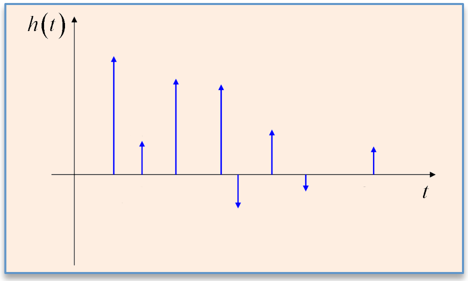
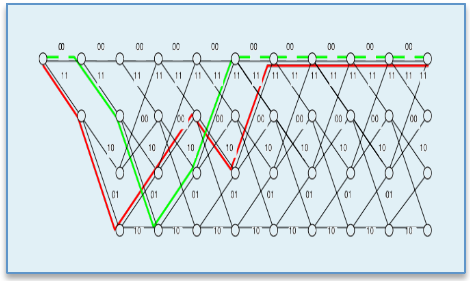
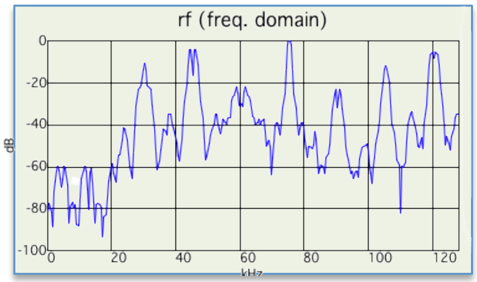
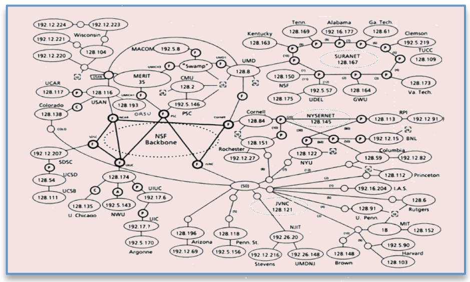








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2012
Lecture #6

- Convolutional codes
- State-machine view & trellis

Error Control Codes for Interplanetary Space Probes

- Early Mariner probes, 1962-1967 (Mars, Venus) – no ECC

- Later Mariner and Viking probes, 1969-1976 (Mars, Venus) – linear block codes, e.g.,

  (32,6,16) bi-orthogonal or Hadamard code
  - codewords comprise: the all-0 word, the all-1 word, and the other codewords all have sixteen 0's, sixteen 1's. The complement of each codeword is a codeword.

