Today:
1. Quick review of DTFS
2. Modulation / Demodulation

1. Discrete Time Fourier Series: DTFS

Recall: If $x[n]$ is periodic with period $P$, it can be expressed as a linear combination of complex exponentials with period $P$.

\[ x[n] = \sum_{k=-\infty}^{\infty} A_k e^{j \Omega_k n} \]

... where $\Omega_k = k \Omega_1$

and $\Omega_1 = \frac{2\pi}{P}$

\[ A_k = \frac{1}{P} \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega_k n} \]

Euler's identities:

1. $e^{j\theta} = \cos \theta + j \sin \theta$
2. $e^{-j\theta} = \cos \theta - j \sin \theta$

Using 1 and 2, it can be shown that:

\[ \cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta}) \]
\[ \sin \theta = \frac{1}{2j} (e^{j\theta} - e^{-j\theta}) \]
Ex. 1

Given: \( x[n] = \sin \left( \frac{2\pi}{3} n \right) \)

Find: \( A_k \), the Fourier series coefficients.

\[ A_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_k n} \quad \text{where} \quad 
\Omega_k = k\frac{2\pi}{N} \quad \text{and} \quad \Omega_0 = \frac{2\pi}{3} \]

So that

\[ A_0 = \frac{1}{3} \sum_{n=0}^{2} x[n] e^{-j\Omega_0 n} = \frac{1}{3} \sum_{n=0}^{2} \sin \left( \frac{2\pi}{3} n \right) e^{j\frac{2\pi}{3} n} \]

\[ = \frac{1}{3} \left[ x[0] \cdot 1 + x[1] \cdot 1 + x[2] \cdot 1 \right] = \frac{1}{3} \left[ \sin \left( \frac{2\pi}{3} (0) \right) + \sin \left( \frac{2\pi}{3} (1) \right) + \sin \left( \frac{2\pi}{3} (2) \right) \right] \]

\[ = \frac{1}{3} \left[ 0 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right] = 0 \]

\[ A_1 = \frac{1}{3} \sum_{n=0}^{2} x[n] e^{j\frac{2\pi}{3} n} = \frac{1}{3} \sum_{n=0}^{2} \sin \left( \frac{2\pi}{3} n \right) e^{j\frac{2\pi}{3} n} \]

\[ = \frac{1}{3} \left[ \sin (0) e^{0} + \sin \left( \frac{2\pi}{3} (1) \right) e^{j\frac{2\pi}{3}} + \sin \left( \frac{2\pi}{3} (2) \right) e^{j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ \sin (0) e^{0} + \sin \left( \frac{2\pi}{3} (1) \right) e^{j\frac{2\pi}{3}} + \sin \left( \frac{2\pi}{3} (2) \right) e^{j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ \sin (0) e^{0} + \sin \left( \frac{2\pi}{3} (1) \right) e^{j\frac{2\pi}{3}} + \sin \left( \frac{2\pi}{3} (2) \right) e^{j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ 0 + \frac{\sqrt{3}}{2} e^{j\frac{2\pi}{3}} + \frac{\sqrt{3}}{2} e^{j\frac{4\pi}{3}} \right] = \frac{1}{2} \]

\[ A_2 = \frac{1}{3} \sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3} n} = \frac{1}{3} \sum_{n=0}^{2} \sin \left( \frac{2\pi}{3} n \right) e^{-j\frac{2\pi}{3} n} \]

\[ = \frac{1}{3} \left[ x[0] e^{0} + x[1] e^{-j\frac{2\pi}{3}} + x[2] e^{-j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ \sin (0) e^{0} + \sin \left( \frac{2\pi}{3} (1) \right) e^{-j\frac{2\pi}{3}} + \sin \left( \frac{2\pi}{3} (2) \right) e^{-j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ \sin (0) e^{0} + \sin \left( \frac{2\pi}{3} (1) \right) e^{-j\frac{2\pi}{3}} + \sin \left( \frac{2\pi}{3} (2) \right) e^{-j\frac{4\pi}{3}} \right] \]

\[ = \frac{1}{3} \left[ 0 + \frac{\sqrt{3}}{2} e^{-j\frac{2\pi}{3}} + \frac{\sqrt{3}}{2} e^{-j\frac{4\pi}{3}} \right] = \frac{i}{2} \]

So that \( A_0 = 0 \), \( A_1 = -\frac{1}{2} \), \( A_2 = \frac{i}{2} \).
(Method 2) By Inspection & Euler's Identity:

Given \( x[n] = \sin\left(\frac{2\pi}{3}n\right) = \frac{1}{2j} \left( e^{\frac{2\pi}{3}n} - e^{\frac{-2\pi}{3}n} \right) \) using Euler's identity.

\[
= \frac{1}{2j} \left( e^{\frac{2\pi}{3}n} - e^{\frac{-2\pi}{3}n} \right) \\
= \frac{j}{2} \left( e^{\frac{2\pi}{3}n} + \frac{1}{2} e^{\frac{(2)2\pi}{3}n} \right)
\]

So that \( A_0 = 0, \ A_1 = -\frac{j}{2}, \ A_2 = \frac{j}{2} \) as before.

