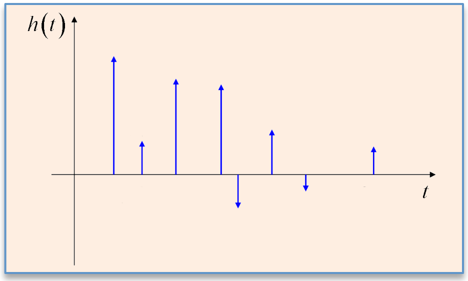
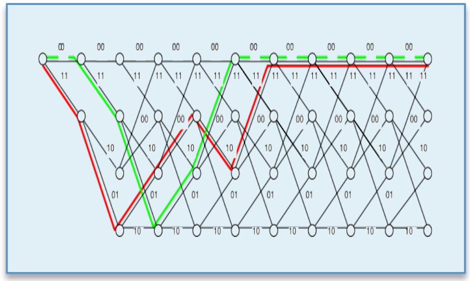
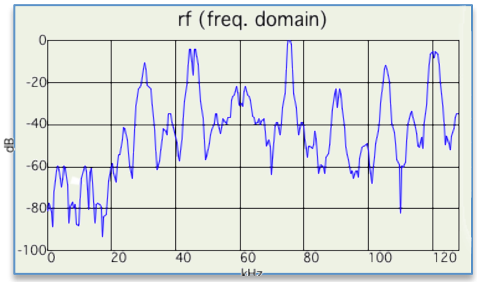
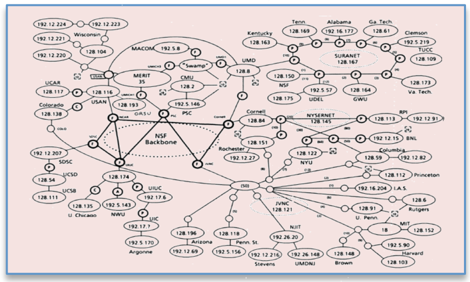




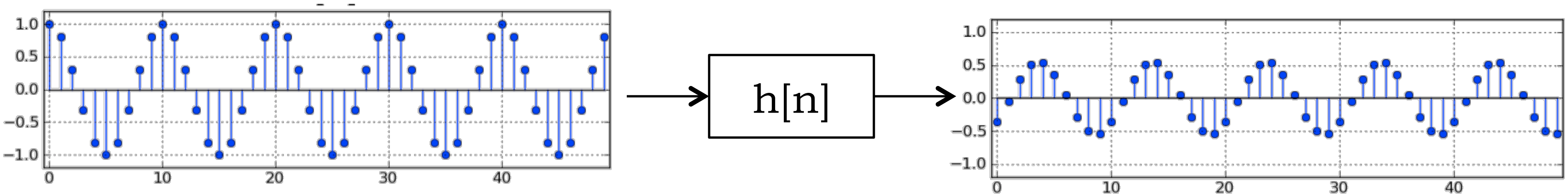
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

# 6.02 Fall 2012
# Lecture #13

- Frequency response
- Filters
- Spectral content

# Sinusoidal Inputs and LTI Systems




A very important property of LTI systems or channels:

If the input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a *sinusoid at the same frequency*, although the amplitude and phase may be altered. The change in amplitude and phase will, in general, depend on the frequency of the input.

# Complex Exponentials as "Eigenfunctions" of LTI System

$x[n]=e^{j\Omega n}$ → h[.] → $y[n]=H(\Omega)e^{j\Omega n}$

Eigenfunction: Undergoes only scaling -- by the **frequency response** $H(\Omega)$ in this case:

$$H(\Omega) \equiv \sum_m h[m]e^{-j\Omega m}$$
$$= \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)$$

This is an infinite sum in general, but is well behaved if h[.] is absolutely summable, i.e., if the system is stable.

We also call $H(\Omega)$ the **discrete-time Fourier transform (DTFT)** of the time-domain function h[.] --- more on the DTFT later.

