Recitation 12

Today:
1. Review of LTI Preliminaries ~ causality, stability.
2. Examples & Graphical evaluation of convolution

Recall:

1. \( h[n] \equiv R\{s[n]\} \) - Unit sample response (aka impulse response)

\[
\begin{align*}
\delta[n] &\rightarrow \text{LT}I \rightarrow h[n] \\
\delta[n-k] &\rightarrow h[n-k] \text{ for } \text{LT}I \text{ system}.
\end{align*}
\]

2. Causal.
   \[ h[n] = 0 \text{ for } n < 0, \text{ system is causal.} \]
   (i.e., output depends only on present and past inputs, not future)

3. Stable (in BIBO sense)
   \[ \text{if } \sum_{k=-\infty}^{\infty} |h[k]| \text{ is finite} \] (i.e., output is bounded for bounded input).

Example 1:
Given the LTI system with unit sample (impulse) response \( h[n] = \alpha^n u[n] \).

(a) Is the system causal?
   Soln: Since \( h[n] = 0 \text{ for } n < 0 \), system is causal.
   \[ \sum_{n=0}^{\infty} \frac{1}{1-\alpha} \]  

(b) Is the system BIBO stable?
   Soln: \[ \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |\alpha^k| = \sum_{k=0}^{\infty} \frac{1}{1-|\alpha|} \text{ for } |\alpha| < 1 \]
   Thus system is BIBO stable if \( |\alpha| < 1 \), and unstable if \( |\alpha| \geq 1 \).
Example 2: The unit sample response $h[n]$ of a LTI-system is given by

$$h[n] = u[n] - u[n-3]$$

If the input to the system is

$$x[n] = \delta[n-1] - \delta[n-3]$$

(a) sketch $h[n]$ and $x[n]$
(b) obtain the output $y[n]$ and sketch it. (3 diff. ways : -)

Solution:

(a) $h[n]$ and $x[n]$

(b) Method 1: Since system is linear and time-invariant, and by the defn of the unit sample (impulse) response,

$$y[n] = h[n-1] - h[n-3]$$

Since $h[n] = u[n] - u[n-3]$, 

$$h[n-1] = u[n-1] - u[n-4]$$
and 

$$h[n-3] = u[n-3] - u[n-6]$$

And thus,

$$y[n] = u[n-1] - u[n-3] - u[n-4] + u[n-6],$$

which is sketched below
Method 2: (Analytical Approach)

\[ x[k] = \delta[k-1] - \delta[k-3] = \begin{cases} 
1 & k = 1 \\
-1 & k = 3 \\
0 & \text{otherwise}
\end{cases} \]

So that:

\[ y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = x[1]h[n-1] + x[3]h[n-3] \]

\[ = h[n-1] - h[n-3] \]

Using values for \( h[n] = u[n] - u[n-3] \) and LTI properties, we obtain:

Method 3: Graphical Mthd.

× Review of the Convolution Sum Operation:

\[ y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \] — convolution sum.

Evaluating Steps:

1. Obtain \( h[-k] \) from time-reversal of \( h[k] \) about origin.

2. Shift \( h[-k] \) by \( n \) to the right to obtain \( h[n-k] \) i.e. \( h[-(k-n)] \).

3. Obtain product \( x[k]h[n-k] \) over all \( k \), at fixed \( n \); sum the product \( \times k \) to produce single output sample at \( n \), i.e. \( y[n] \).

4. Repeat steps 2 and 3 for different values of \( n \) from \(-\infty \) to \( \infty \).

Returning to our example...

\[ y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \] where

\[ h[k] \]

\[ x[k] \]

Step 1: Obtain \( h[-k] \):

\[ h[k] \]

\[ @ n=0 \]

Step 2: Obtain \( h[n-k] \) @ \( n=1 \) shift 1 to the right

\[ y[0] = 0, \quad y[1] = 1 \] (of next page)
\( x[k] = 8[k-1] - 8[k-3] \)

\( h[k] = u[k] - u[k-3] \)

\[
\begin{align*}
&A) \\
&n=0 \\
&y[0] = 0
\end{align*}
\]

\[
\begin{align*}
&B) \\
&n=1 \\
&y[1] = 1
\end{align*}
\]

\[
\begin{align*}
&C) \\
&n=2 \\
&y[2] = 1
\end{align*}
\]

\[
\begin{align*}
&D) \\
&n=3 \\
&y[3] = 0
\end{align*}
\]

\[
\begin{align*}
&E) \\
&n=4 \\
&y[4] = -1
\end{align*}
\]

\[
\begin{align*}
&F) \\
&n=5 \\
&y[5] = -1
\end{align*}
\]

So: \( y(n) = (0, 1, 1, 0, -1, -1) \)

which is identical to what we obtained before.