(Ex 2) Find the Fourier coefficients of the periodic sequence \( x[n] \) shown below.

\[ x[n] \]

-4 -3 -2 -1 0 1 2 3 4 5

\( n \)

Soln: It can be seen that \( x[n] \) is the periodic extension of \( \{0, 1\} \) with fundamental period

\( P = 2 \), i.e. \( \Omega_1 = \frac{2\pi}{P} = \frac{2\pi}{2} = \pi \)

So that \( e^{i\pi k} = e^{i\pi k} = (-1)^k \)

Thus \( A_0 = \frac{1}{2} \sum_{n=0}^{N-1} x[n] e^{i\Omega_0 n} = \frac{1}{2} \sum_{n=0}^{N-1} x[n] (-1)^n = \frac{1}{2} \sum_{n=0}^{N-1} x[n] = \frac{1}{2} \left( x[0] + x[1] \right) = \frac{1}{2} (0 + 1) = \frac{1}{2} \)

\( A_1 = \frac{1}{2} \sum_{n=0}^{N-1} x[n] e^{i\Omega_1 n} = \frac{1}{2} \sum_{n=0}^{N-1} x[n] (-1)^n = \frac{1}{2} \left\{ x[0](-1)^0 + x[1](-1)^1 \right\} = \frac{1}{2} (0(1) + 1(-1)) = \frac{1}{2} \)

So \( A_0 = \frac{1}{2}, \ A_1 = -\frac{1}{2} \)
2. Modulation

Consider the baseband bandlimited signal $x[n]$, given by

$$x[n] = \sum_{k=-k_x}^{k_x} A_k e^{j\Omega_k n}$$

with the spectrum below, where $\Omega_k = k\Omega_0$ as before.

Then,

$$y[n] = x[n] \cos(k_c\Omega_0 n)$$

$$= \left[ \sum_{k=-k_x}^{k_x} A_k e^{j\Omega_k n} \right] \left[ \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n} \right]$$

using Euler's identity to write the cosine term.

$$= \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k+k_c)\Omega_0 n} + \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k-k_c)\Omega_0 n}$$

So that the spectrum of $y[n]$ is the replication and shifting of the baseband signal $x[n]$ at $\pm k_c$, scaled by $\frac{1}{2}$.
Example 3

\[ x[n] = \cos \left( \frac{2\pi}{3} n \right) \]

\[ \text{carrier} = \cos (2\pi n) \]

\[ \text{Required: sketch spectra of } x[n], \text{carrier, } y[n]. \]

\[ \text{Solution: } \]

1. \[ x[n] = \cos \left( \frac{2\pi}{3} n \right) \sim \text{fundamental frequency } \Omega_1 = \frac{2\pi}{3} \]

2. \[ \text{carrier} = \cos (2\pi n) = \cos \left( \frac{2\pi}{3} \cdot 3n \right) \sim k_c = 3, \quad \Omega_c = \frac{2\pi}{3} \]

From 1, \[ x[n] = \cos \left( \frac{2\pi}{3} n \right) = \frac{1}{2} e^{\frac{2\pi}{3} n} + \frac{1}{2} e^{-\frac{2\pi}{3} n} \]

Spectrum at \( k = \pm 1 \), magnitude \( \frac{1}{2} \).

No imaginary component.

From 2, \[ \text{carrier} = \cos \left( \frac{2\pi}{3} \cdot 3n \right) = \frac{1}{2} e^{\frac{2\pi}{3} \cdot 3n} + \frac{1}{2} e^{-\frac{2\pi}{3} \cdot 3n} \]

Spectrum at \( k_c = 3 \), magnitude \( \frac{1}{2} \).

No imaginary component.

3. \[ y[n] = \cos \left( \frac{2\pi}{3} n \right) \cos \left( \frac{2\pi}{3} \cdot 3n \right) = \left( \frac{1}{2} e^{\frac{2\pi}{3} n} + \frac{1}{2} e^{-\frac{2\pi}{3} n} \right) \left[ \frac{1}{2} e^{\frac{2\pi}{3} \cdot 3n} + \frac{1}{2} e^{-\frac{2\pi}{3} \cdot 3n} \right] \]

\[ = \frac{1}{4} e^{\frac{2\pi}{3} \cdot (4n)} + \frac{1}{4} e^{-\frac{2\pi}{3} \cdot (4n)} + \frac{1}{4} e^{\frac{2\pi}{3} \cdot (2n)} + \frac{1}{4} e^{-\frac{2\pi}{3} \cdot (2n)} \]

Spectrum at \( k = -4, -2, 2, 4 \).

No imaginary component.
3. **Demodulation**

Consider the received signal with the spectrum:

\[ t(n) = \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k+k_x)\Omega_1 n} + \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k-k_x)\Omega_1 n} \]

Then \[ z(n) = t(n) \cos(k_c \Omega_1 n) \]

\[ = \left[ \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k+k_x)\Omega_1 n} + \frac{1}{2} \sum_{k=-k_x}^{k_x} A_k e^{j(k-k_x)\Omega_1 n} \right] \left[ \frac{1}{2} e^{j k_c n} + \frac{1}{2} e^{-j k_c n} \right] \]

\[ = \frac{1}{4} \sum_{k=-k_x}^{k_x} A_k e^{j(k+2k_c)\Omega_1 n} + \frac{1}{4} \sum_{k=-k_x}^{k_x} A_k e^{j(k-2k_c)\Omega_1 n} \]

Thus the spectrum of \( z \) is as shown below:

After Low-Pass Filtering (LPF) of gain 2, and cutoff frequency \( \pm k_x \), we get the desired signal.