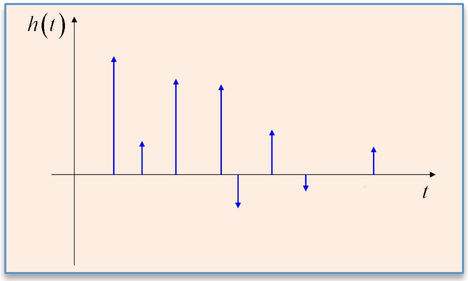
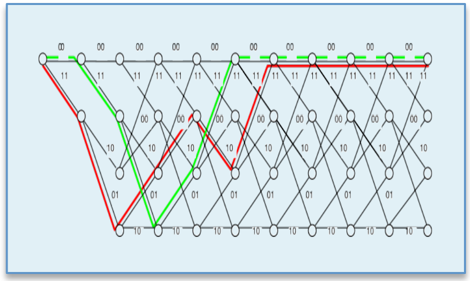
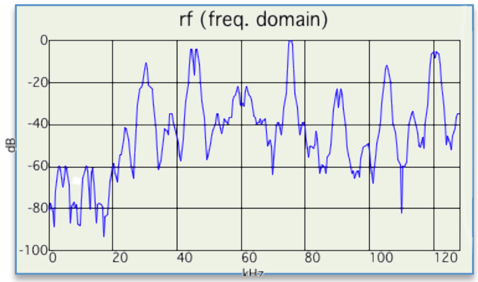
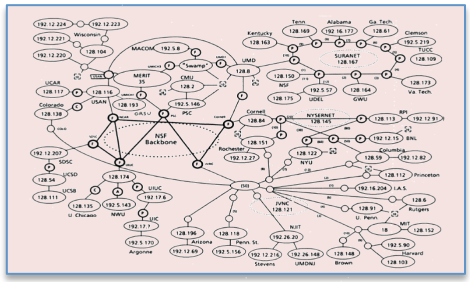








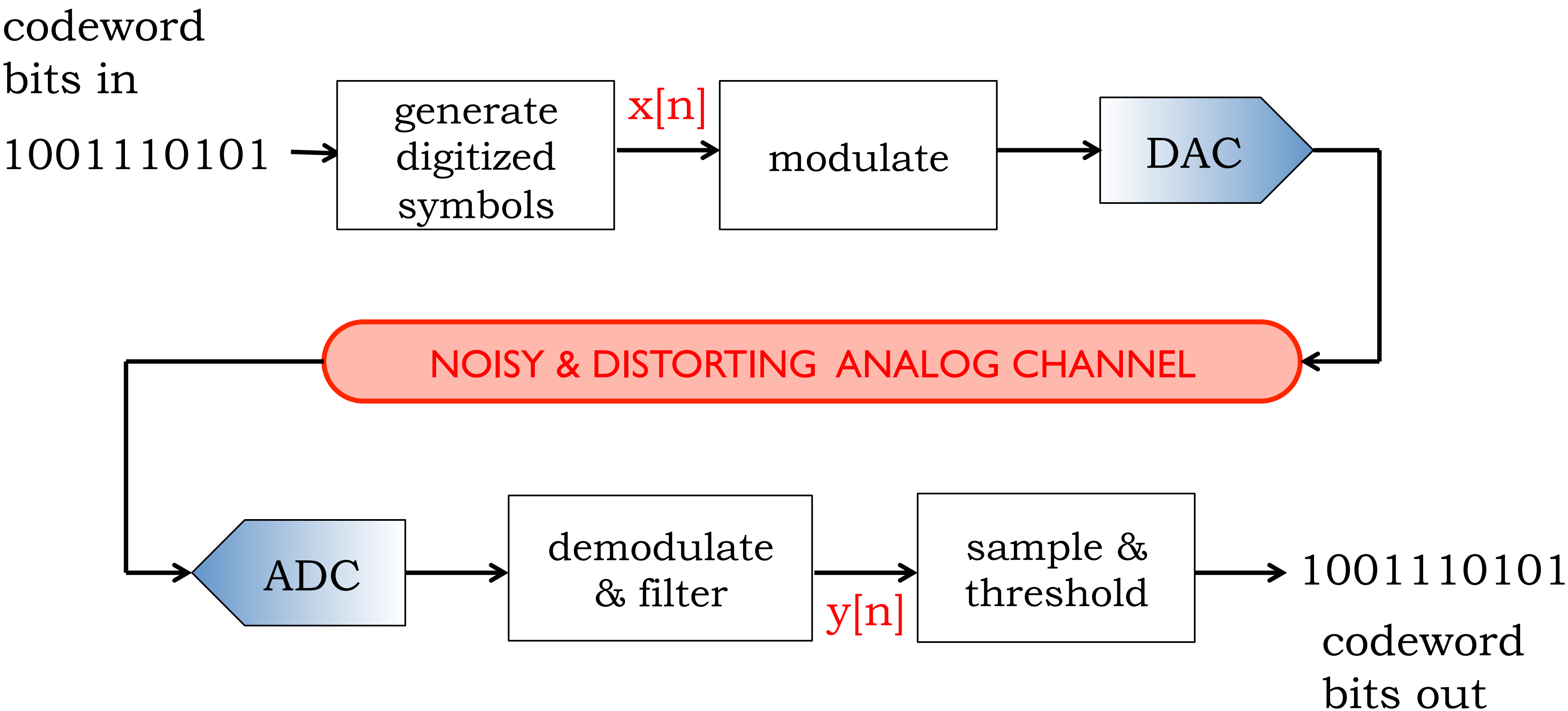
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

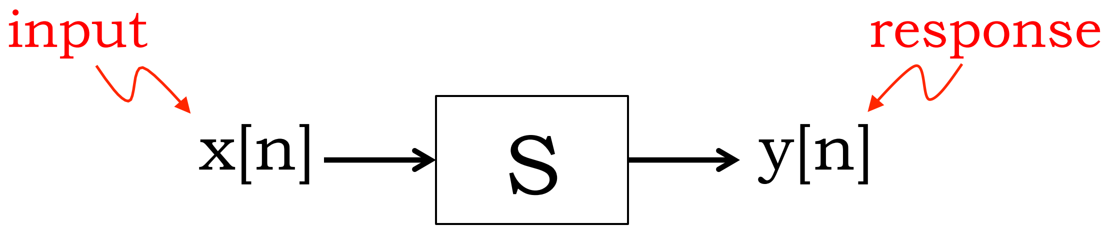
6.02 Fall 2012
Lecture #10

- Linear time-invariant (LTI) models
- Convolution

Modeling Channel Behavior



The Baseband** Channel



A discrete-time signal such as x[n] or y[n] is described by an infinite sequence of values, i.e., the time index n takes values in $-\infty$ to $+\infty$. The above picture is a snapshot at a particular time n.