Bi-orthogonal Codes

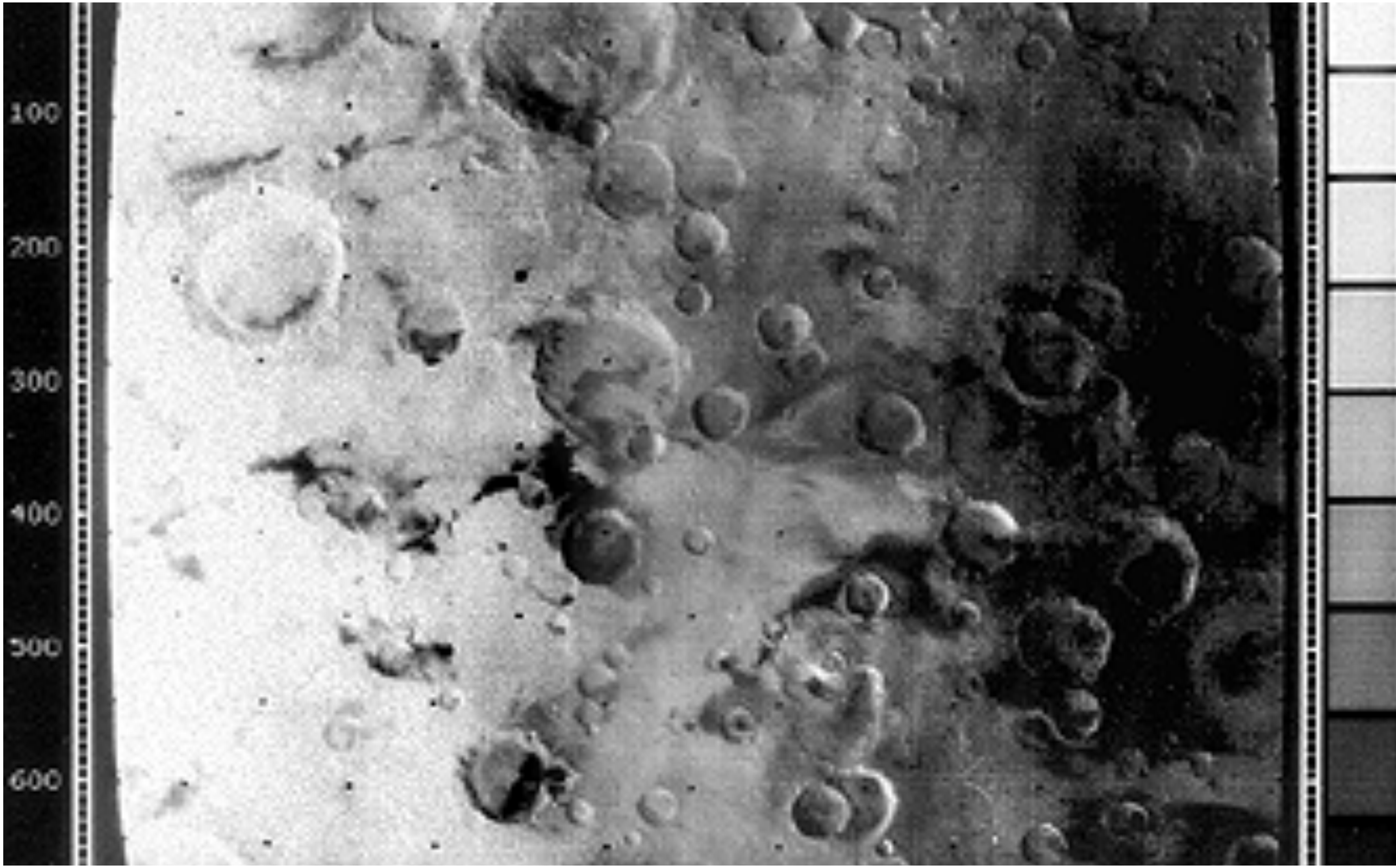
- e.g., used on Mariner 9 (1971, Mars orbit) to correct picture transmission errors.
  - Data word length: k=6 bits, for 64 grayscale values.
  - Usable block length n around 30 bits. Could have done 5-repetition code, but comparable rate with better error correction from a [32, 6, 16] Hadamard code.
  - Used through the 1980's.
- The efficient decoding algorithm was an important factor in the decision to use this code. The circuitry used was called the "Green Machine".
- More generally for such codes,

  n=2^(k-1), d=2^(k-2)

Mariner 9 (400 million km trip)

- "Spacecraft control was through the central computer and sequencer which had an onboard memory of 512 words. The command system was programmed with 86 direct commands, 4 quantitative commands, and 5 control commands. Data was stored on a digital reel-to-reel tape recorder. The 168 meter 8-track tape could store 180 million bits recorded at 132 kbits/s. Playback could be done at 16, 8, 4, 2, and 1 kbit/s using two tracks at a time. Telecommunications were via dual S-band 10 W/20 W transmitters and a single receiver through the high gain parabolic antenna, the medium gain horn antenna, or the low gain omnidirectional antenna." (NASA)
-

