Recitation 11

10-20-2011

Today:
LTI Systems & Convolution.

Review & Defn:

(A) System ~ mathematical model of a physical process relating input to output.
(excitation) (response)

$$x \longrightarrow \boxed{\text{System}} \longrightarrow y$$

input (excitation) — response (output)

If x and y are discrete-time (DT) signals (or sequences), then system is DT System.

(B) Causality

~ causal if $y[k=n_0]$ depends only on input $x[k \leq n_0]$, i.e. output depends only on present and past inputs, not future!

Thus, all memoryless systems are causal (but not vice-versa)!

<cf. current Wall Street>

(C) Linearity

~ linear if satisfies
If
① $\mathcal{R}\{x_1\} = y_1$ and $\mathcal{R}\{x_2\} = y_2$, then $\mathcal{R}\{x_1 + x_2\} = y_1 + y_2$
and
② $\mathcal{R}\{ax\} = ay$ $\forall$ signals x, scalars a.

If ①, ② not satisfied, system is nonlinear.

Combining ① and ②,

linear ~ $\boxed{\mathcal{R}\{a_1 x_1 + a_2 x_2\} = a_1 y_1 + a_2 y_2}$

(eg1) Is $y = x^2$ linear?

NO — If $X = ax$

Then $y = (ax)^2 = a^2x^2 \neq \underbrace{ax^2 = ay}$

(eg2) Is $y = \cos x$ linear?

NO — If $y_1 = \cos x_1$

and $y_2 = \cos x_2$

Is $(y_1 + y_2) = \cos(x_1 + x_2)$? Absolutely not!

(D) Time-Invariance

~ If time-shift (delay) in input causes same delay in output,

ie. $\mathcal{R}\{x[n-k]\} = y[n-k] \quad \forall\ k \in \mathbb{Z}$ (integer).

(E) Stability (In bounded-input, bounded-output (BIBO) sense).

BIBO Stable if for any bounded input x (ie $|x| \leq k_1$)

the output y is also bounded (ie $|y| \leq k_2$), $k_1, k_2 \in \mathbb{R}$ (real).

(F) LTI System

~ If linear AND also time-invariant, then LTI.

Response of LTI System

① Impulse Response: $h[n]$ (aka Unit Sample Response)

~ the response of the system to input $\delta[n]$.

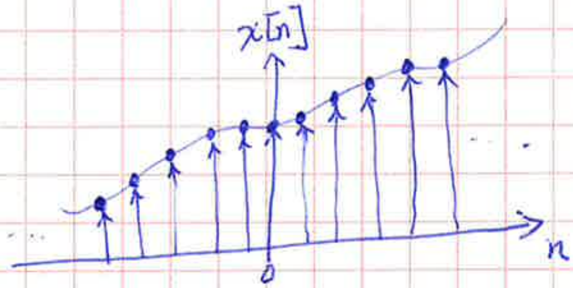
$$\boxed{h[n] = \mathcal{R}\{\delta[n]\}} \quad \text{— (i)}$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \qquad \longrightarrow h[n-k]$$ for time-invariant systems!

② Response to arbitrary input: $y[n]$

Consider input $x[n]$ shown →
{perhaps obtained by sampling $x(t)$}



$x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

So that

$$y[n] = \mathcal{R}\{x[n]\}$$

$$= \mathcal{R}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} \quad \text{— ②}$$

(A): B'cos system is linear, we can write $\mathcal{R}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{R}\{\delta[n-k]\}$

(B): B'cos system is time-invariant, $\mathcal{R}\{\delta[n-k]\} = h[n-k]$

So that substituting in ②,

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]} \quad \text{— ③}$$

(3) Convolution Sum :

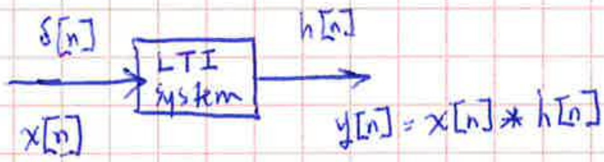
Eqn (3) defines the convolution of two sequences $x[n]$ and $h[n]$,

i.e. $$y[n] = \underbrace{x[n] * h[n]}_{\text{better } (x*h)[n]} \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$ — convolution sum.

Important (Fundamental) Conclusion

(∗) Thus, the output of a LTI system is the convolution of the input $x[n]$ and impulse response $h[n]$ of the system!

$$\boxed{y[n] = x[n] * h[n]} \quad \text{— (4)}$$



Response of LTI system.

(4) Properties of the Convolution Sum:

(A) Commutative: $x[n] * h[n] = h[n] * x[n]$

i.e. $$\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$ one may be easier to evaluate than the other!

(B) Associative: $\left(x[n] * h_1[n]\right) * h_2[n] = x[n] * \left(h_1[n] * h_2[n]\right)$

useful for cascaded (serial) system.

(C) Distributive: $x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$

Parallel systems.

LTI system with $h[n]$ response

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n]$$

(5) Step Response : $s[n]$

~response of the unit step defined as $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k] \quad \text{Recall } u[n-k] = \begin{cases} 1 & n \geq k \quad (\text{ie } k \leq n) \\ 0 & n-k < 0 \quad (\text{ie } k > n) \end{cases}$$

ie.

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k]} \text{ — (5)}$$

, using the relationship to the right above!

Using (5), it can be seen that

$$s[n] - s[n-1] = h[n]$$

~~Hence~~ ie.

$$\boxed{h[n] = s[n] - s[n-1]} \text{ — (6)}$$

(6) Convolution Sum Operation :

Given: ~~y[n]~~ $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ — convolution sum.

Evaluation Operation Steps:

(A) Time-reverse $h[k]$ to obtain $h[-k]$ about origin

(B) Shift $h[k]$ (function of k) by n (fixed parameter) to the right to obtain $h[n-k]$ ie $h[-(k-n)]$

(C) Obtain product $x[k]h[n-k]$, over all k, (D) Sum the product $\forall k$ to produce single output sample $y[n]$ (at fixed n)

(E) Repeat A,B,C,D for a different n over all $-\infty$ to ∞