In the diagram above, the sequence of *output* values y[.] is the *response* of system S to the *input* sequence x[.]

The system is causal if y[k] depends only on x[j] for $j \le k$

**From before the modulator till after the demodulator & filter

Time Invariant Systems

Let y[n] be the response of S to input x[n].

If for all possible sequences x[n] and integers N

$$x[n-N] \longrightarrow \boxed{S} \longrightarrow y[n-N]$$

then system S is said to be *time invariant* (TI). A time shift in the input sequence to S results in an identical time shift of the output sequence.

In particular, for a TI system, a shifted unit sample function $\delta[n-N]$ at the input generates an identically shifted unit sample response $h[n-N]$ at the output.

Linear Systems

Let $y_1[n]$ be the response of S to an arbitrary input $x_1[n]$ and $y_2[n]$ be the response to an arbitrary $x_2[n]$.

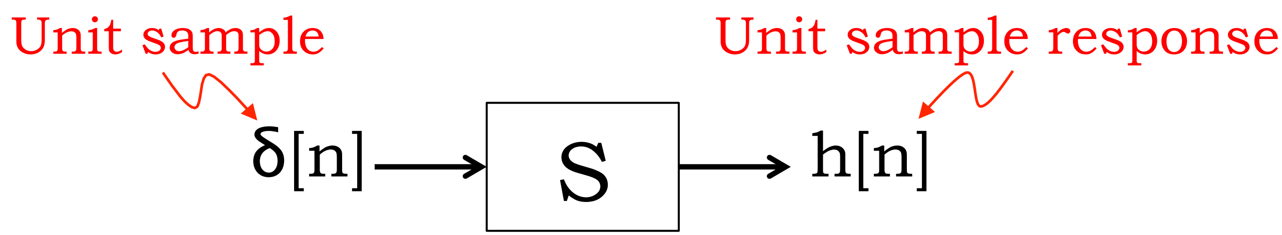
If, for arbitrary scalar coefficients *a* and *b*, we have:

$$ax_1[n]+bx_2[n] \longrightarrow \boxed{\;S\;} \longrightarrow ay_1[n]+by_2[n]$$

then system S is said to be *linear*. If the input is the weighted sum of several signals, the response is the *superposition* (i.e., same weighted sum) of the response to those signals.

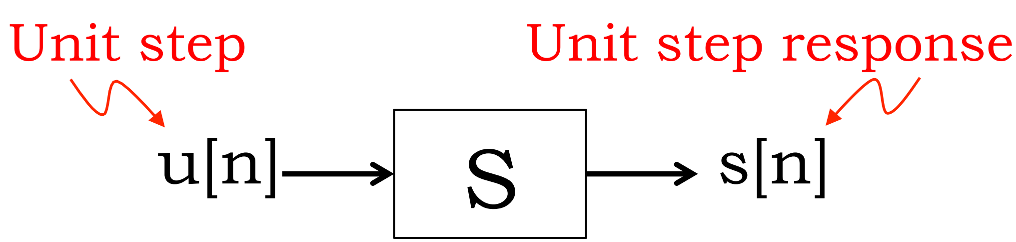
One key consequence: If the input is identically 0 for a linear system, the output must also be identically 0.

Unit Sample and Unit Step Responses

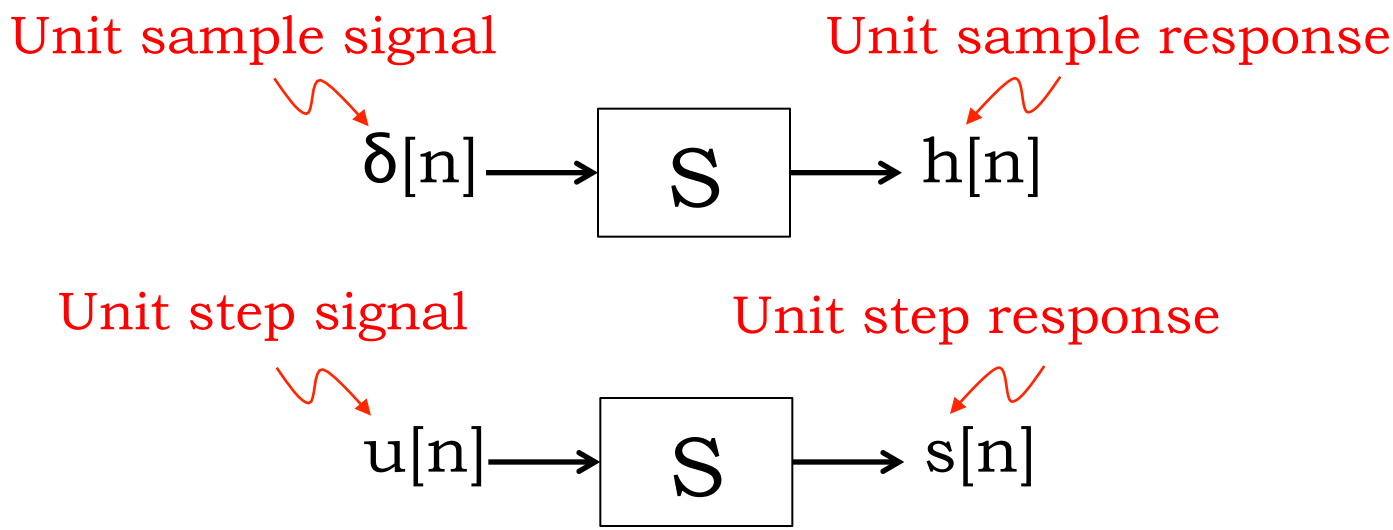


The *unit sample response* of a system S is the response of the system to the unit sample input. We will always denote the unit sample response as h[n].