# From Complex Exponentials to Sinusoids

$$\cos(\Omega n) = (e^{j\Omega n} + e^{-j\Omega n})/2$$

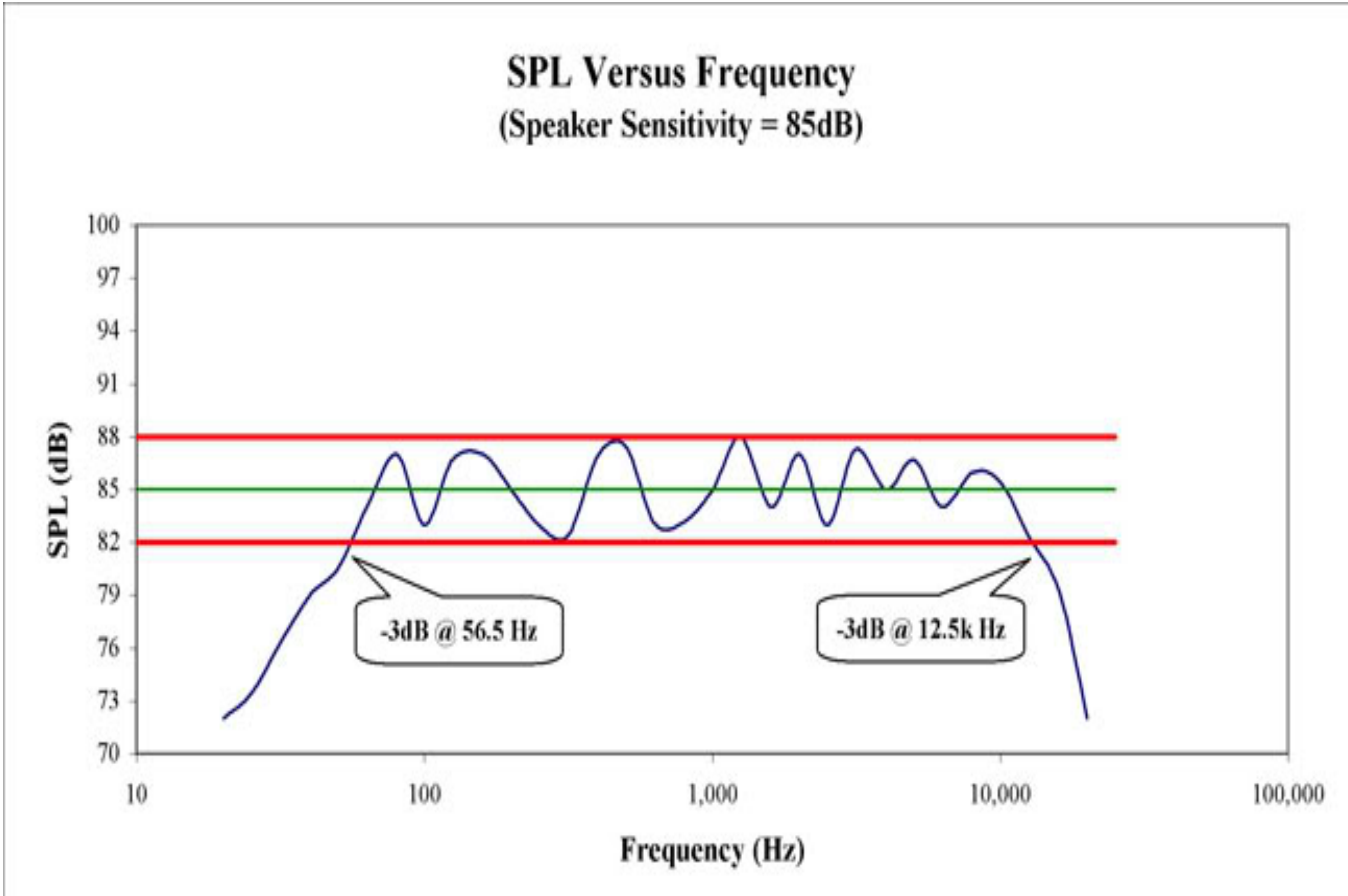
So response to a cosine input is:

$$A\cos(\Omega_0 n + \varnothing_0) \longrightarrow \boxed{H(\Omega)} \longrightarrow |\mathbf{H}(\Omega_0)|\, A\cos(\Omega_0 n + \varnothing_0 + \angle\mathbf{H}(\Omega_0))$$

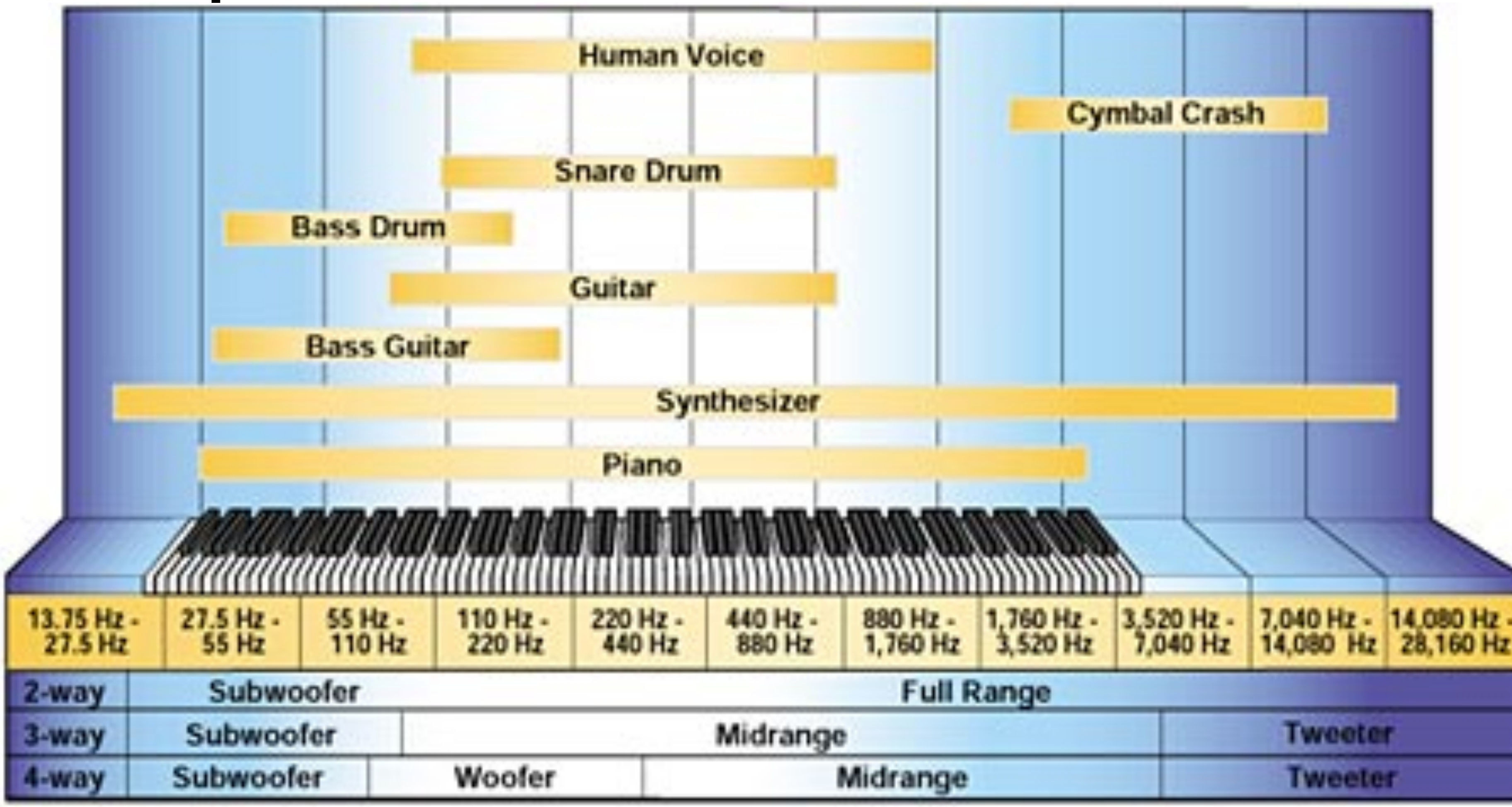
(Recall that we only need vary Ω in the interval [–π,π].)

