Recitation 6

Today:
- channel coding (deletion & correction)
- detection & parity-check codes
- correction & repetition (repetition) codes
- linear block codes and properties

Recall: Model of Digital Communication System:

Thm. Shannon's Noisy Channel Theorem: (Error-free transmission possible!)

For every channel, we can associate a "channel capacity" C.
There exist error control codes such that information can be transmitted across the channel at rates less than C with arbitrarily low bit error rate.

(1) Coding Gain: ECC can be energy efficient!

Waterfall Curves

BER vs. SNR plots.
Consider, want to send \( \{\text{Apple (A)}, \text{Banana (B)}\} \).

**Case I:**
\[
\begin{align*}
A &= 00 \\
B &= 11
\end{align*}
\]
\( d_H = 1 \)

Cannot detect anything!

**Case II:**
\[
\begin{align*}
A &= 000 \\
B &= 111
\end{align*}
\]
\( d_H = 2 \)

Can detect single error \((10, B)\)

Cannot correct!

**Case III:**
\[
\begin{align*}
A &= 0000 \\
B &= 1111
\end{align*}
\]
\( d_H = 3 \)

Can detect up to 2 errors

Can correct single error

Can continue like this to see that

\[
\begin{align*}
t_{\text{detect}} &\leq d_{\text{min}} - 1 \\
t_{\text{correct}} &\leq \left\lfloor \frac{d_{\text{min}} - 1}{2} \right\rfloor
\end{align*}
\]
Minimum Distance: if a block code is the minimum Hamming distance between all distinct pairs of codewords in \( C \).

\[
d_{\text{min}} = \min \left( d_H(\mathbf{c}_i, \mathbf{c}_j) \right) \quad \forall \mathbf{c}_i \neq \mathbf{c}_j
\]

For an error to go undetected, it must change the symbol values in at least \( d_{\text{min}} \)
coordinates for one codeword to look like another in \( C \).

For error to be undetectable, must change by 2 coordinates (2-Hits)

\[
i \cdot d_{\text{min}} = 2
\]

Then: A code with \( d_{\text{min}} \) can detect all error patterns of weight less than or equal to

\[
t_{\text{detect}} \leq d_{\text{min}} - 1
\]

The length-4 binary repetition code has two codewords (0000) and (1111).

How many errors can it detect?

\[
t_{\text{detect}} = d_{\text{min}} - 1 = 4 - 1 = 3 \text{ errors at most}
\]

\[
i.e. \text{ can detect all error patterns of weight 3!}
\]

ECC: One error detected:

- **ARQ** - retransmit (feedback exists, reliability premium over BW)
- **FEC** - attempt correction (no feedback mechanism, BW essential)
- **Muting** - tag and pass onto sink (delay critical, but can live with some errors goaudio)
Example: Given: bit error rate $p$

Question: For following coding schemes, the probability of undetected error $P(\text{undetected error}) = ?$

1. No code, word length = 4
   - Answer: No code $\Rightarrow$ can't detect any error.
   - So: $P(\text{undetected error}) = P(1 \text{ error}) + P(2 \text{ errors}) + P(3 \text{ errors}) + P(4 \text{ errors})$
     
     $= C_1^4 p(1-p)^3 + C_2^4 p^2(1-p)^2 + C_3^4 p^3(1-p) + C_4^4 p^4$
     
     $= \sum_{i=1}^{4} C_i^4 p^i (1-p)^{4-i}$

2. Even parity, word length = 4
   - Answer: Even parity $\Rightarrow$ can detect 1 error and 3 errors, cannot detect 2 errors, and 4 errors.
   - So: $P(\text{undetected error}) = P(2 \text{ errors}) + P(4 \text{ errors})$
     
     $= C_2^4 p^2(1-p)^2 + C_4^4 p^4$

For replication coding with word length of $c$, bit error rate $p$.

$P(\text{decoding error}) = \begin{cases} \sum_{i=\lceil \frac{c}{2} \rceil}^{c} \binom{c}{i} p^i (1-p)^{c-i} & \text{if } c \text{ odd} \\ \sum_{i=\lceil \frac{c}{2} \rceil + 1}^{c} \binom{c}{i} p^i (1-p)^{c-i} + \frac{1}{2} \binom{c}{\lceil \frac{c}{2} \rceil} p^{\lceil \frac{c}{2} \rceil} (1-p)^{\frac{c}{2}} & \text{if } c \text{ even} \end{cases}$

Proof: if $c$ is odd, replication coding can NOT correct $\lceil \frac{c}{2} \rceil$ bit errors or more bit errors.

So: $P(\text{decoding error}) = \sum_{i=\lceil \frac{c}{2} \rceil}^{c} \binom{c}{i} p^i (1-p)^{c-i}$

if $c$ is even, replication coding can NOT correct $\lceil \frac{c}{2} \rceil$ bit errors or more bit error, and can NOT correct half of $\frac{c}{2}$ bit error.

So: $P(\text{decoding error}) = \sum_{i=\lceil \frac{c}{2} \rceil + 1}^{c} \binom{c}{i} p^i (1-p)^{c-i} + \frac{1}{2} \binom{c}{\lceil \frac{c}{2} \rceil} p^{\lceil \frac{c}{2} \rceil} (1-p)^{\frac{c}{2}}$

$\lceil \frac{c}{2} \rceil \text{ errors}$ or more errors, $\frac{c}{2} \text{ errors}$