Similarly, the *unit step response* s[n]:



Relating h[n] and s[n] of an LTI System



$$\delta[n] = u[n] - u[n-1] \quad \Rightarrow \quad h[n] = s[n] - s[n-1]$$

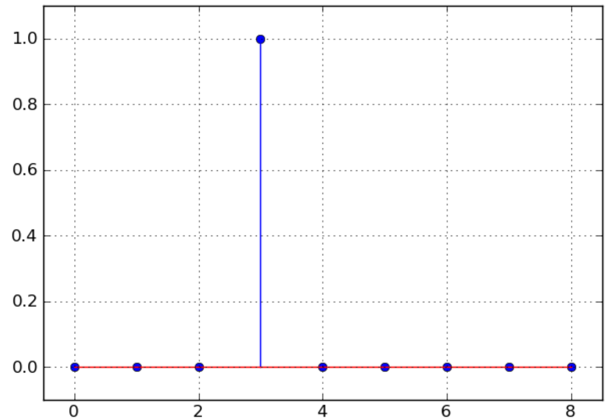
from which it follows that $$s[n] = \sum_{k=-\infty}^{n} h[k]$$

(assuming $s[-\infty] = 0$, e.g., a causal LTI system; more generally, a "right-sided" unit sample response)

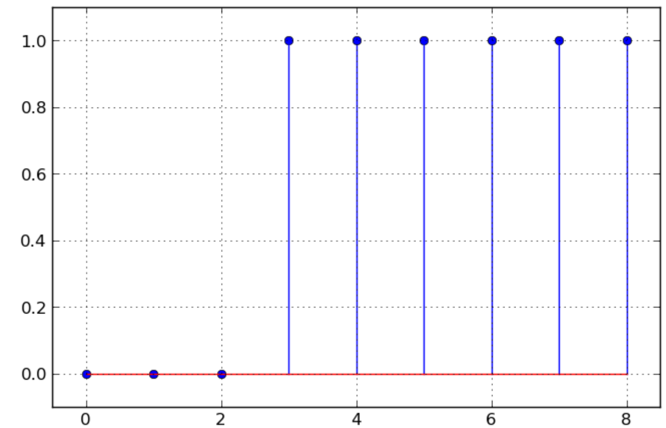
h[n]

s[n]