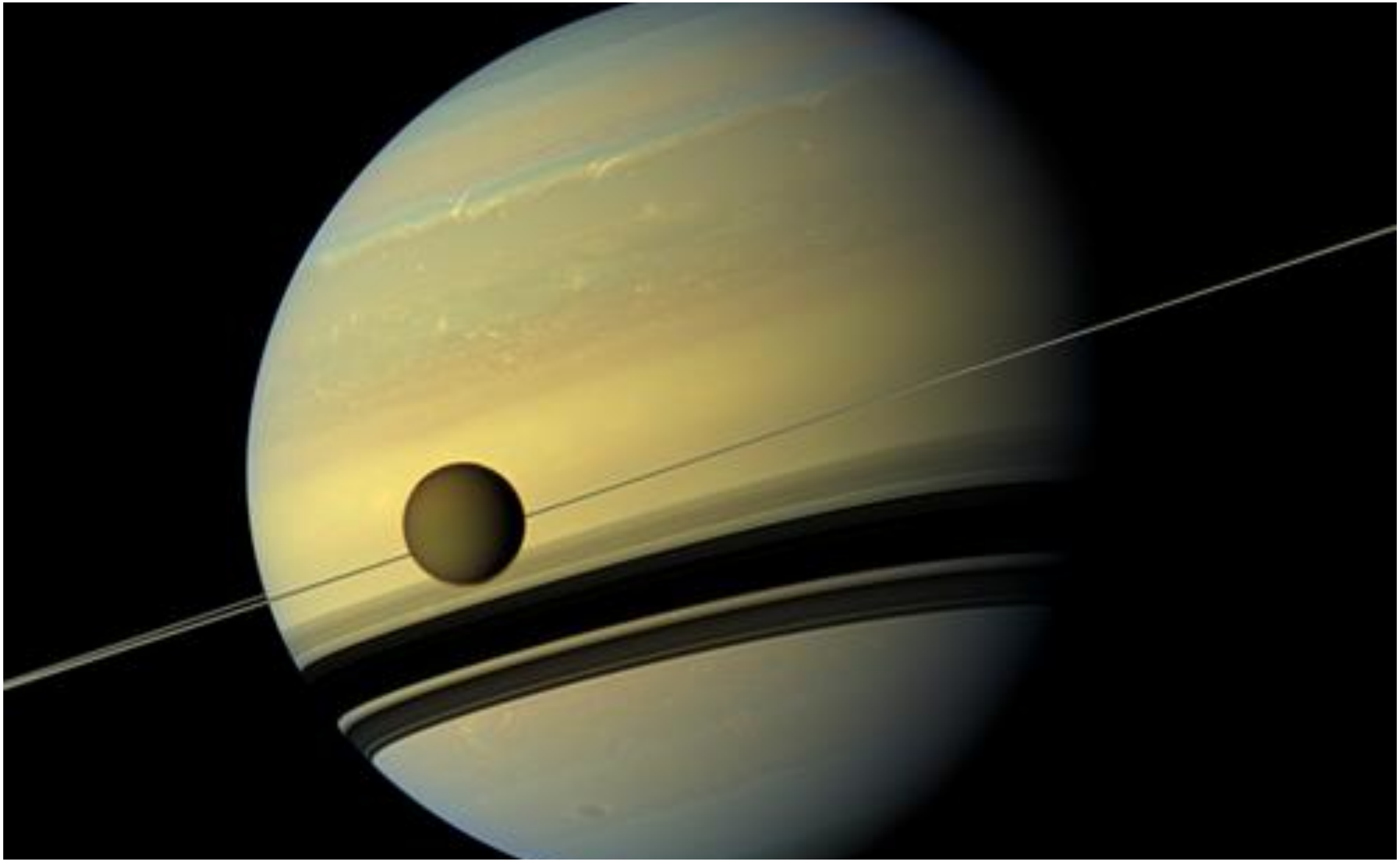
7329 images, e.g.:

