11-8-2011

Recitation 16

Today:

① Quick review of DTFS

② Modulation / Demodulation

1. Discrete-Time Fourier Series : DTFS

⊛ **Recall**: If $x[n]$ periodic with period P, can be expressed as linear combination of complex exponentials with period P.

i.e.

$$\boxed{x[n] = \sum_{k=\langle P\rangle} A_k e^{j\Omega_k n}}$$

— synthesis equation

... where $\Omega_k = k\Omega_1$ and $\Omega_1 = \frac{2\pi}{P}$

and

$$\boxed{A_k = \frac{1}{P}\sum_{n=\langle P\rangle} x[n]\cdot e^{-j\Omega_k n}}$$

— analysis equation

⊛ **Euler's identities**:

$$\begin{cases} e^{j\theta} = \cos\theta + j\sin\theta & \text{—(1)} \\ e^{-j\theta} = \cos\theta - j\sin\theta & \text{—(2)} \end{cases}$$

Using ① and ②, it can be shown that

$$\boxed{\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \text{and} \quad \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}}$$

Ex. 1 Given: $x[n] = \sin\left(\frac{2\pi}{3} n\right)$

Find: A_k, the Fourier series coefficients.

Soln:

Method 1 Direct Application of analysis equation:

$$A_k = \frac{1}{P} \sum_{n=\langle P \rangle} x[n] \, e^{-j\Omega_k n} \quad \text{where} \quad \Omega_k = k\Omega_1 \quad \text{and} \quad \Omega_1 = \frac{2\pi}{3} \quad \left(x[n] = \sin\left(\overset{\Omega_1}{\frac{2\pi}{3}} n\right)\right)$$

So that

$$\underset{(k=0)}{A_0} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\Omega_0 n} = \frac{1}{3}\sum_{n=0}^{2} x[n] \cdot e^{-j(0)n} \qquad \text{ie } P = \frac{2\pi}{\Omega_1} = \frac{2\pi}{(2\pi/3)} = \boxed{3}$$

$$= \frac{1}{3}\sum_{n=0}^{2} x[n] \cdot 1 = \frac{1}{3}\left\{x[0] + x[1] + x[2]\right\} = \frac{1}{3}\left\{\sin\left(\frac{2\pi}{3}(0)\right) + \sin\left(\frac{2\pi}{3}(1)\right) + \sin\left(\frac{2\pi}{3}(2)\right)\right\}$$

$$= \frac{1}{3}\left\{0 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right\} = \boxed{0}$$

$$\underset{(k=1)}{A_1} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\Omega_1 n} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}n} = \frac{1}{3}\left\{x[0] e^{(0)} + x[1] e^{-j\frac{2\pi}{3}(1)} + x[2] e^{-j\frac{2\pi}{3}(2)}\right\}$$

$$= \frac{1}{3}\left\{\sin(0) e^{(0)} + \sin\left(\frac{2\pi}{3}\right)(1) \cdot e^{-j\frac{2\pi}{3}} + \sin\frac{4\pi}{3} e^{-j\frac{4\pi}{3}}\right\} = \boxed{\frac{-j}{2}}$$

$$\underset{(k=2)}{A_2} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\Omega_2 n} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}(2)n} = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-j\frac{4\pi}{3}n}$$

$$= \frac{1}{3}\left\{x[0] e^{(0)} + x[1] e^{-j\frac{4\pi}{3}(1)} + x[2] e^{-j\frac{4\pi}{3}(2)}\right\}$$

$$= \frac{1}{3}\left\{\underset{(0)}{\sin(0)}\, \underset{(1)}{e^{(0)}} + \underset{\frac{\sqrt{3}}{2}}{\sin\frac{2\pi}{3}} \cdot e^{-j\frac{4\pi}{3}} + \underset{-\frac{\sqrt{3}}{2}}{\sin\frac{4\pi}{3}}\, \underset{e^{j\frac{4\pi}{3}}}{e^{-j\frac{8\pi}{3}}}\right\} = \boxed{\frac{j}{2}}$$

So that $A_0 = 0$, $A_1 = -\frac{j}{2}$, $A_2 = \frac{j}{2}$

Method 2: By Inspection & Euler's Identity:

Given $x[n] = \sin\left(\frac{2\pi}{3}n\right) = \frac{1}{2j}\left(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}\right)$ using Euler's identity.