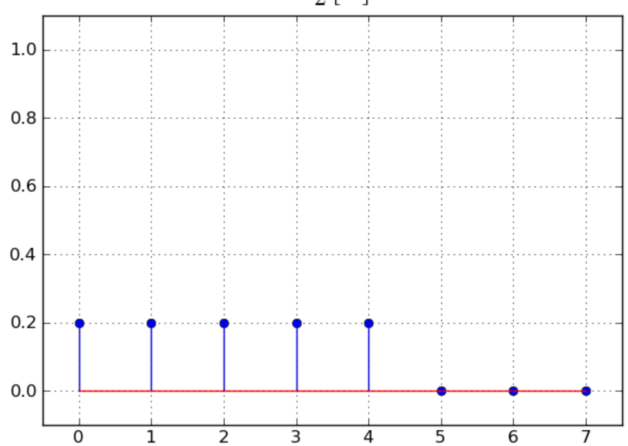
$h_1[n]$



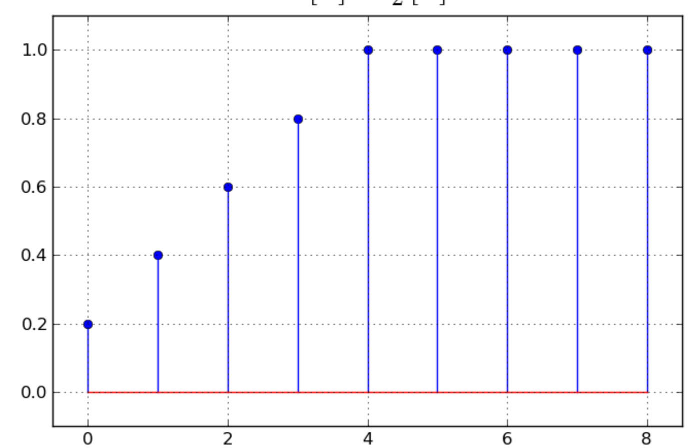
$u[n] * h_1[n]$



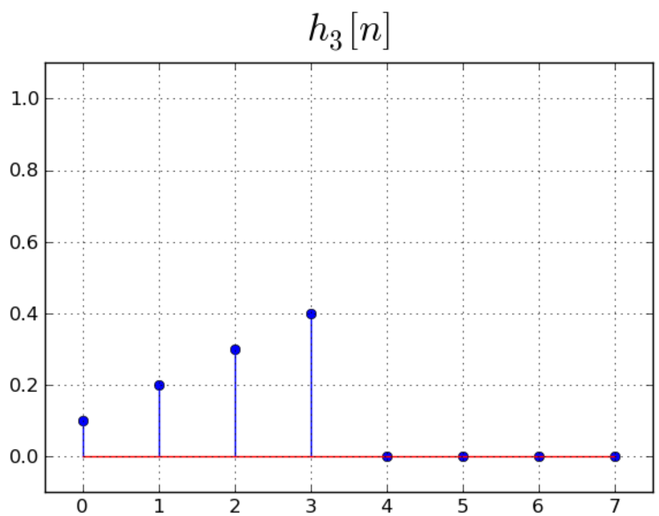
$h_2[n]$

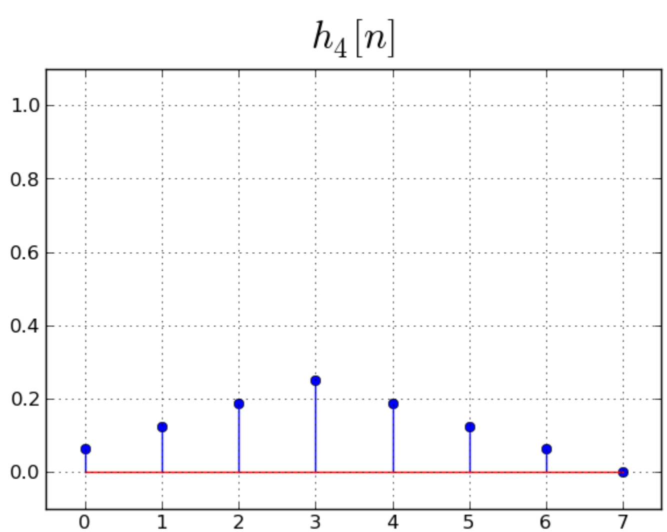


$u[n] * h_2[n]$

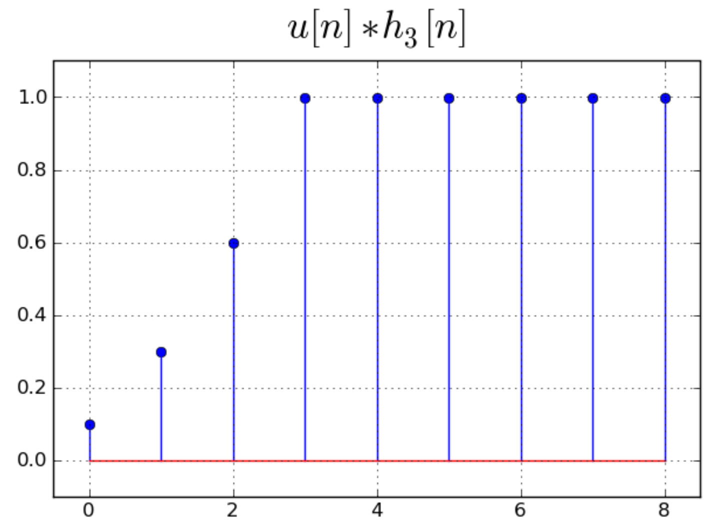


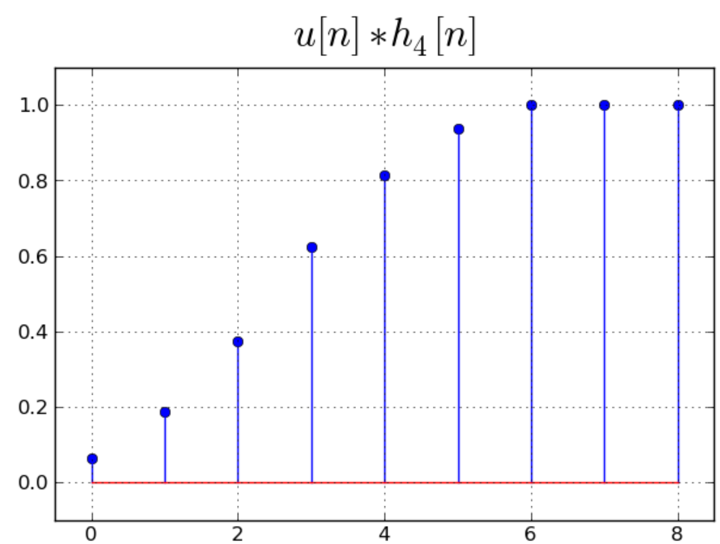
h[n]



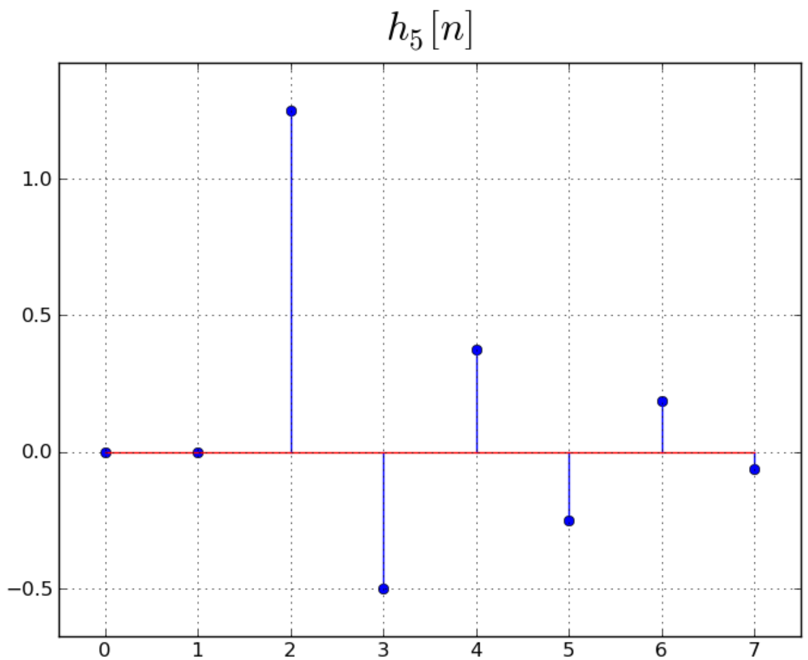


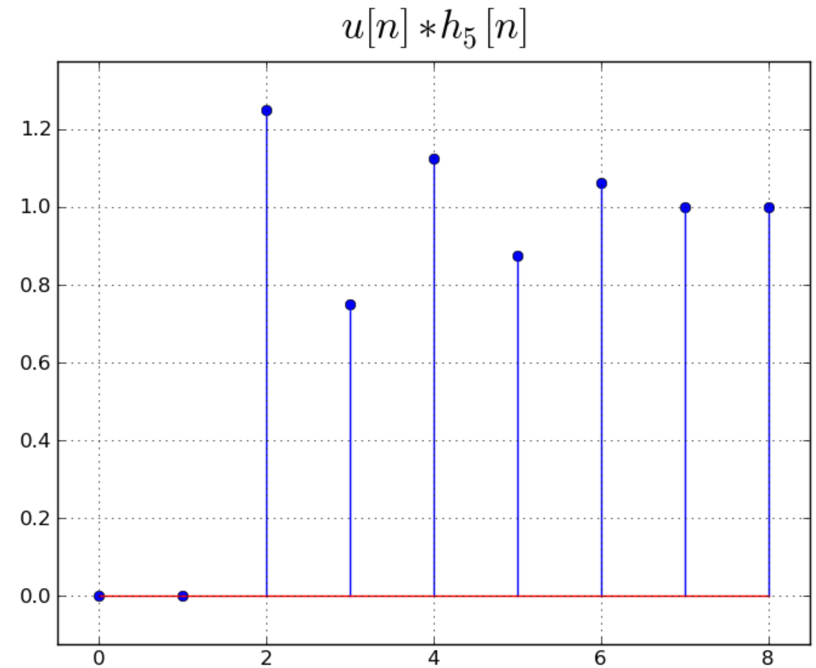
s[n]