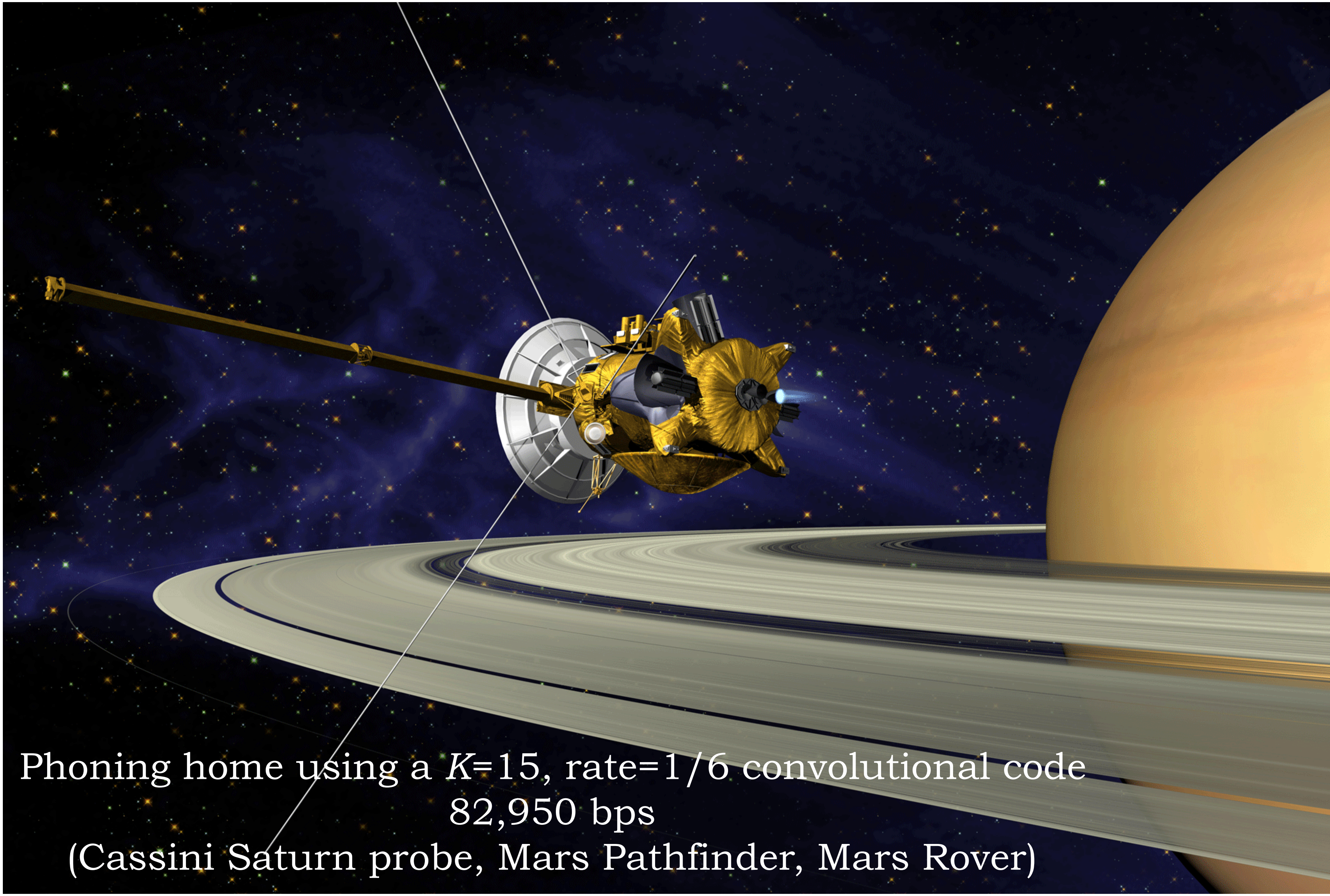


More powerful codes needed for higher data rates with limited transmitter power

- Space probe may have a 20W transmitter to cover tens of billions of kilometers!
  - Part of the secret is the antenna --- directs the beam to produce the same received intensity as an omnidirectional antenna radiating in the megawatts
  - Also "cryogenically-cooled low-noise amplifiers, sophisticated receivers, and data coding and error-correction schemes. These systems can collect, detect, lock onto, and amplify a vanishingly small signal that reaches Earth from the spacecraft, and can extract data from the signal virtually without errors." (JPL quote)
- Convolutional codes with Viterbi decoding – Voyager (1977) onwards, Cassini, Mars Exploration Rover, …

Saturn and Titan from Cassini, August 29, 2012




Phoning home using a K=15, rate=1/6 convolutional code
82,950 bps
(Cassini Saturn probe, Mars Pathfinder, Mars Rover)

Convolutional Codes (Peter Elias, 1955)

- Like the block codes discussed earlier, send parity bits computed from blocks of message bits
  - Unlike block codes, generally don't send message bits, send only the parity bits! (i.e., "non-systematic")
  - The code rate of a convolutional code tells you how many parity bits are sent for each message bit. We'll mostly be talking about rate 1/*r* codes, i.e., r parity bits/message bit.
  - Use a sliding window to select which message bits are participating in the parity calculations. The width of the window (in bits) is called the code's constraint length **K**.

$$p_0[n] = x[n] + x[n-1] + x[n-2]$$

$$p_1[n] = x[n] + x[n-2]$$

Addition mod 2 (aka XOR)