This gives rise to an easy experimental way to determine the frequency response of an LTI system.

# Loudspeaker Frequency Response



# Spectral Content of Various Sounds




http://forum.blu-ray.com/showthread.php?t=150915

# Connection between CT and DT

The continuous-time (CT) signal

$$x(t) = \cos(\omega t) = \cos(2\pi f t)$$

sampled every T seconds, i.e., at a sampling frequency of $f_s = 1/T$, gives rise to the discrete-time (DT) signal

$$x[n] = x(nT) = \cos(\omega n T) = \cos(\Omega n)$$

So $\Omega = \omega T$

and $\Omega = \pi$ corresponds to $\omega = \pi/T$ or $f = 1/(2T) = f_s/2$

# Properties of $H(\Omega)$

Repeats periodically on the frequency ($\Omega$) axis, with period $2\pi$, because the input $e^{j\Omega n}$ is the same for $\Omega$ that differ by integer multiples of $2\pi$. So only the interval $\Omega$ in $[-\pi,\pi]$ is of interest!

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$\Omega = 0$, i.e., $e^{j\Omega n} = 1$, corresponds to a constant (or "DC", which stands for "direct current", but now just means constant) input, so $H(0)$ is the "DC gain" of the system, i.e., gain for constant inputs.

$H(0) = \sum h[m]$ --- show this from the definition!

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$\Omega = \pi$ or $-\pi$, i.e., $Ae^{j\Omega n}=(-1)^n A$, corresponds to the highest-frequency variation possible for a discrete-time signal, so $H(\pi)=H(-\pi)$ is the high-frequency gain of the system.

$H(\pi) = \sum (-1)^m h[m]$ --- show from definition!

# Symmetry Properties of H(Ω)

$$H(\Omega) \equiv \sum_m h[m] e^{-j\Omega m}$$
$$= \sum_m h[m]\cos(\Omega m) - j\sum_m h[m]\sin(\Omega m)$$
$$= C(\Omega) - jS(\Omega)$$

For real h[n]:

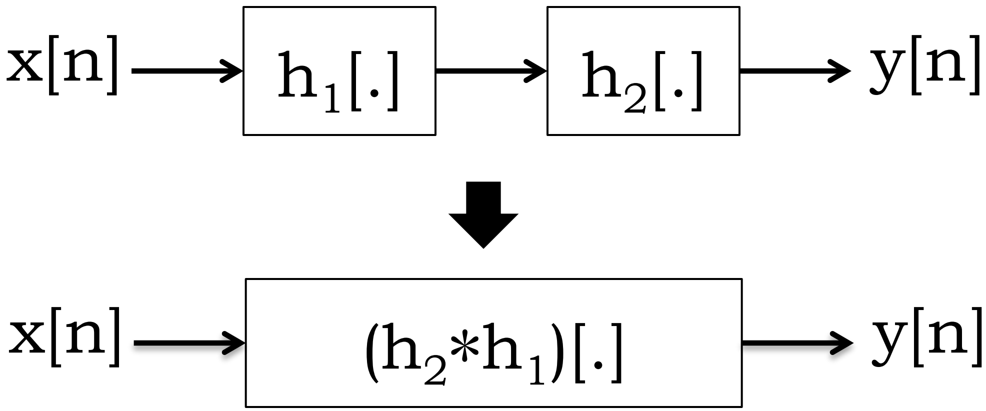
**Real part** of H(Ω) & **magnitude** are EVEN functions of Ω.

**Imaginary part** & **phase** are ODD functions of Ω.

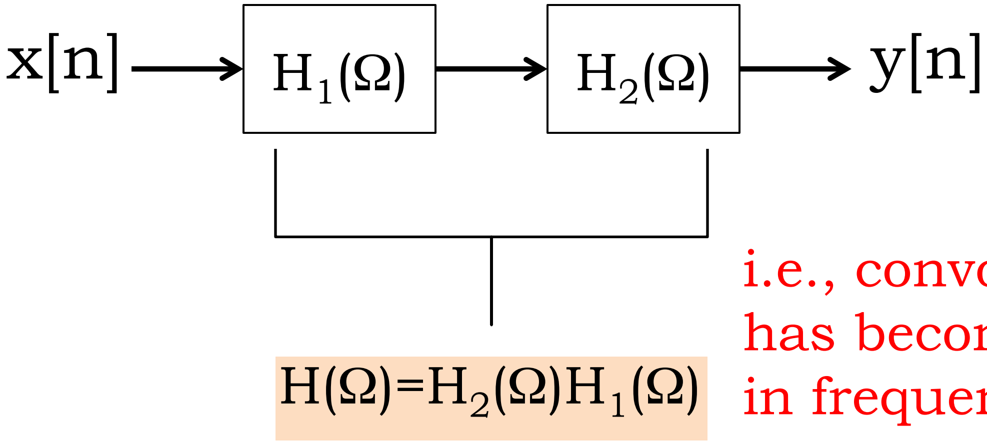
For real and *even* h[n] = h[–n], H(Ω) is purely real.