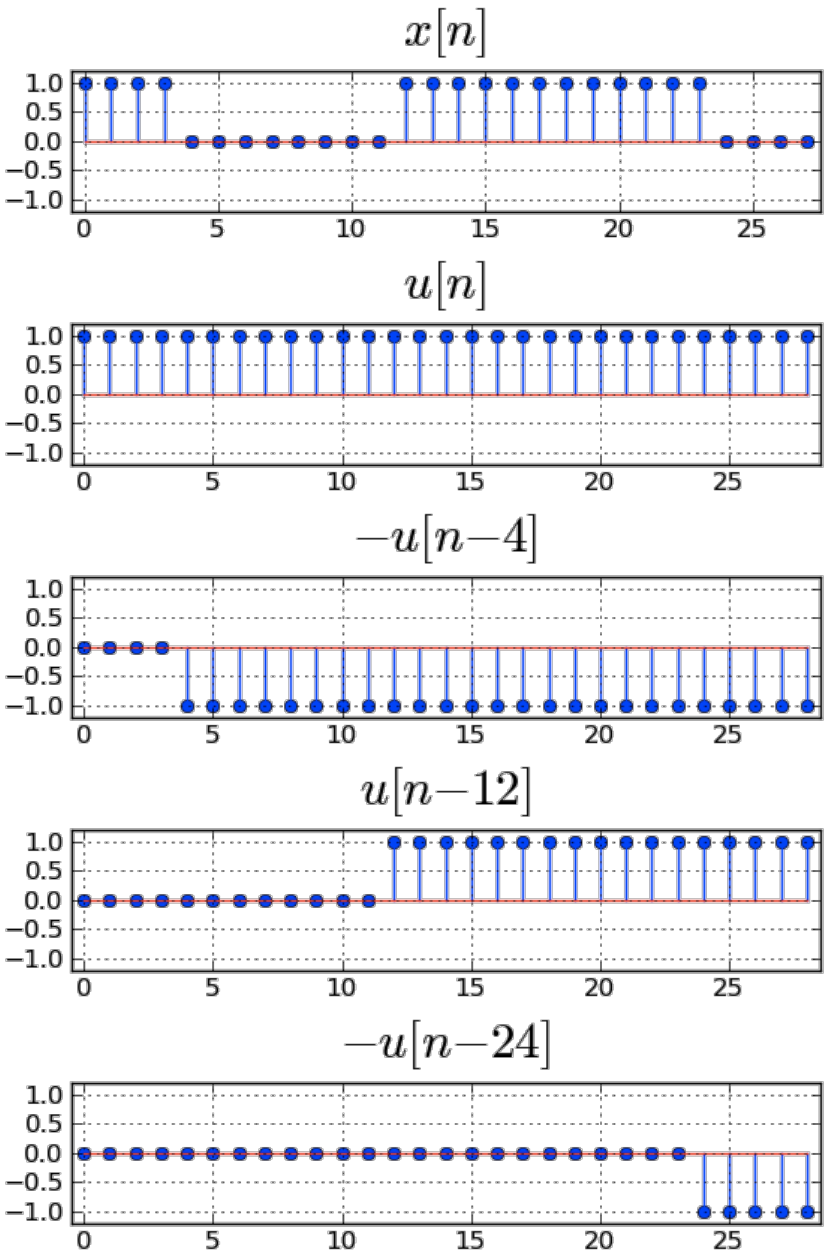




h[n] s[n]







Unit Step Decomposition

"Rectangular-wave" digital signaling waveforms, of the sort we have been considering, are easily decomposed into time-shifted, scaled unit steps --- each transition corresponds to another shifted, scaled unit step.

e.g., if x[n] is the transmission of 1001110 using 4 samples/bit:

$$
\begin{aligned}
x[n] & \\
&= u[n] \\
&\quad - u[n-4] \\
&\quad + u[n-12] \\
&\quad - u[n-24]
\end{aligned}
$$

