx = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\sqrt{E_b}}{\sqrt{2}\cdot\sigma}\right)$$

where $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-x^2}\,dx$.

Since $\frac{N_0}{2} = \sigma^2$, we obtain $\boxed{p(e|s_1) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$

(5) BER/PER Calculations

Example 1: Given SNR = 10 dB, packet size = 1000 bits.

(A) BER = ?

(B) Packet Error Rate (PER) = ?

Soln: Recall $SNR_{dB} = 10\log_{10}\left(\frac{E_b}{N_0}\right)$

So that $\frac{E_b}{N_0} = 10^{SNR/10}$

Given SNR = 10dB, $\frac{E_b}{N_0} = 10^{10/10} = 10$

(A) $BER = \frac{1}{2}\,erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$

$= \frac{1}{2}\,erfc\left(\sqrt{10}\right) = \frac{1}{2}\,erfc(3.16) = \underline{3\times 10^{-6}}$

(B) $p(\text{correct bit}) \triangleq P_c = 1 - p(\text{Bit in error})$

ie $P_c = 1 - BER$

A packet is correct when all 1000-bits in the packet are correct (for our example).

ie. $P(\text{Packet Correct}) = \left[P(\text{Correct Bit})\right]^{1000} = \left[1 - BER\right]^{1000}$

$\text{Packet Error Rate} = 1 - P(\text{Packet Correct}) = 1 - (1-BER)^{1000}$

Recall that for $|x| <<< 1$, $(1-x)^N \approx 1 - Nx$.

So for very small BER, $(1-BER)^{1000} \approx 1 - 1000 * BER$.

So that $PER \simeq 1 - (1 - 1000 * BER) = 1000 * BER = 1000 * (3\times 10^{-6})$

$= \underline{3\times 10^{-3}}$

Example 2: Now consider the case where

$SNR = 0\,dB$, for same packet size = 1000 bits.

(A) $BER = ?$

(B) $PER = ?$

Soln: $\frac{E_b}{N_o} = 10^{SNR/10} = 10^0 = 1$

(A) So that $BER = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_o}}\right) = \frac{1}{2}\,\mathrm{erfc}(\sqrt{1}) = \underline{0.0787}$

(B) $PER = 1 - (1 - BER)^{1000} = 1 - (1 - 0.0787)^{1000} \approx 1$ which is so unacceptable!

Note that we didn't use the approximation b/cos the BER in this case is quite high!

Comparing Ex1 to Ex2, we see that changes in SNR can yield unacceptable changes in PER especially.