For real and *odd* h[n] = –h[–n], H(Ω) is purely imaginary.

# Convolution in Time <---> Multiplication in Frequency

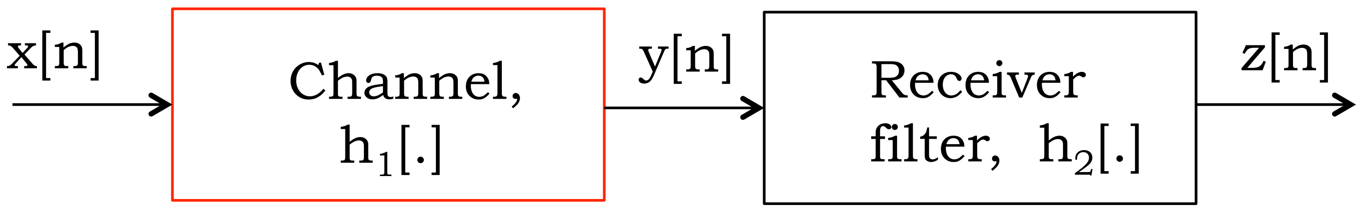



In the frequency domain (i.e., thinking about input-to-output frequency response):




i.e., convolution in time has become multiplication in frequency!

# Example: "Deconvolving" Output of Channel with Echo




Suppose channel is LTI with

$$h_1[n]=\delta[n]+0.8\delta[n-1]$$

$$H_1(\Omega) = ?? = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

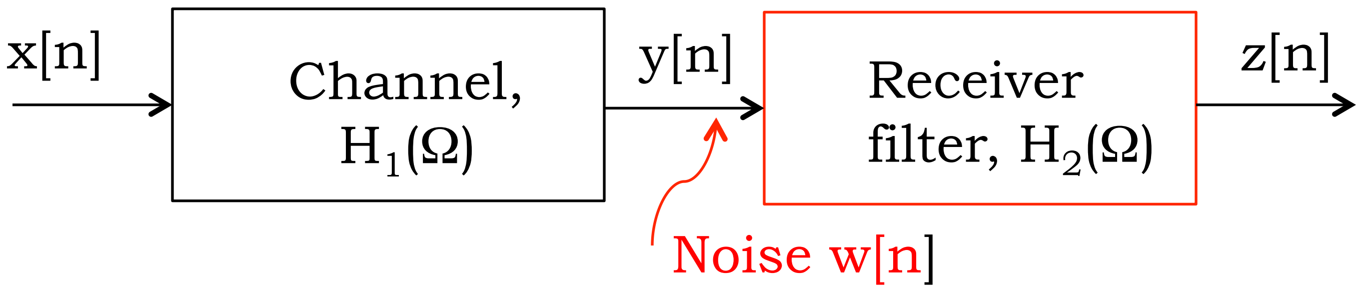
$$= 1+ 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$$ *EVEN function of* $\Omega$;

$$<H_1(\Omega) = \arctan\,[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$$ *ODD* .

# A Frequency-Domain view of Deconvolution




Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

$\Rightarrow$ $H_2(\Omega)=1/H_1(\Omega)$ "Inverse filter"

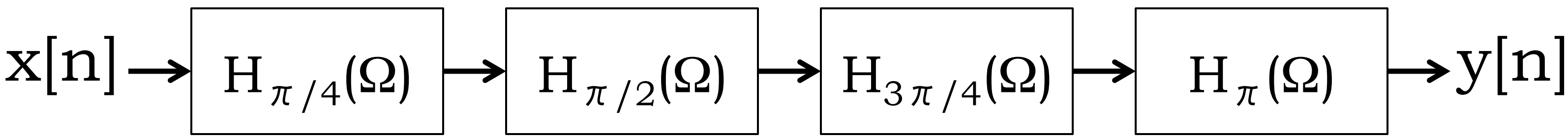
$= (1/|H_1(\Omega)|).\ \exp\{-j<H_1(\Omega)\}$