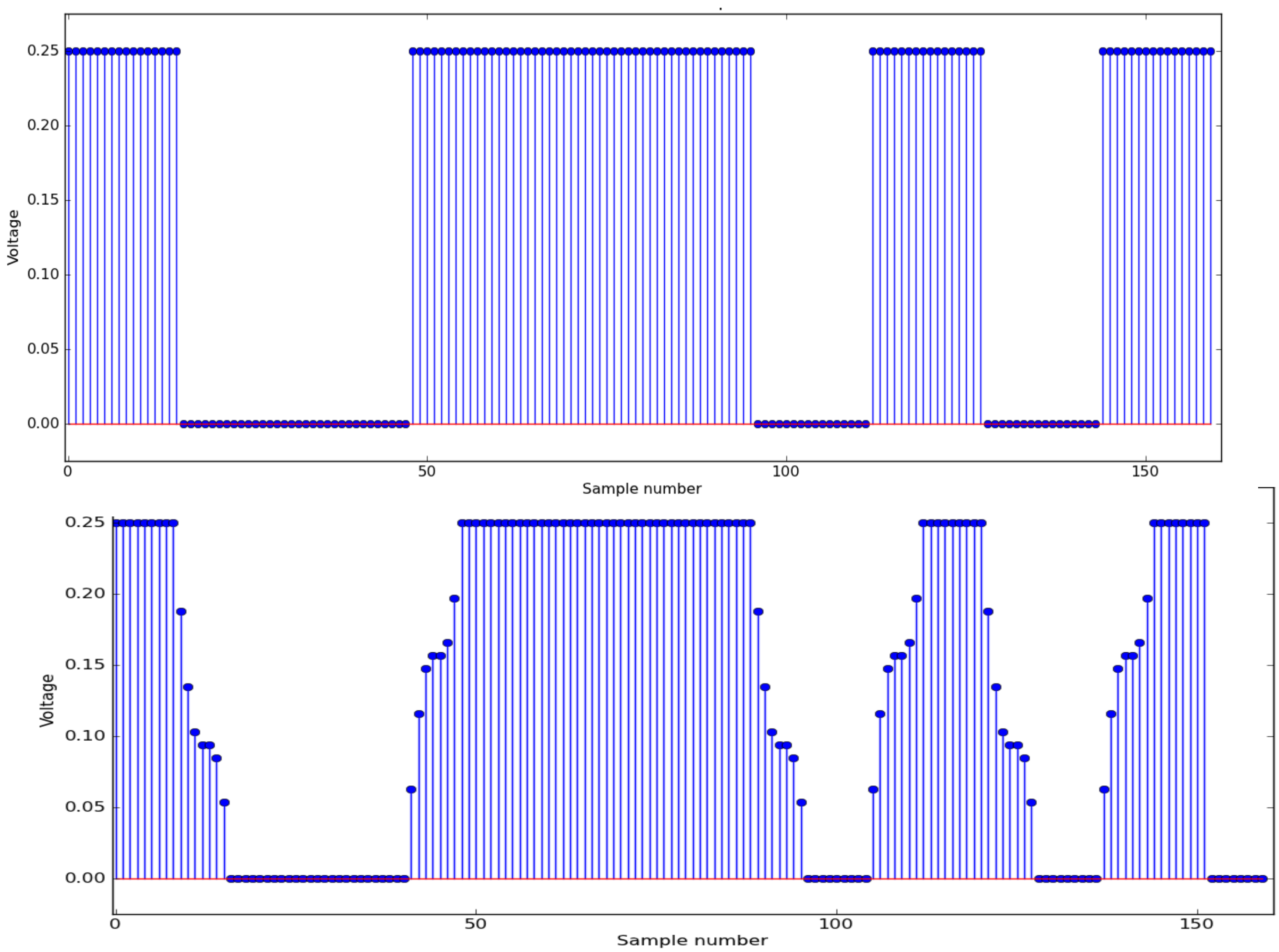
… so the corresponding response is

$$x[n] = u[n] - u[n-4] + u[n-12] - u[n-24]$$

$$y[n] = s[n] - s[n-4] + s[n-12] - s[n-24]$$

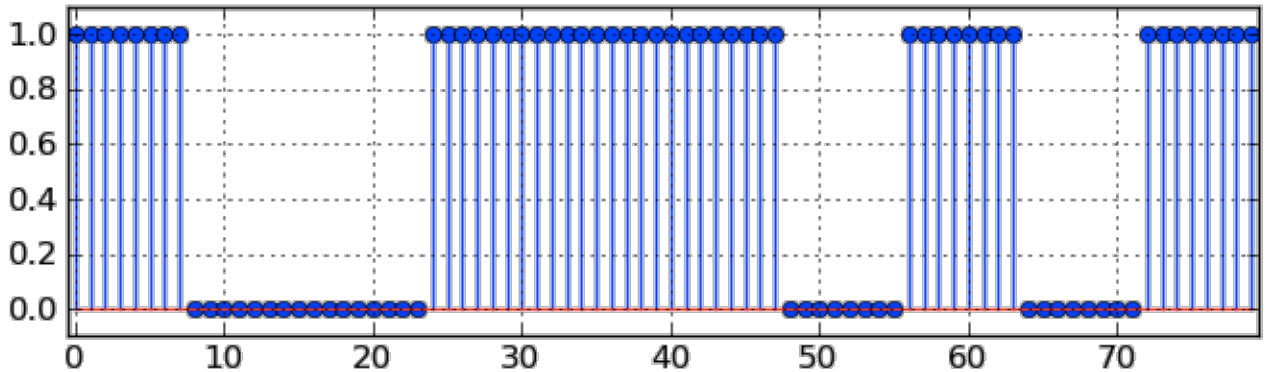
Note how we have invoked linearity and time invariance!

Example


0.25
0.20
0.15
0.10
0.05
0.00
Voltage
0
50
100
150
Sample number
0.25
0.20
0.15
0.10
0.05
0.00
Voltage
0
50
100
150
Sample number

