FAQ : Frequency Decomposition and LTI Systems

The idea of this note is to address the question: "What does it mean to use the frequency response of the system $H(e^{j\Omega})$, to predict the output $y[n]$ given input $x[n]$?", see Figure 1 below. Well after applying frequency decomposition of $x[n]$ into complex exponentials

$$e^{j(\frac{2\pi k}{N})n} \qquad k = 0, 1, \cdots, N-1 \text{ (periodic in } N)$$

we can use linearity of the LTI system to gain some intuition. Send each (complex) sinusoid component $e^{j(\frac{2\pi k}{N})n}$ individually through the channel, and the output is

$$H(e^{j(\frac{2\pi k}{N})})e^{j(\frac{2\pi k}{N})n}.$$

The signal $e^{j(\frac{2\pi k}{N})n}$ really *does not* change; it is only "weighted" by the frequency response $H(e^{j\Omega})$ at the corresponding sinusoid frequency $\Omega = \frac{2\pi k}{N}$.

Why use complex exponentials $e^{j(\frac{2\pi k}{N})n}$? Why not just use $\cos((\frac{2\pi k}{N})n)$ and $\sin((\frac{2\pi k}{N})n)$? We really care about cos and sin's, which can be expressed in terms of complex sinusoid as follows

$$\begin{aligned} \cos\left(\left(\frac{2\pi k}{N}\right)n\right) &= \frac{1}{2}e^{j(\frac{2\pi k}{N})n} + \frac{1}{2}e^{-j(\frac{2\pi k}{N})n}, \\ \sin\left(\left(\frac{2\pi k}{N}\right)n\right) &= \frac{1}{2j}e^{j(\frac{2\pi k}{N})n} - \frac{1}{2j}e^{-j(\frac{2\pi k}{N})n}. \end{aligned}$$

Essentially, we need complex arithmetic to enable us "to work" the frequency decomposition equation (see Lecture 14, Slide 10) to compute the frequency coefficients a_k.

What does it mean to have negative frequency $\Omega = -\frac{2\pi k}{N}$? If it so pleases you, use periodicity and think of

$$\Omega = \Omega + 2\pi, \quad \text{or} \quad \Omega = -\frac{2\pi k}{N} + 2\pi = \frac{2\pi(N-k)}{N},$$

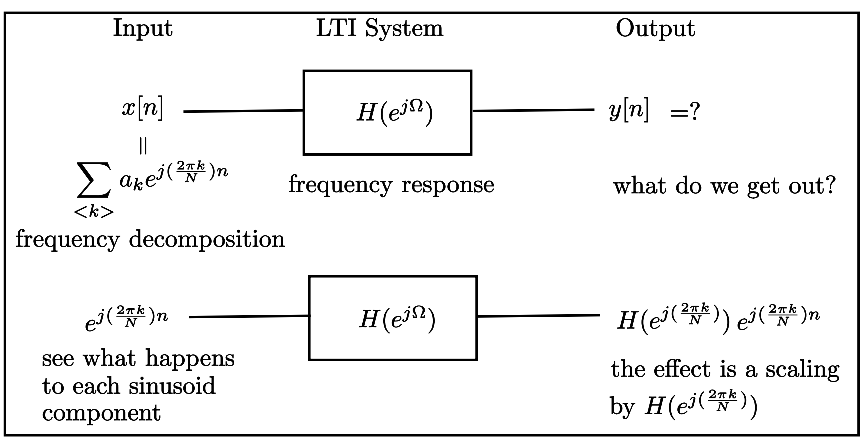

Figure 1: Figuring out frequency response

which is a positive frequency because recall that $k < N$. Alternatively, you can try to interpret the importance of negative frequencies, as that so we can express both cos's and sin's in terms of complex sinusoids (see previous discussion).

