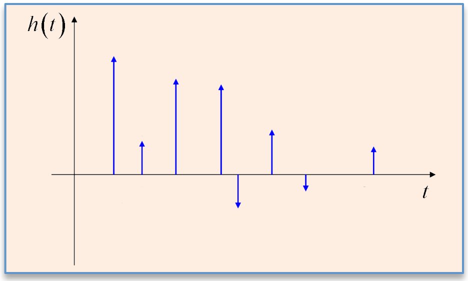
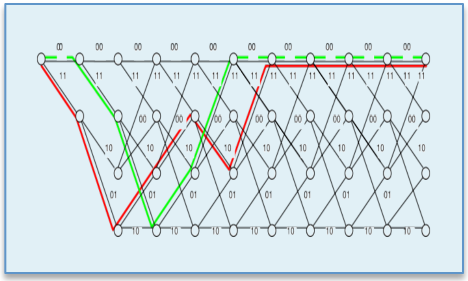
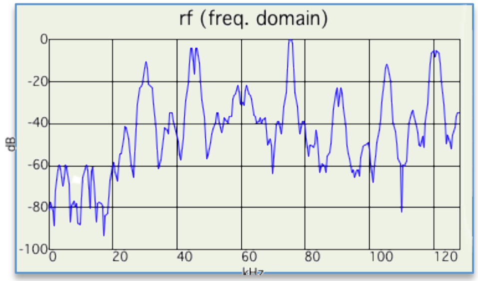
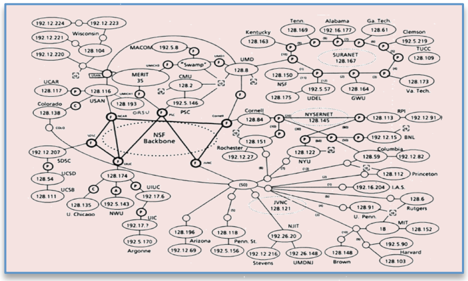








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2012
Lecture #12

- Bounded-input, bounded-output stability
- Frequency response

Bounded-Input Bounded-Output (BIBO) Stability

What ensures that the infinite sum

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

is well-behaved?

One important case: If the unit sample response is *absolutely summable,* i.e.,

$$\sum_{m=-\infty}^{\infty} |h[m]| < \infty$$

and the input is *bounded,* i.e., $|x[k]| \leq M < \infty$

Under these conditions, the convolution sum is well-behaved, and the *output* is guaranteed to be *bounded.*

The absolute summability of h[n] is necessary and sufficient for this bounded-input bounded-output (BIBO) stability.

Time now for a Frequency-Domain Story

in which
convolution
is transformed to
multiplication,
and other
good things
happen

A First Step

Do periodic inputs to an LTI system, i.e., x[n] such that

$$x[n+P] = x[n] \text{ for all } n, \text{ some fixed } P$$

(with P usually picked to be the smallest positive integer for which this is true) yield periodic outputs? If so, of period P?

Yes! --- use Flip/Slide/Dot.Product to see this easily: sliding by P gives the same picture back again, hence the same output value.

Alternate argument: Since the system is TI, using input x delayed by P should yield y delayed by P. But x delayed by P is x again, so y delayed by P must be y.

But much more is true for Sinusoidal Inputs to LTI Systems

Sinusoidal inputs, i.e.,

$$x[n] = \cos(\Omega n + \theta)$$

yield sinusoidal outputs at the same 'frequency' Ω rads/sample.

And observe that such inputs are not even periodic in general!

Periodic if and only if $2\pi/\Omega$ is rational, =P/Q for some integers P(>0), Q. The smallest such P is the period.

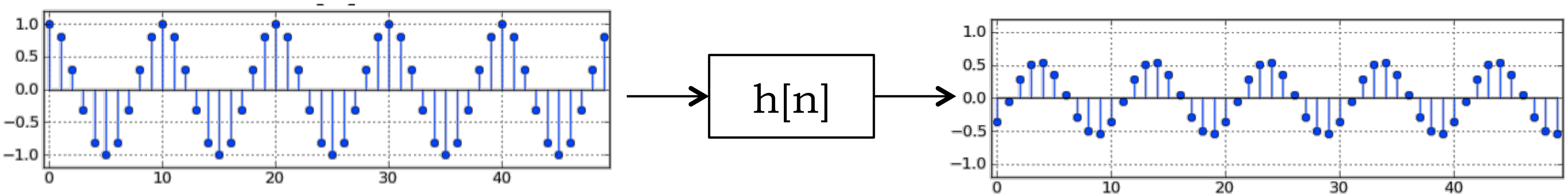
Nevertheless, we often refer to $2\pi/\Omega$ as the 'period' of this sinusoid, whether or not it is a periodic discrete-time sequence. This is the period of an underlying continuous-time signal.