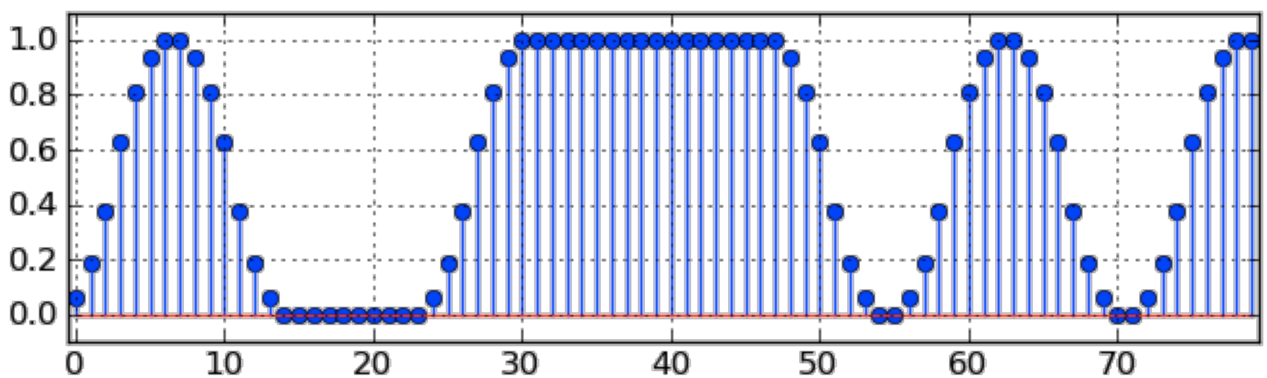
Transmission Over a Channel

$x[n]$ at 8 samples/bit

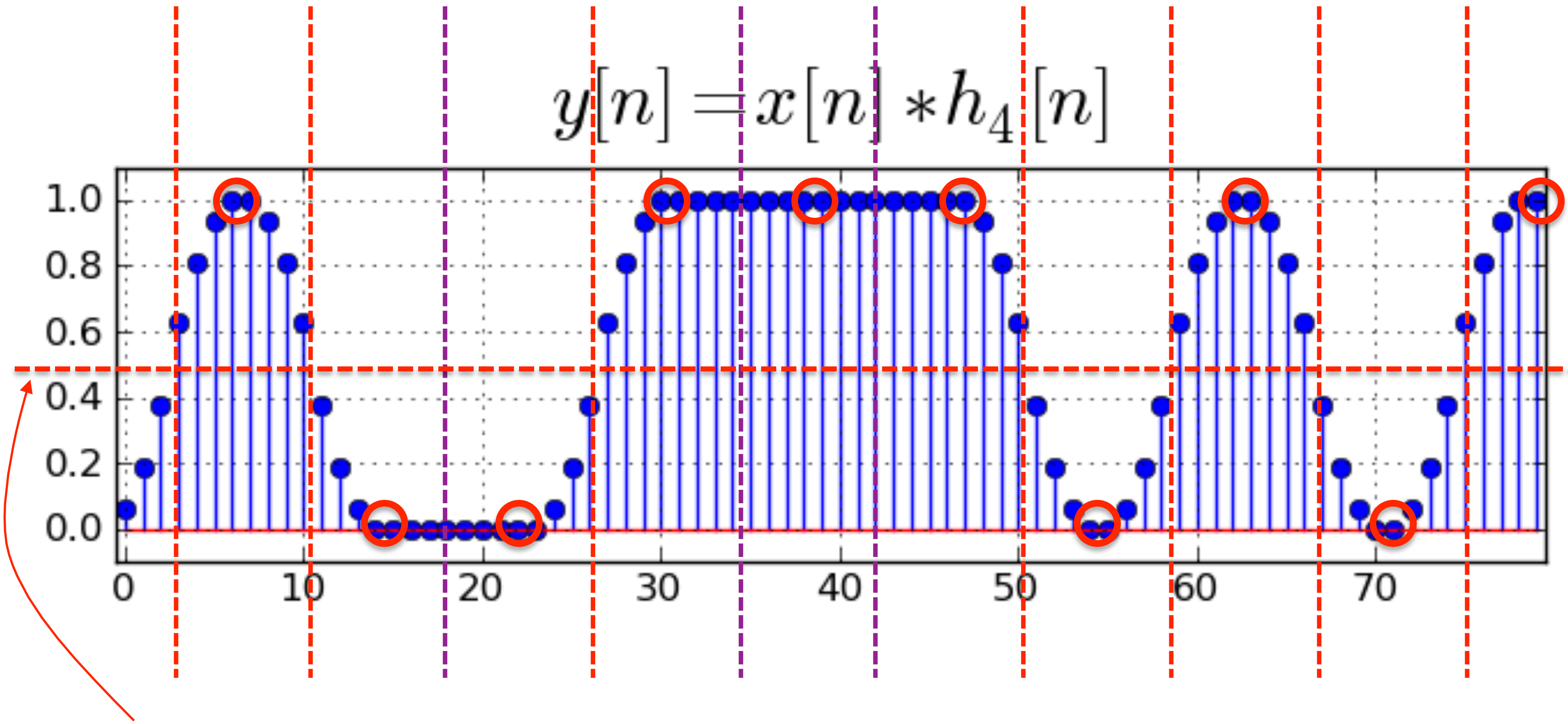


Ignore this notation for now, will explain shortly

$y[n] = x[n] * h_4[n]$



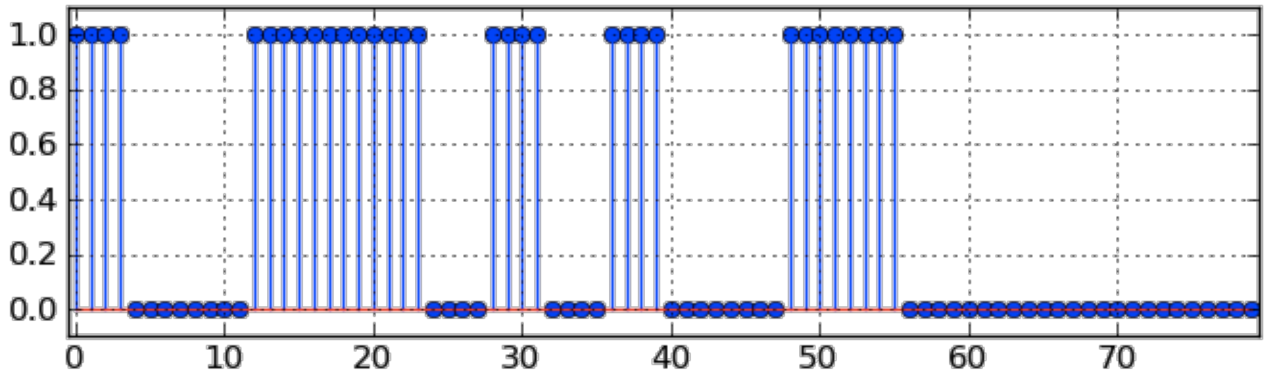
Receiving the Response



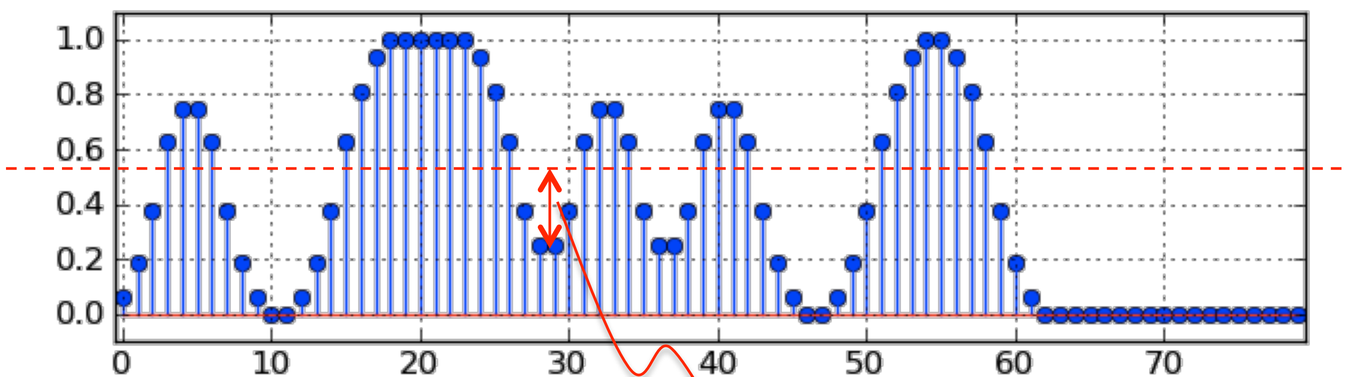
Digitization threshold = 0.5V

Faster Transmission

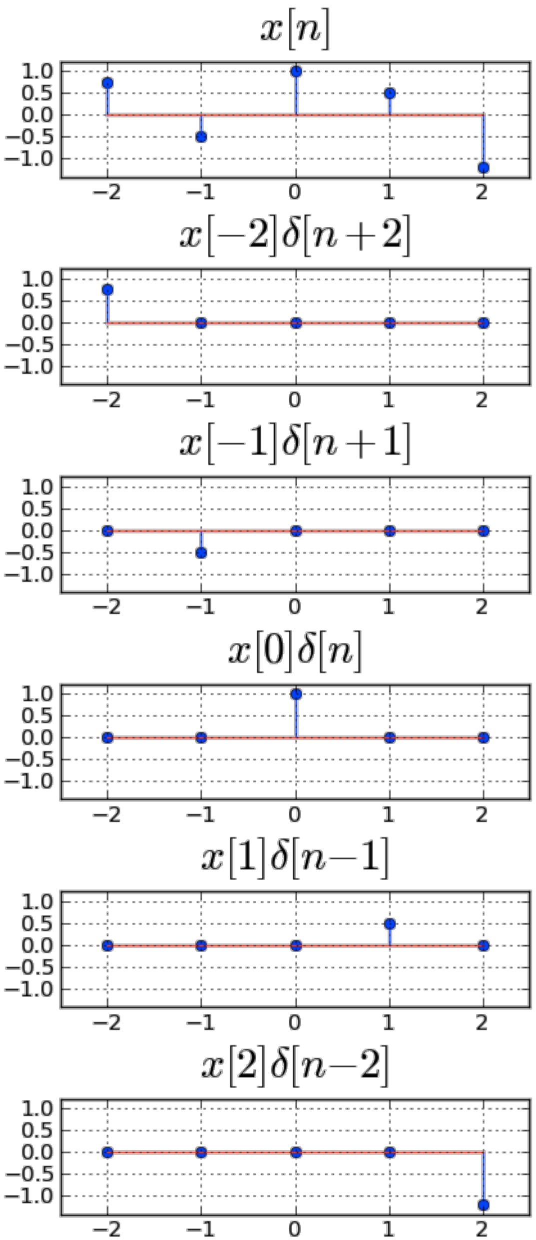
$x[n]$ at 4 samples/bit



$y[n] = x[n] * h_4[n]$



Noise margin? 0.5 − y[28]



Unit Sample Decomposition

A discrete-time signal can be decomposed into a sum of time-shifted, scaled unit samples.

Example: in the figure, x[n] is the sum of x[-2]δ[n+2] + x[-1]δ[n+1] + … + x[2]δ[n-2].

In general:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

For any particular index, only one term of this sum is non-zero

Modeling LTI Systems

If system S is both linear and time-invariant (LTI), then we can use the unit sample response to predict the response to *any* input waveform x[n]:

Sum of shifted, scaled unit samples

Sum of shifted, scaled responses

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{S} \longrightarrow y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CONVOLUTION SUM

Indeed, the unit sample response h[n] completely characterizes the LTI system S, so you often see

$$\mathrm{x[n]} \longrightarrow \boxed{\mathrm{h[.]}} \longrightarrow \mathrm{y[n]}$$

Convolution

Evaluating the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

for all n defines the output signal y in terms of the input x and unit-sample response h. Some constraints are needed to ensure this infinite sum is well behaved, i.e., doesn't "blow up" --- we'll discuss this later.

We use $*$ to denote convolution, and write y=x*h. We can then write the value of y at time n, which is given by the above sum, as $y[n] = (x*h)[n]$. We could perhaps even write $y[n] = x*h[n]$