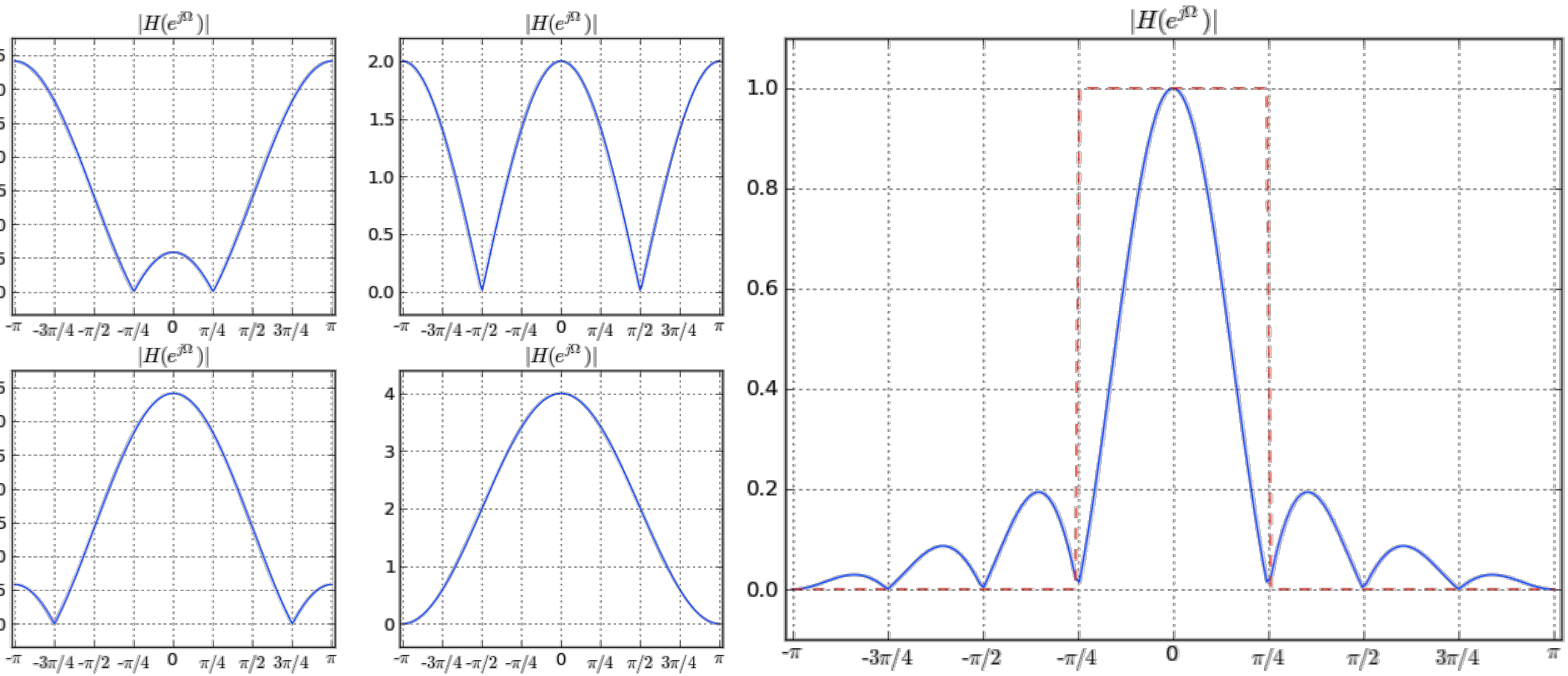
Inverse filter at receiver does very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

# A 10-cent Low-pass Filter

Suppose we wanted a low-pass filter with a cutoff frequency of $\pi/4$?