$$= \frac{1}{2j}\left(e^{j\frac{2\pi}{3}n} - e^{+j\frac{4\pi}{3}}\right)$$

$-\frac{2\pi}{3} + 2\pi$

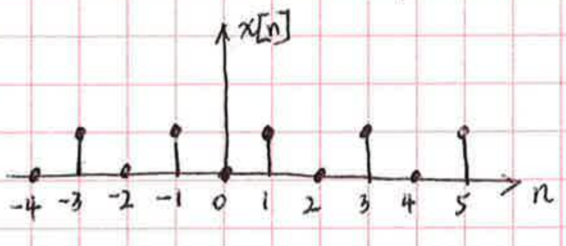
$$= \frac{-j}{2}\left(e^{j\frac{2\pi}{3}n}\right) + \frac{j}{2}\left(e^{j(2)\frac{2\pi}{3}n}\right)$$

$$= \underbrace{0}_{k=0} \underbrace{- \frac{j}{2}e^{j\frac{2\pi}{3}n}}_{k=1} + \underbrace{\frac{j}{2}e^{j(2)\frac{2\pi}{3}n}}_{k=2}$$

So that

$A_0 = 0, \quad A_1 = \frac{-j}{2}, \quad A_2 = \frac{j}{2}$ as before.

Ex 2: Find the Fourier coefficients of the periodic sequence $x[n]$ shown below.



Soln: It can be seen that $x[n]$ is the periodic extension of $\{0, 1\}$ with fundamental period $P = 2$. ie $\Omega_1 = \frac{2\pi}{P} = \frac{2\pi}{2} = \pi$

So that $e^{j\Omega_k} = e^{j\pi k} = (-1)^k$

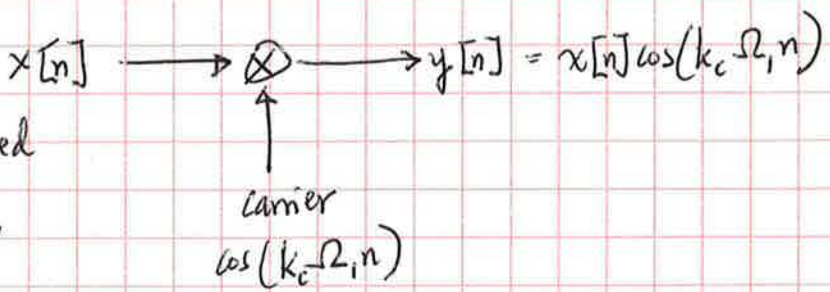
Thus $\underset{(k=0)}{A_0} = \frac{1}{2}\sum_{n=0}^{1} x[n]e^{j\Omega_{k=0}n} = \frac{1}{2}\sum_{n=0}^{1} x[n](-1)^{(0)n} = \frac{1}{2}\sum_{n=0}^{1} x[n] = \frac{1}{2}\left\{x[0] + x[1]\right\} = \frac{1}{2}(0+1) = \frac{1}{2}$

$\underset{(k=1)}{A_1} = \frac{1}{2}\sum_{n=0}^{1} x[n]e^{j\Omega_{k=1}n} = \frac{1}{2}\sum_{n=0}^{1} x[n](-1)^{(1)n} = \frac{1}{2}\left\{x[0](-1)^0 + x[1](-1)^1\right\} = \frac{1}{2}\left((0)(1) + (1)(-1)\right) = \frac{-1}{2}$

So $A_0 = 1/2, \quad A_1 = -1/2$

2. Modulation

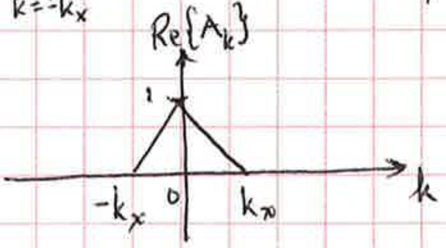
Consider

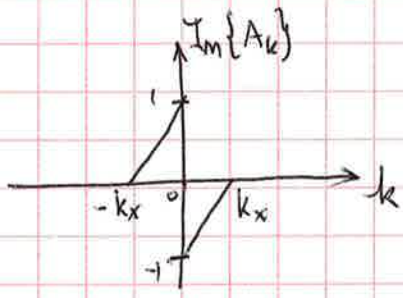


the baseband bandlimited signal $x[n]$, given by

Ω_1 ~ fundamental freq.
k_c ~ carrier period.

$$x[n] = \sum_{k=-k_x}^{k_x} A_k e^{j\Omega_k n}$$ with the spectrum below, where $\Omega_k = k\Omega_1$ as before.