Parity Bit Equations

- A convolutional code generates sequences of parity bits from sequences of message bits by a convolution operation:

$$p_i[n] = \left(\sum_{j=0}^{K-1} g_i[j]\, x[n-j] \right) \bmod 2$$

- *K* is the constraint length of the code
  - The larger *K* is, the more times a particular message bit is used when calculating parity bits
    → greater redundancy
    → *better error correction possibilities (usually, though not always)*

- g_i is the *K*-element generator for parity bit p_i.
  - Each element $g_i[j]$ is either 0 or 1
  - More than one parity sequence can be generated from the same message; the simplest choice is to use 2 generator polynomials

Transmitting Parity Bits

- We'll transmit the parity sequences, not the message itself
  - As we'll see, we can recover the message sequences from the parity sequences
  - Each message bit is "spread across" K elements of each parity sequence, so the parity sequences are better protection against bit errors than the message sequence itself

- If we're using multiple generators, construct the transmit sequence by interleaving the bits of the parity sequences:

$$xmit = p_0[0], p_1[0], p_0[1], p_1[1], p_0[2], p_1[2], \ldots$$

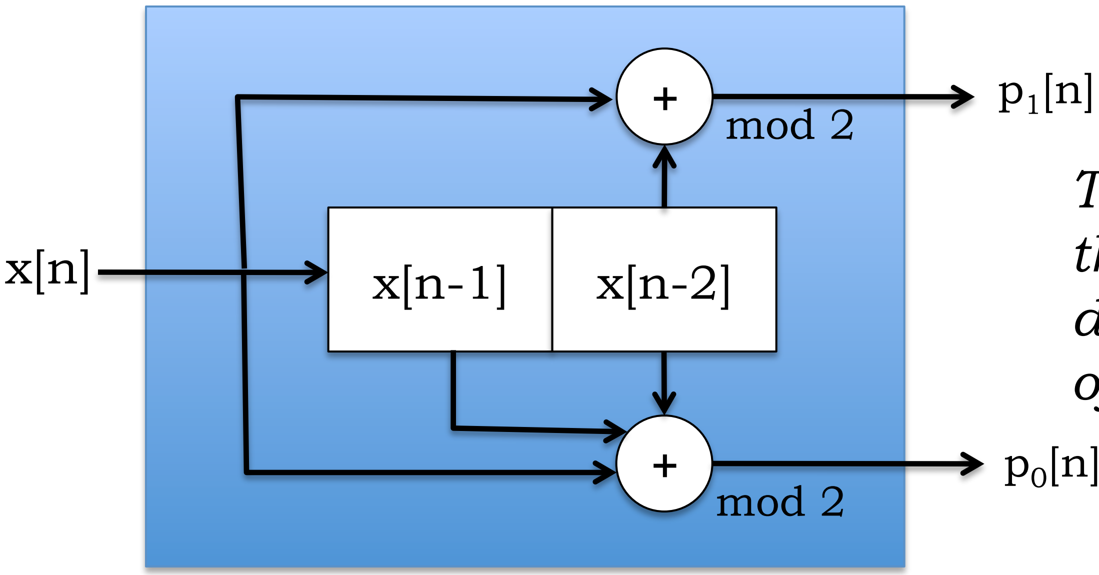
- Code rate is 1/number_of_generators
  - 2 generators → rate = ½
  - Engineering tradeoff: using more generators improves bit-error correction but decreases rate of the code (the number of message bits/s that can be transmitted)

Example

- Using two generators:
  - g_0 = 1, 1, 1, 0, 0, … abbreviated as 111 for K=3 code
  - g_1 = 1, 0, 1, 0, 0, … abbreviated as 110 for K=3 code
- Writing out the equations for the parity sequences:
  - $p_0[n] = x[n] + x[n-1] + x[n-2]$
  - $p_1[n] = x[n] + x[n-2]$
- Let x[n] = [1,0,1,1,…]; x[n]=0 when n<0:
  - $p_0[0] = (1 + 0 + 0) \bmod 2 = 1$, $p_1[0] = (1 + 0) \bmod 2 = 1$
  - $p_0[1] = (0 + 1 + 0) \bmod 2 = 1$, $p_1[1] = (0 + 0) \bmod 2 = 0$
  - $p_0[2] = (1 + 0 + 1) \bmod 2 = 0$, $p_1[2] = (1 + 1) \bmod 2 = 0$
  - $p_0[3] = (1 + 1 + 0) \bmod 2 = 0$, $p_1[3] = (1 + 0) \bmod 2 = 1$
- Transmit: 1, 1, 1, 0, 0, 0, 0, 1, …