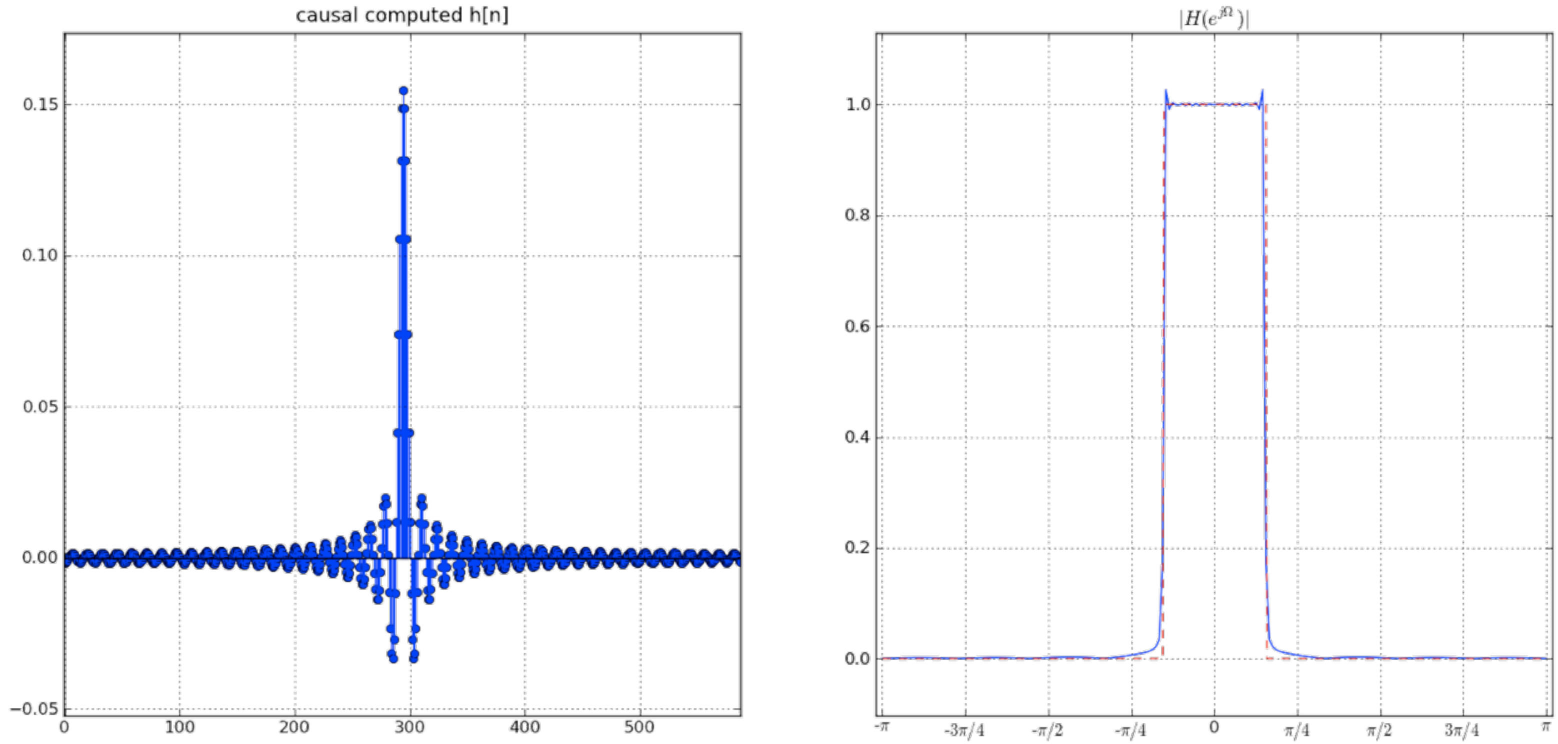


# To Get a Filter Section with a Specified Zero-Pair in $H(\Omega)$

- Let $h[0] = h[2] = 1$,    $h[1] = \mu$,    all other $h[n] = 0$
- Then $H(\Omega) = 1 + \mu e^{-j\Omega} + e^{-j2\Omega} = e^{-j\Omega}(\mu + 2\cos(\Omega))$
- So $|H(\Omega)| = |\mu + 2\cos(\Omega)|$, with zeros at $\pm \arccos(-\mu/2)$

# The $4.99 version of a Low-pass Filter, h[n] and H(Ω)



# Determining h[n] from H(Ω)

$$H(\Omega) = \sum_{m} h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) $2\pi$ interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega = \int_{<2\pi>} \sum_{m} h[m] e^{-j\Omega(m-n)} d\Omega$$

$$= 2\pi \cdot h[n]$$

$\Rightarrow$

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

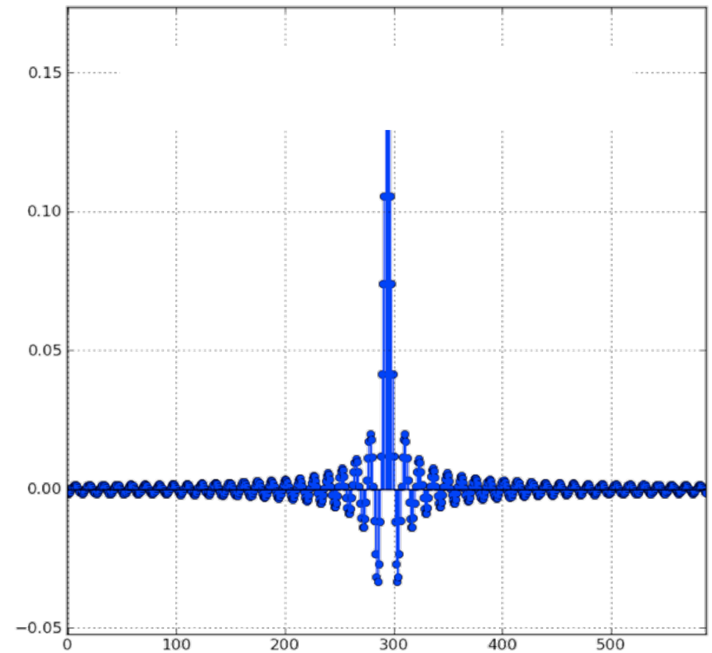
# Design **ideal lowpass filter** with cutoff frequency $\Omega_C$ and $H(\Omega)$=1 in passband

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_C n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_C / \pi \qquad , \quad n = 0$$




DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n))

# Exercise: Frequency response of h[n-D]

Given an LTI system with unit sample response h[n] and associated frequency response $H(\Omega)$,

determine the frequency response $H_D(\Omega)$ of an LTI system whose unit sample response is

