Recitation 8

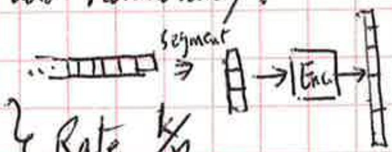
10-4-2011

Today: ① Convolutional Codes
② Hamming Codes, if time permits.

Announcement: Quiz 1 Review ✓
Office Hours ✓

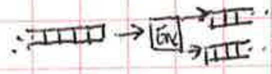
Recall:

CC ~ Elias, 1955 – use of shift-registers to add redundancy!



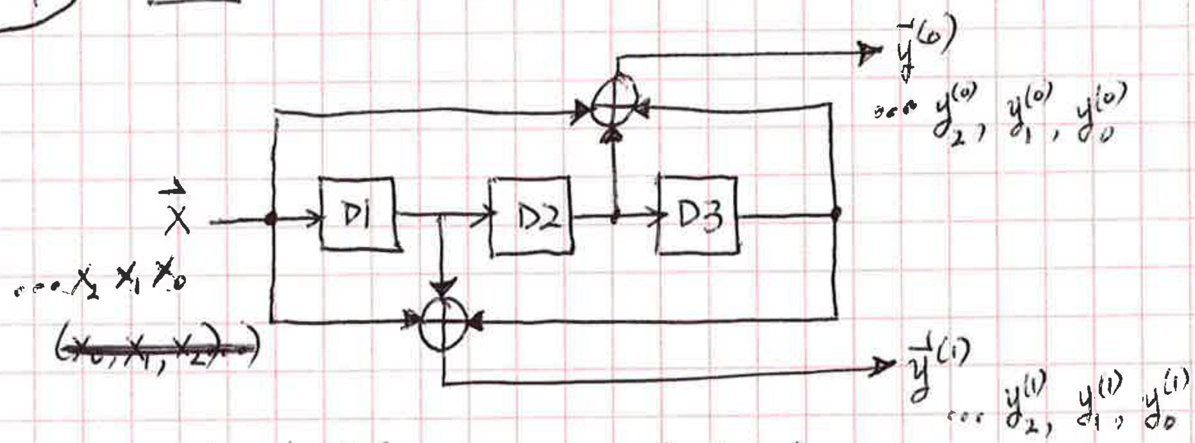
* Block codes: – Segment data stream into k-length fixed "blocks"
– Encode blocks into length-n codewords. } Rate k/n

Convolutional Codes: – Encode entire data stream (possibly infinite) into single codeword



– Encoder outputs n-bits for every k-input bits ~ rate k/n.

Example 1 Consider rate-½ encoder shown below:



Assuming encoder initialized to all-zeros, encode the info sequence given by $\vec{x} = (1\,0\,1\,1\,0)$

Time i	$x[i]$	Current state D1 D2 D3	Next state D1 D2 D3	$y^{(0)}$	$y^{(1)}$
0	1	0 0 0	1 0 0	1	1
1	0	1 0 0	0 1 0	0	1
2	1	0 1 0	1 0 1	0	1
3	1	1 0 1	1 1 0	0	1
4	0	1 1 0	0 1 1	1	1
5	0	0 1 1	0 0 1	0	1
6	0	0 0 1	0 0 0	1	1

End!

Use commas to separate bits output at same time.

output $\vec{y} = (11, 01, 01, 01, 11, 01, 11)$

Effective rate $= \frac{5}{14} < \frac{1}{2}$ – code rate.

Moral of story ~ Encode longer bit sequences!

Linear: If inputs $\vec{x}_1$ and $\vec{x}_2$ yield outputs $\vec{y}_1$ and $\vec{y}_2$ respectively, then input $(\vec{x}_1 + \vec{x}_2)$ yields output $(\vec{y}_1 + \vec{y}_2)$ ~ also a codeword.

Impulse Response: (aka "generator sequence") $\vec{g}_j^{(i)}$

(eg) for our previous encoder, taps at

$$\vec{g}^{(0)} = (1\,0\,1\,1)$$

$$\vec{g}^{(1)} = (1\,1\,0\,1)$$

Example 2 Use the generator sequences to encode the message in (eg 1)

$$\vec{x} = (1\,0\,1\,1\,0).$$

Soln: Define D-transforms:

$$X(D) = (1 + \quad + D^2 + D^3) \sim (1\,0\,1\,1\,0)$$

$$G^{(0)}(D) = (1 + \quad + D^2 + D^3) \sim (1\,0\,1\,1)$$

$$G^{(1)}(D) = (1 + D + \quad + D^3) \sim (1\,1\,0\,1)$$

(or) alternatively ~

$(10110)(1011)$

$$\begin{array}{r} 1011 \\ 1011 \\ 1011 \\ \hline 1000101 \end{array}$$

etc.

Then,

$$Y^{(0)}(D) = X(D)\cdot G^{(0)}(D) = (1 + D^2 + D^3)(1 + D^2 + D^3) = 1 + \cdots + D^4 + \cdots + D^6 = (1000101)$$

and

$$Y^{(1)}(D) = X(D)\cdot G^{(1)}(D) = (1 + D^2 + D^3)(1 + D + D^3) = 1 + D + D^2 + D^3 + D^4 + D^5 + D^6 = (1111111)$$

Then $\vec{y}^{(0)} = (1\,0\,0\,0\,1\,0\,1)$

and $\vec{y}^{(1)} = (1\,1\,1\,1\,1\,1\,1)$

and thus $\underline{\vec{y} = (11, 01, 01, 01, 11, 01, 11)}$, as above.