Shift-Register View

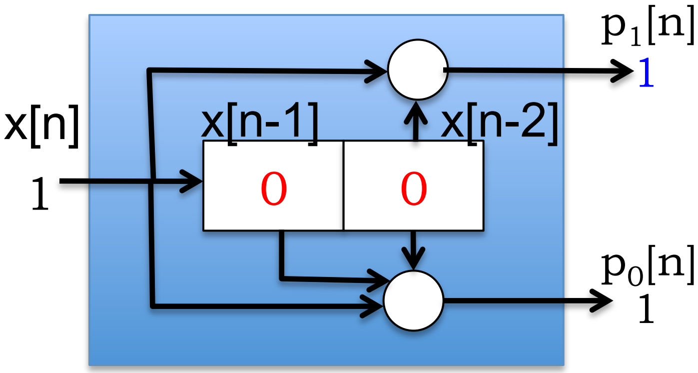
- One often sees convolutional encoders described with a block diagram like the following:



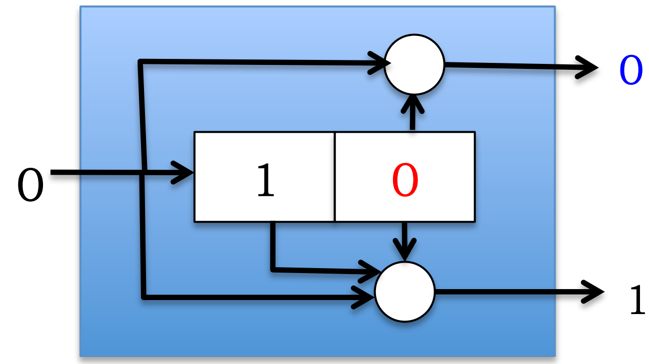
The values in the registers define the ***state*** *of the encoder*

- Message bit in, parity bits out
  - Input bits arrive one-at-a-time from the left
  - The box computes the parity bits using the incoming bit and the K-1 previous message bits
  - At the end of the bit interval, the saved message bits are *shifted right* by one, and the incoming bit moves into the left position.

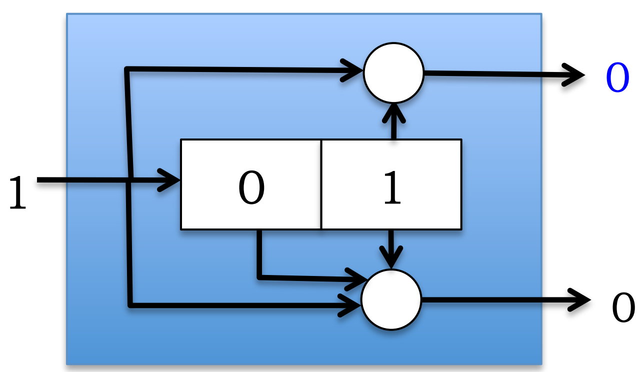
Example: Transmit message 1011



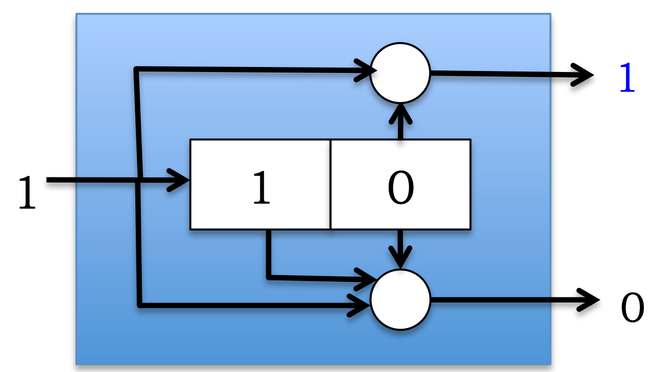
Processing x[0]