Convolution

Evaluating the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

for all n defines the output signal y in terms of the input x and unit-sample response h. Some constraints are needed to ensure this infinite sum is well behaved, i.e., doesn't "blow up" --- we'll discuss this later.

We use $*$ to denote convolution, and write y=x∗h. We can thus write the value of y at time n, which is given by the above sum, as $y[n] = (x * h)[n]$

Instead you'll find people writing $y[n] = x[n] * h[n]$, where the poor index n is doing double or triple duty. This is **awful** notation, but a super-majority of engineering professors (including at MIT) will inflict it on their students.

Don't stand for it!

Properties of Convolution

$$(x*h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

The second equality above establishes that convolution is commutative:

$$x*h = h*x$$

Convolution is associative:

$$x*(h_1*h_2) = \left(x*h_1\right)*h_2$$

Convolution is distributive:

$$x*\left(h_1+h_2\right) = (x*h_1)+(x*h_2)$$

Series Interconnection of LTI Systems

$$x[n] \longrightarrow \boxed{h_1[.]} \xrightarrow{w[n]} \boxed{h_2[.]} \longrightarrow y[n]$$

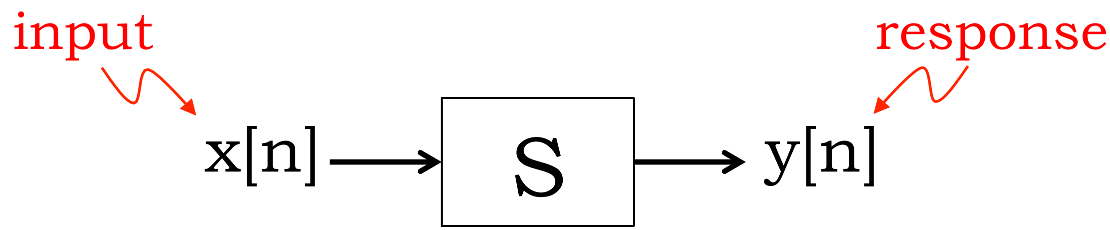
$$y = h_2 * w = h_2 * \left(h_1 * x\right) = \left(h_2 * h_1\right) * x$$

$$x[n] \longrightarrow \boxed{(h_2 * h_1)[.]} \longrightarrow y[n]$$

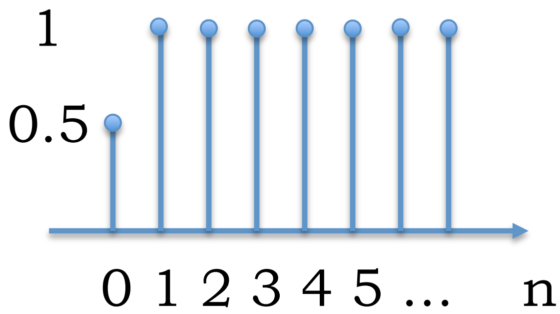
$$x[n] \longrightarrow \boxed{(h_1 * h_2)[.]} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{h_2[.]} \longrightarrow \boxed{h_1[.]} \longrightarrow y[n]$$

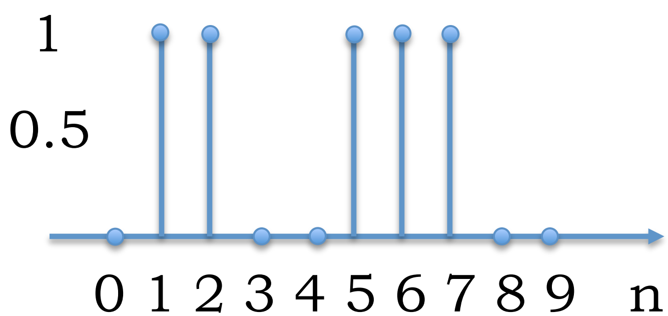
Spot Quiz



Unit step response: s[n]



x[n]



Find y[n]:

1. Write x[n] as a function of unit steps

2. Write y[n] as a function of unit step responses

3. Draw y[n]