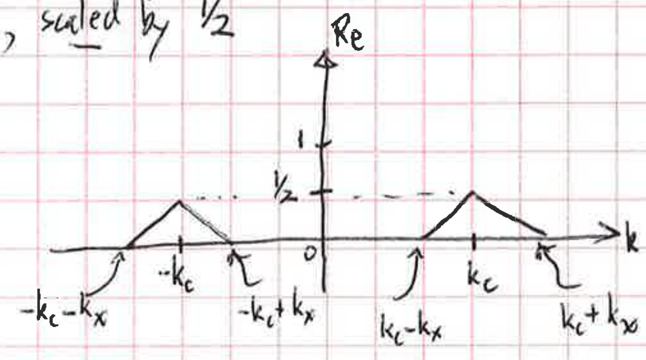


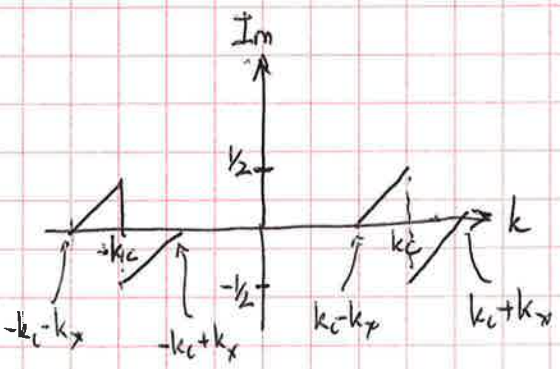
Then, $y[n] = x[n]\cos(k_c\Omega_1 n)$

$$= \left[\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n}\right]\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk\Omega_1 n}\right]$$ using Euler's identity to write the cosine term.

$$= \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}$$

So that the spectrum of $y[n]$ is the replication and shifting of the baseband signal $x[n]$ at $\pm k_c$, scaled by ½





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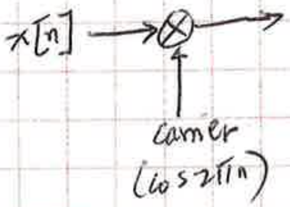
Example 3



Given $x[n] = \cos\left(\frac{2\pi}{3}n\right)$

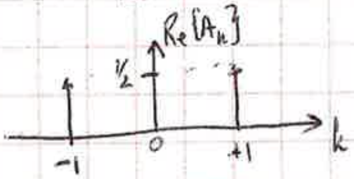
carrier ~ $\cos(2\pi n)$

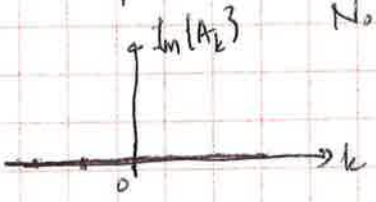
Required: Sketch spectra of $x[n]$, carrier, $y[n]$.

Soln: ① $x[n] = \cos\left(\frac{2\pi}{3}n\right)$ ~ fundamental frequency $\Omega_1 = \frac{2\pi}{3}$ ~~[illegible]~~

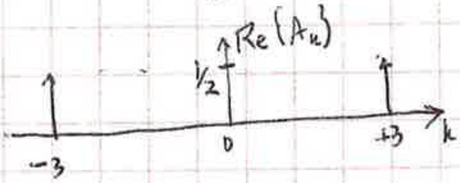
② carrier $= \cos(2\pi n) = \cos\left(\frac{2}{3}\pi \cdot 3n\right)$ ~ $k_c = 3$, $\Omega_1 = \frac{2\pi}{3}$

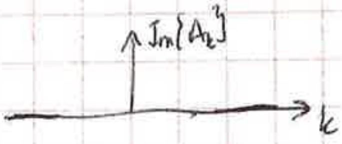
From ①, $x[n] = \cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n}$ spectrum at $k = \pm 1$, magnitude $\frac{1}{2}$. No imaginary component!