Processing x[1]



Processing x[2]



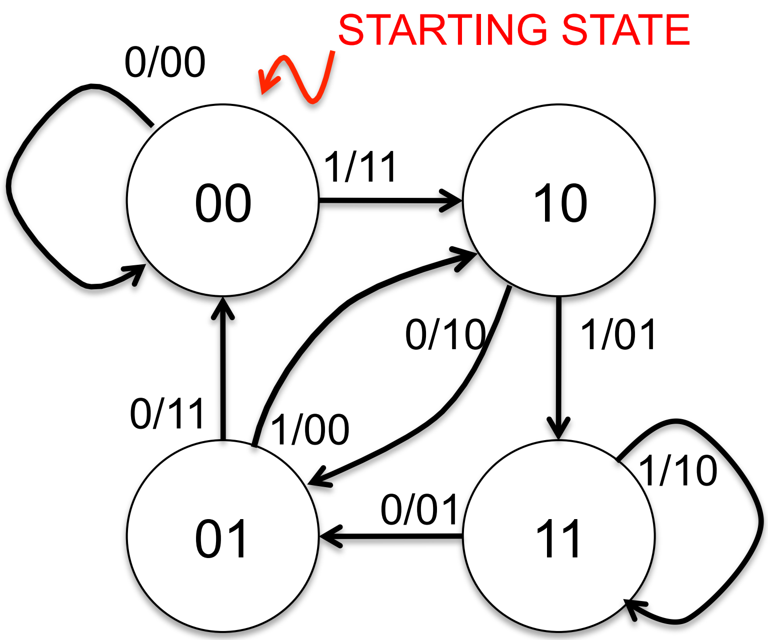
Processing x[3]

$p_0[n] = x[n] + x[n-1] + x[n-2]$

$p_1[n] = x[n] + x[n-2]$

Xmit seq: 1, 1, 1, 0, 0, 0, 0, 1, …
(codeword)

State-Machine View



$p_0[n] = x[n] + x[n-1] + x[n-2]$

$p_1[n] = x[n] + x[n-2]$

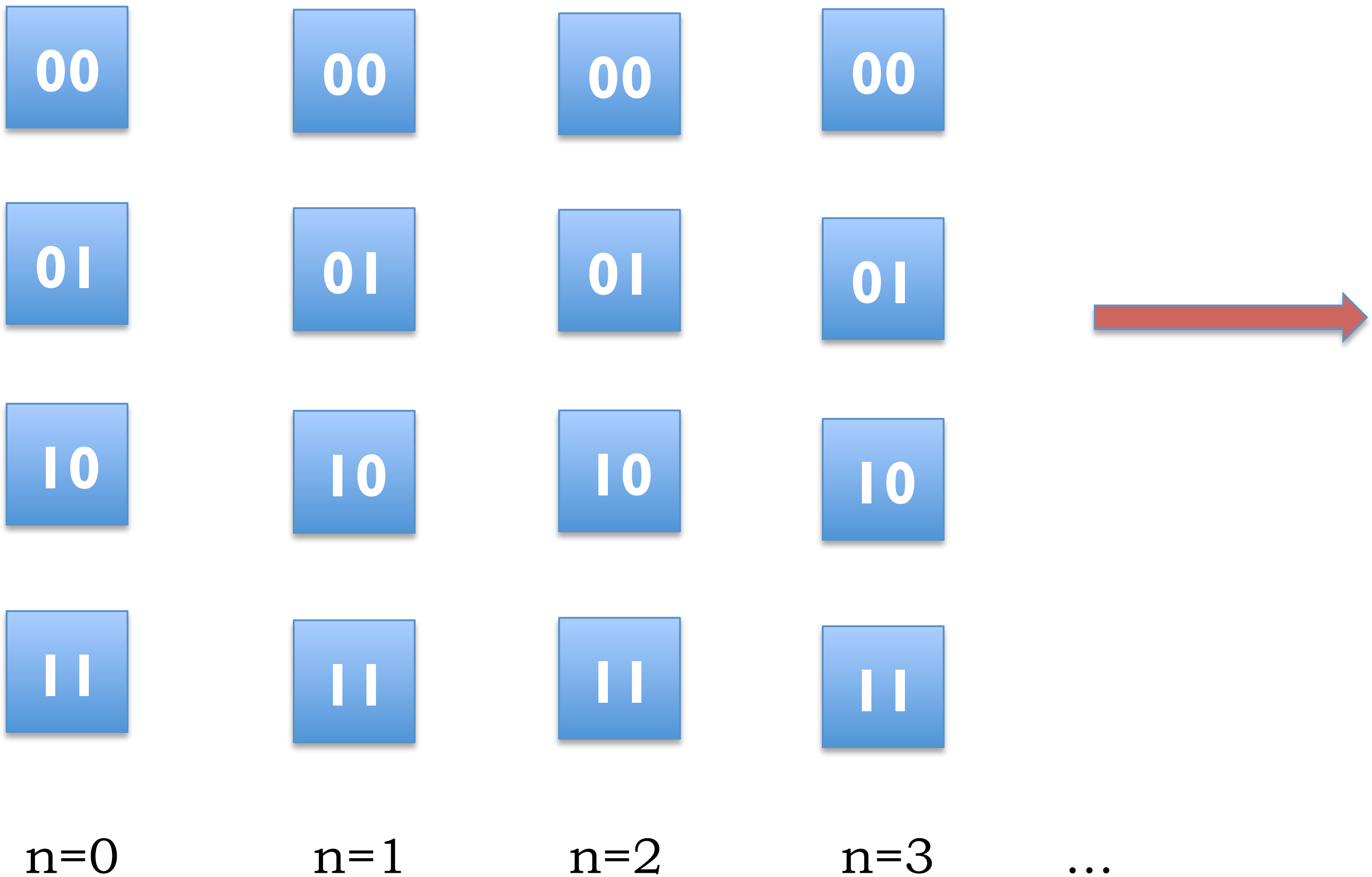
(Generators: $g_0 = 111$, $g_1 = 101$)