$H(e^{j\Omega})$ is a complex number. What does it mean to be weighted by a complex number? As an example, let's say $H(e^{j(\frac{2\pi k}{N})}) = 1 + j$. So passing $\cos((\frac{2\pi k}{N})n)$ into the system, we will get out

$$\frac{1+j}{2} e^{j(\frac{2\pi k}{N})n} + \frac{1-j}{2} e^{-j(\frac{2\pi k}{N})n}.$$

Expressing $1+j$ in terms of magnitude and phase components, we get $1+j = \sqrt{2} e^{j\frac{\pi}{4}}$ (think of the real-imaginary 2D plane). So plugging in above we get

$$\frac{1}{\sqrt{2}} e^{j((\frac{2\pi k}{N})n + \frac{\pi}{4})} + \frac{1}{\sqrt{2}} e^{-(j(\frac{2\pi k}{N})n + \frac{\pi}{4})} = \sqrt{2} \cos\left((\frac{2\pi k}{N})n + \frac{\pi}{4} \right).$$

Hence, we see that the complex weighting $1+j$ gives the $\cos((\frac{2\pi k}{N})n)$ sinusoid an $|1+j| = \sqrt{2}$ amplitude "boost", and a phase-shift of $\pi/4$.

As an exercise, try to repeat the above argument when $H(e^{j(\frac{2\pi k}{N})})$ is an arbitrary complex number $H(e^{j(\frac{2\pi k}{N})}) = a + jb$. Also try to work out the similar argument for the sin case.

What is conjugate symmetry $H(e^{j\Omega}) = H(e^{-j\Omega})$? Why is this property required? The previous discussion (subtly) requires conjugate symmetry to work (can you see why?). In other words, we need conjugate symmetry to show that when putting in a $\cos((\frac{2\pi k}{N})n)$ signal (or $\sin((\frac{2\pi k}{N})n)$ signal), we get out the same $\cos((\frac{2\pi k}{N})n)$ (or $\sin((\frac{2\pi k}{N})n)$) sinusoid with amplitude "boosts" and phase shifts.

Without conjugate symmetry, we will get the output of a real sinusoid to be complex, which is absurd because we assumeed that the channel unit sample response $h[n]$ is also real.

What is the lowest/highest frequencies in the system? They are

$$\begin{aligned} e^{j(\frac{2\pi 0}{N})n} &= 1, 1, 1, 1, \cdots \\ e^{j(\pi n)} &= 1, -1, 1, -1, \cdots \end{aligned}$$

(highest frequency $\Omega = \pi$ is right in the middle of the frequency range $0 \leq \Omega \leq 2\pi$ because of periodicity). The response of $H(e^{j\Omega})$ at these frequencies are

$$\begin{aligned} H(e^{j(\frac{2\pi 0}{N})n}) &= \sum_{n=0}^{N-1} h[n], \\ H(e^{j(\frac{2\pi 0}{N})n}) &= h[0] - h[1] + h[2] - h[3] \cdots + (-1)^{N-1} h[N-1]. \end{aligned}$$

Why does changing the 0-th term $h[0]$ preserve "wiggles"? Consider $h[n] = 1, 1, 1$ and thus $H(e^{j\Omega}) = 1 + (e^{j\Omega})^{-1} + (e^{j\Omega})^{-2}$. This has zeros at $\Omega = e^{j\frac{2\pi}{3}}$ and $e^{-j\frac{2\pi}{3}}$.

Now change $h[0] = 2$, then now $H(e^{j\Omega}) = 2 + (e^{j\Omega})^{-1} + (e^{j\Omega})^{-2}$ evaluates to 1 at $\Omega = e^{j\frac{2\pi}{3}}$ and $e^{-j\frac{2\pi}{3}}$. Changing $h[0] = 1$ to $h[0] = 2$ has now changed $H(e^{j\Omega})$ from "crossing" the horizontal axis at the zeros, to "crossing" the horizontal line displaced a distance of 1 away from the zero.
The similar "crossing" behavior, implies preservation of "wiggles".

The theory of signals and systems is actually more general that what is presented here. Here we assumed that the unit sample response $h[n]$ is real, which is ok for 6.02 purposes.