Examples

$\cos(3\pi n/4)$ has frequency $3\pi/4$ rad/sample, and period 8; shifting by integer multiples of 8 yields the same sequence back again, and no integer smaller than 8 accomplishes this.

$\cos(3n/4)$ has frequency ¾ rad/sample, and is not periodic as a DT sequence because $8\pi/3$ is irrational, but we could still refer to $8\pi/3$ as its 'period', because we can think of the sequence as arising from sampling the periodic continuous-time signal $\cos(3t/4)$ at integer t.

Sinusoidal Inputs and LTI Systems



A very important property of LTI systems or channels:

> If the input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

Let's prove this to be true … but use complex exponentials instead, for clean derivations that take care of sines and cosines (or sinusoids of arbitrary phase) simultaneously.

A related simple case: real discrete-time (DT) exponential inputs also produce exponential outputs of the same type

- Suppose $x[n] = r^n$ for some real number r
- $$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]r^{n-m}$$

$$= \left(\sum_{m=-\infty}^{\infty} h[m]r^{-m} \right) r^n$$

- i.e., just a scaled version of the exponential input

Complex Exponentials

A complex exponential is a complex-valued function of a single argument – an angle measured in radians. Euler's formula shows the relation between complex exponentials and our usual trig functions:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$

In the complex plane, $e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$ is a point on the unit circle, at an angle of φ with respect to the positive real axis. cos and sin are projections on real and imaginary axes, respectively.

Increasing φ by 2π brings you back to the same point!
So any function of $e^{j\varphi}$ only needs to be studied for φ in [-π, π] .

Useful Properties of $e^{j\varphi}$

When $\varphi = 0$:

$$e^{j0} = 1$$

When $\varphi = \pm\pi$:

$$e^{j\pi} = e^{-j\pi} = -1$$

$$e^{j\pi n} = e^{-j\pi n} = (-1)^n$$

(More properties later)

Frequency Response

$$A(\cos\Omega n + j\sin\Omega n) = Ae^{j\Omega n} \longrightarrow \boxed{h[.]} \longrightarrow y[n]$$

Using the convolution sum we can compute the system's response to a complex exponential (of frequency Ω) as input:

$$y[n] = \sum_m h[m]x[n-m]$$

$$= \sum_m h[m]Ae^{j\Omega(n-m)}$$

$$= \left(\sum_m h[m]e^{-j\Omega m}\right)Ae^{j\Omega n}$$

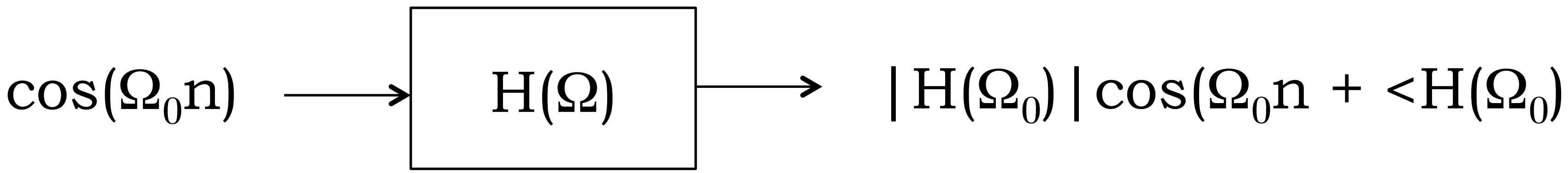
$$= H(\Omega)\cdot x[n]$$

where we've defined the *frequency response* of the system as

$$\boxed{H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}}$$

Back to Sinusoidal Inputs

Invoking the result for complex exponential inputs, it is easy to deduce what an LTI system does to sinusoidal inputs:



This is IMPORTANT

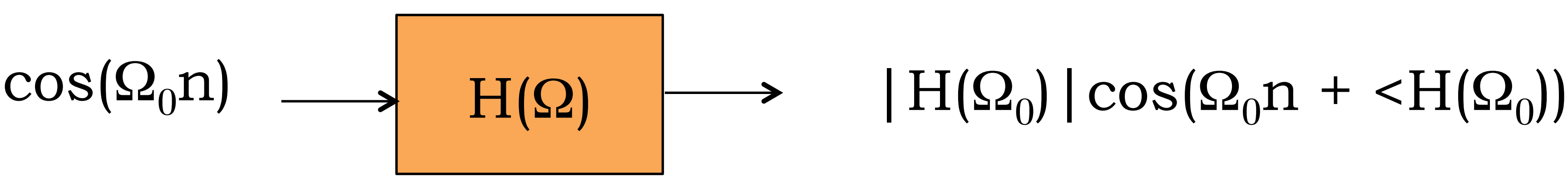
From Complex Exponentials to Sinusoids

$$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n)})/2$$

So response to this cosine input is

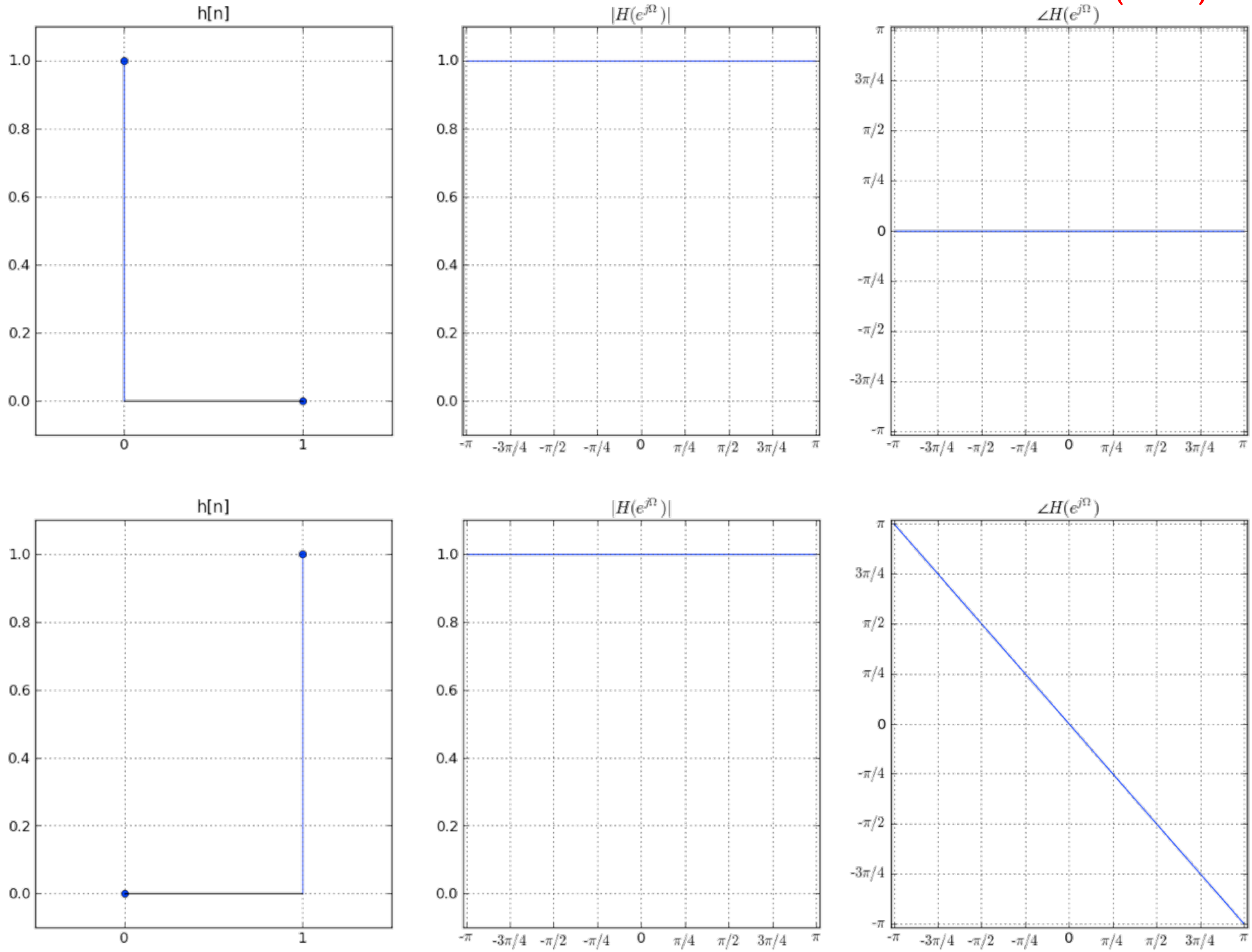
$$(H(\Omega)e^{j\Omega n} + H(-\Omega)e^{-j\Omega n)})/2 = \text{Real part of } H(\Omega)e^{j\Omega n}$$

$$= \text{Real part of } |H(\Omega)|e^{j(\Omega n + <H(\Omega))}$$



Example h[n] and H(Ω)

Sometimes written as $H(e^{j\Omega n})$



Frequency Response of "Moving Average" Filters