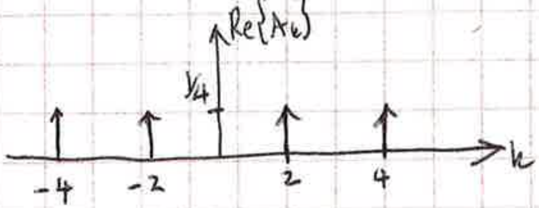
From ②, carrier $= \cos\left(\frac{2\pi}{3} \cdot 3n\right) = \frac{1}{2}e^{j\left(\frac{2\pi}{3} \cdot 3n\right)} + \frac{1}{2}e^{-j\left(2\frac{\pi}{3} \cdot 3n\right)}$, spectrum at $k_c = \pm 3$, magnitude $= \frac{1}{2}$. No Imaginary component

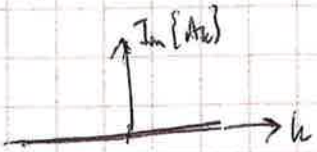




③ $y[n] = \cos\left(\frac{2\pi}{3}n\right) \cdot \cos\left(\frac{2\pi}{3} 3n\right) = \left(\frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n}\right)\left[\frac{1}{2}e^{j\frac{2\pi}{3} \cdot 3n} + \frac{1}{2}e^{-j\frac{2\pi}{3} \cdot 3n}\right]$

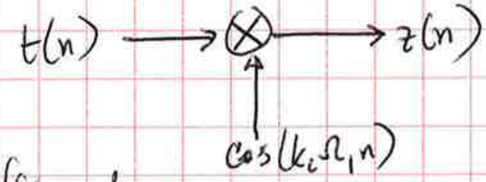
$= \frac{1}{4}e^{j\frac{2\pi}{3}(4)n} + \frac{1}{4}e^{-j\frac{2\pi}{3}(2)n} + \frac{1}{4}e^{j\frac{2\pi}{3}(2)n} + \frac{1}{4}e^{-j\frac{2\pi}{3}(4)n}$ spectrum at $k = -4, -2, +2, +4$





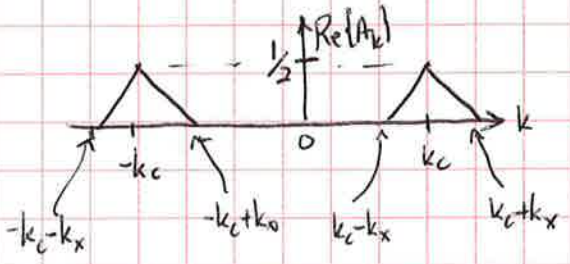
No Imaginary component

3. Demodulation



Consider the received signal with the ff spectrum:

$$t(n) = \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}$$



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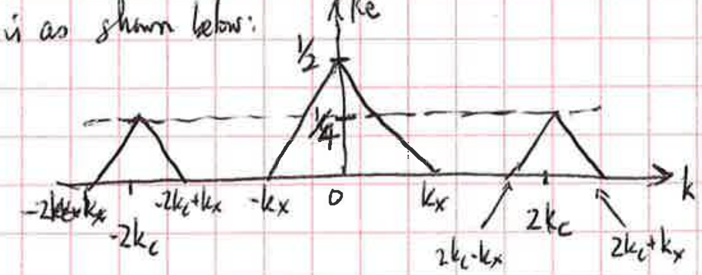
Then $z(n) = t(n)\cos(k_c\Omega_1 n)$

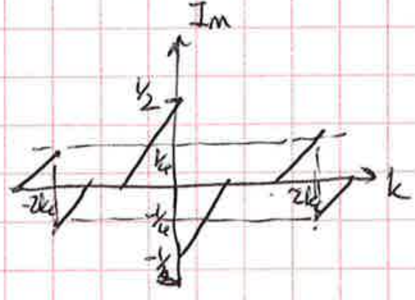
$$= \left[\frac{1}{2}\sum_{k=-k_x}^{k} A_k e^{j(k+k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_1 n}\right]\cdot\left[\frac{1}{2}e^{jk_c\Omega_1 n} + \frac{1}{2}e^{-jk_c\Omega_1 n}\right]$$

$$= \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k+2k_c)\Omega_1 n} + \underbrace{\frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n} + \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n}} + \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k-2k_c)\Omega_1 n}$$

$$= \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k+2k_c)\Omega_1 n} + \frac{1}{2}\sum_{k=-k_x}^{k_x} A_k e^{jk\Omega_1 n} + \frac{1}{4}\sum_{k=-k_x}^{k_x} A_k e^{j(k-2k_c)\Omega_1 n}$$

Thus the spectrum of z is as shown below:





After Low-Pass filtering (LPF) of gain 2, and cutoff frequency $\pm k_x$, we get the desired baseband signal.