Constraint length K:

~ max number of output bits that can be affected by any input bit.

For m memory elements, each bit in input data sequence can affect at most (m+1) bits, hence length of g = (m+1); ie. m determines extent to which an input bit directly affects the output data streams.

$\boxed{K \triangleq 1 + \max m}$ ~ not universal definition, but used in military & industry!

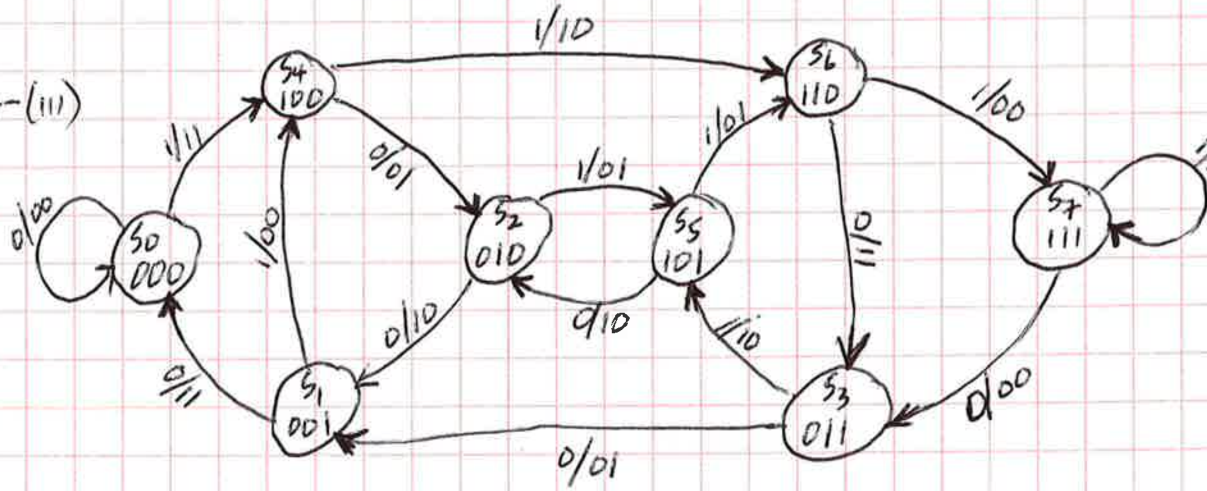
max m ~ maximal memory order ~ length of longest input shift register.

State Diagram:

For the encoder in example 1 with $\vec{g}^{(0)} = (1011)$ and $\vec{g}^{(1)} = (1101)$, use the state diagram to encode $\vec{x} = (10110)$ as before.

Soln:

Eight states (000)–(111)
$D_1 D_2 D_3$



$\vec{y} = (11, 01, 01, 01, 11$ — then input zeros back to S_0-000 — $01, 11)$ ✓

Systematic Codes:

An (n,k) linear code systematic if 'first' k-bits are uncoded data bits.

So that
The generator matrix $G = [\underbrace{I}_{\substack{k\times k \\ \text{identity} \\ \text{matrix}}} | P]$

The parity-check matrix $H = [P^T | I]$

Thus, $GH^T = [I|P]\begin{bmatrix} P \\ I \end{bmatrix} = P + P = \emptyset$

So, for any codeword $\vec{c} = \vec{m}G$,

$$\boxed{\vec{c}H^T = \vec{m}GH^T = \vec{m}(GH^T) = \emptyset}$$

Syndrome:

Given received codeword $\vec{r}$,

syndrome is $\vec{s} = \vec{r}H^T$

If $\vec{s} = 0$, no error.

If single-bit error,

$\vec{r} = \vec{c} + \vec{e}_i$ where $\vec{e}_i = [0\ 0\ 0 \cdots 1 \cdots 0\ 0]$ ← i^{th} position

So $\vec{s} = \vec{r}H^T = (\vec{c} + \vec{e}_i)H^T = \vec{e}_i H^T$

error in ith bit position

Example: Hamming Code.

H.C. — $(2^m-1,\ 2^m-m-1)$ linear blockcode, $d_{min}=3$

— Parity-check matrix (H) contains all non-zero m-bit binary vectors!

For $m=3$, $n=7$, $k=4$ $\Rightarrow$ (7,4) HC

(*) Suppose, given generator matrix

$$G_{4\times7}=\left[\begin{array}{cccc|ccc} 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 0&0&1&0&1&0&1\\ 0&0&0&1&1&1&1 \end{array}\right]$$

$$\qquad\qquad I \qquad\qquad P$$

Construct parity-check matrix H;

$$H_{3\times7}=[P^T \,|\, I] = \left[\begin{array}{cccc|ccc} 1&0&1&1&1&0&0\\ 1&1&0&1&0&1&0\\ 0&1&1&1&0&0&1 \end{array}\right]$$

~ note matrix contains all non-zero 3-bit vectors!

Q: If received vector $\vec{r}=(1\,0\,1\,1\,1\,1\,0)$, decode original sent message!

syndrome $\vec{s}=\vec{r}H^T = (1\,0\,1\,1\,|\,1\,1\,0)\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\\1&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} = (0\,1\,0)$

"sixth" bit position.

Thus, corrected $\hat{r}=(1\,0\,1\,1\,|\,1\,\boxed{0}\,0)$ ~ corrected.

Original ~~estimated~~ message $=(1\,0\,1\,1)$ ~ since systematic!