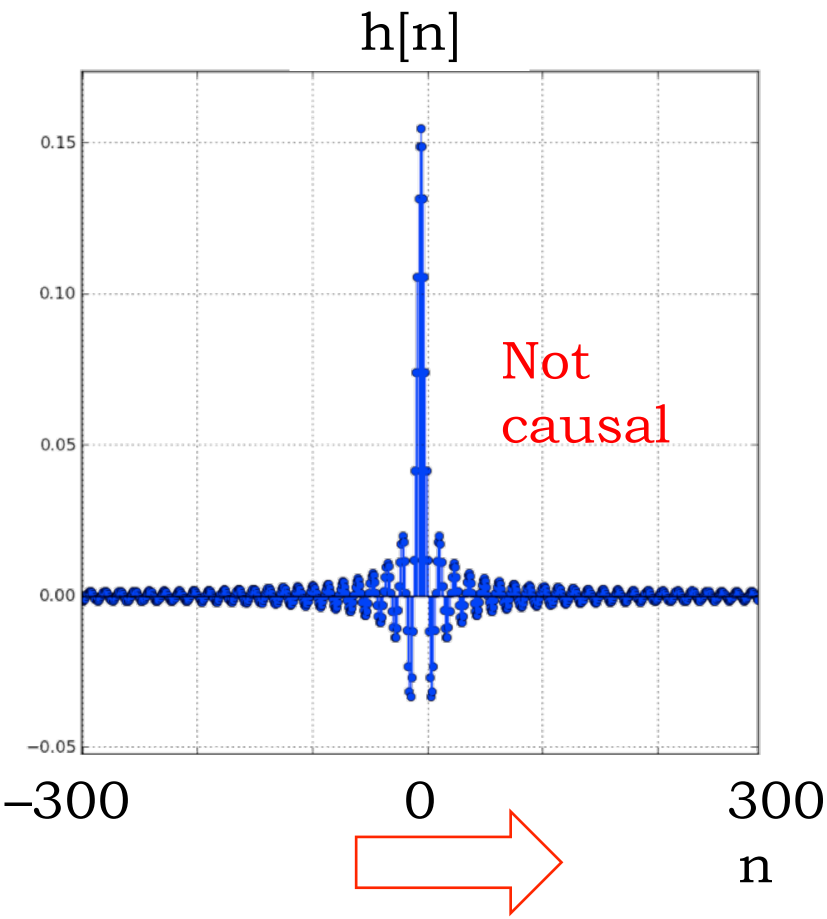
$$h_D[n] = h[n-D].$$

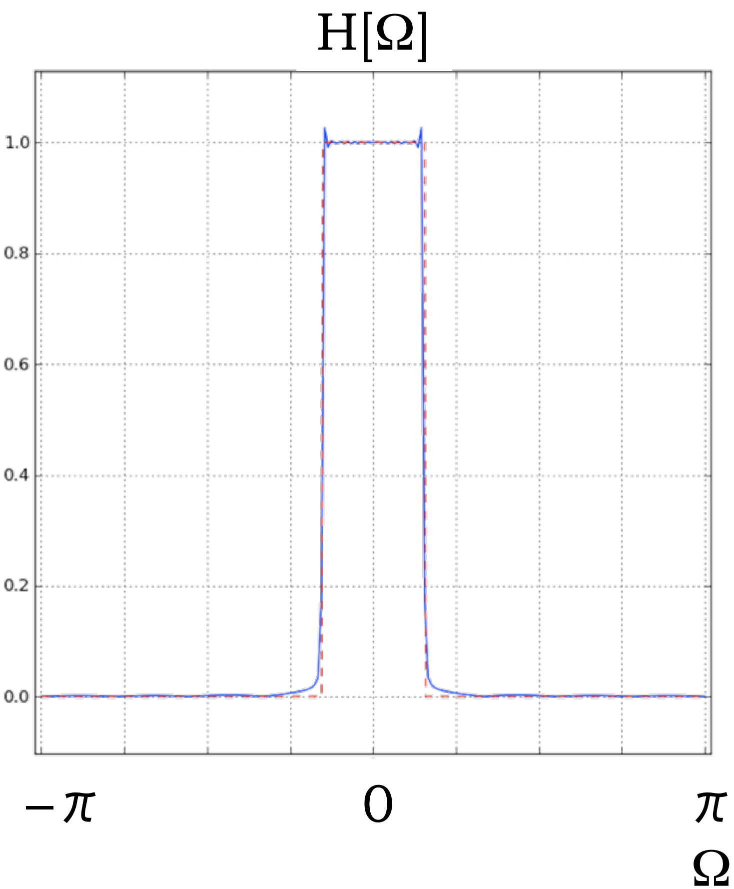
Answer: $H_D(\Omega) = \exp\{-j\Omega D\}.H(\Omega)$

so : $|H_D(\Omega)| = |H(\Omega)|$ , i.e., magnitude unchanged

$\angle H_D(\Omega) = -\Omega D + \angle H(\Omega)$ , i.e., linear phase term added