The state machine is the *same* for all K=3 codes. Only the p_i labels change depending on number and values for the generator polynomials.

- Example: K=3, rate-½ convolutional code
- There are 2^{K-1} states
- States labeled with (x[n-1], x[n-2]) value
- Arcs labeled with $x[n]/p_0[n]p_1[n]$
- msg=101100; xmit = 11 10 00 01 01 11

Trellis View

- State machine unfolded in time (fill in details using notes as guide, for the example considered here!)



The Parity Stream forms a Linear Code

- Smallest-weight nonzero codeword has a weight that (locally in time) plays a role analogous to d, the minimum Hamming distance. It's called the free distance (fd) of the convolutional code.

- What is fd for our example?

Encoding & Decoding Convolutional Codes

- Transmitter (aka Encoder)
  - Beginning at starting state, processes message bit-by-bit
  - For each message bit: makes a state transition, sends $p_0 p_1 \ldots$
  - Pad message with K-1 zeros to ensure return to starting state

- Receiver (aka Decoder)
  - Doesn't have direct knowledge of transmitter's state transitions; only knows (possibly corrupted) received parity bits, p_i
  - Must find most likely sequence of transmitter states that could have generated the received parity bits, p_i
  - If BER < ½, P(more errors) < P(fewer errors)
  - *When BER < ½, maximum-likelihood message sequence is the one that generated the codeword (here, sequence of parity bits) with the smallest Hamming distance from the received codeword (here, parity bits)*
  - I.e., find nearest valid codeword *closest* to the received codeword – Maximum-likelihood (ML) decoding

In the absence of noise …

- Decoding is trivial:

$$p_0[n] = x[n] + x[n-1] + x[n-2]$$

$$p_1[n] = x[n] + x[n-2]$$

- Can you see how to recover the input x[.] from the parity bits p[.] ?

- In the presence of errors in the parity stream, message bits will get corrupted at about the same rate as parity bits, with this simple-minded recovery.

Spot Quiz!

Consider the convolutional code given by

$p_0[n] = x[n] + x[n-2] + x[n-3]$

$p_1[n] = x[n] + x[n-1] + x[n-2]$

$p_2[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$

1. Constraint length, K, of this code = _____

2. Code rate = ____

3. Coefficients of the generators = ____, ____, ____

4. No. of states in state machine of this code = ____