# e.g.: Approximating an ideal lowpass filter

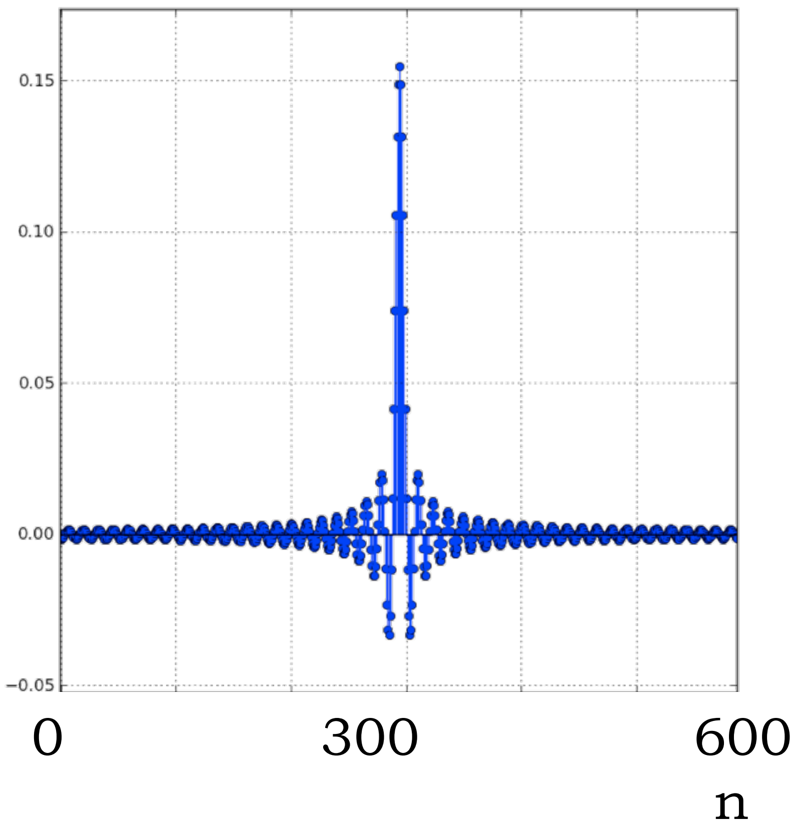


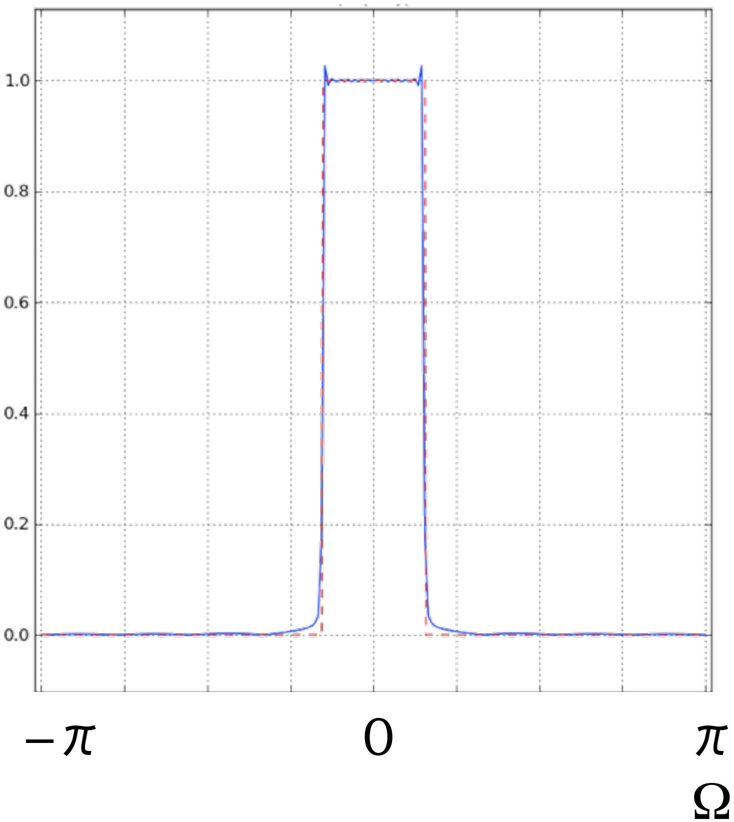

Idea: shift h[n] right to get causal LTI system.
Will the result still be a lowpass filter?

# Causal approximation to ideal lowpass filter

$h_C[n] = h[n-300]$




$|H_C[\Omega]|$




Determine $\angle H_C(\Omega)$

# DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad H(\Omega) = \sum_n h[n] e^{-j\Omega n}$$

Any contiguous interval of length $2\pi$

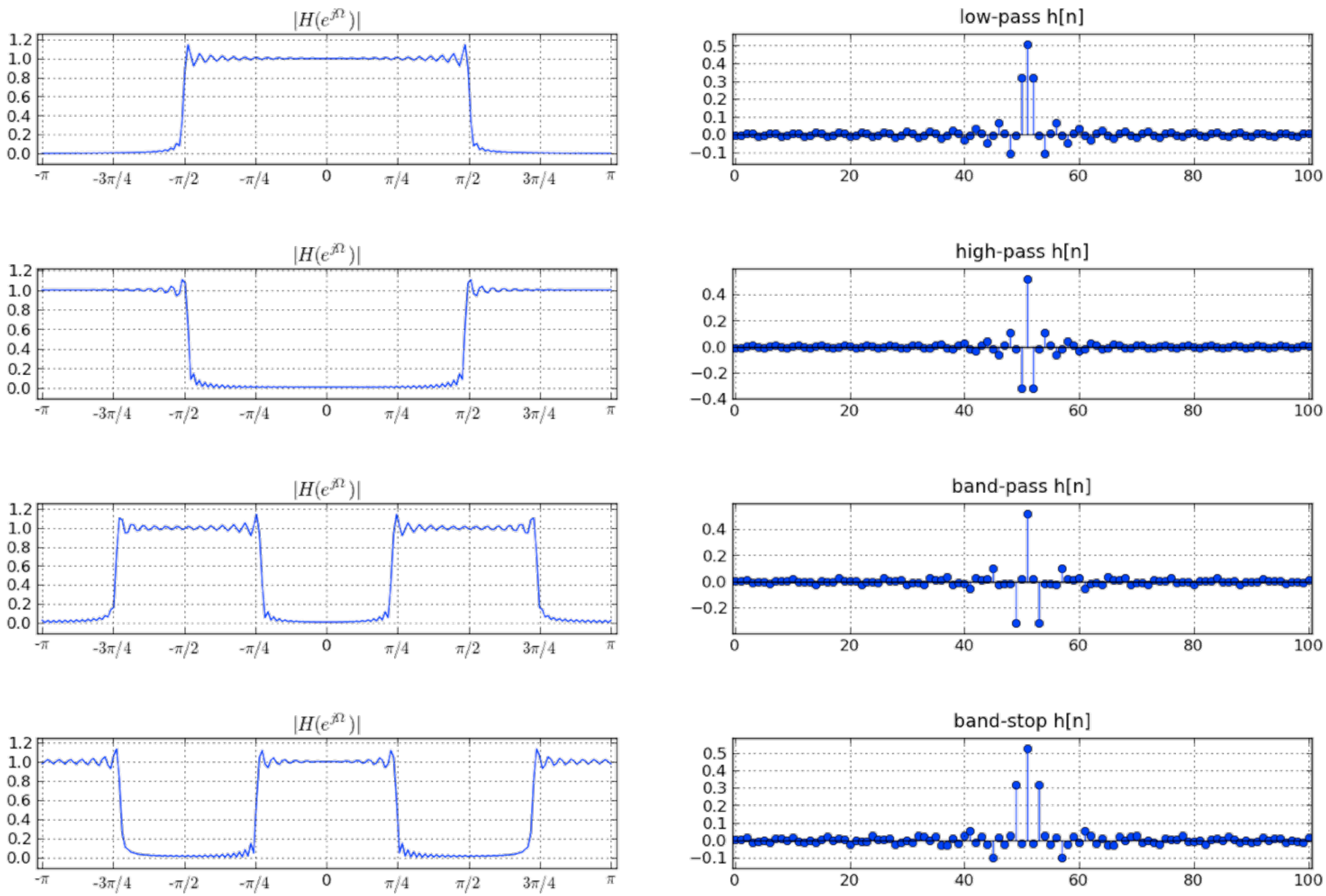
then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all $\Omega$ in $[-\pi, \pi]$.

$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

# Useful Filters



# Frequency Response of Channels