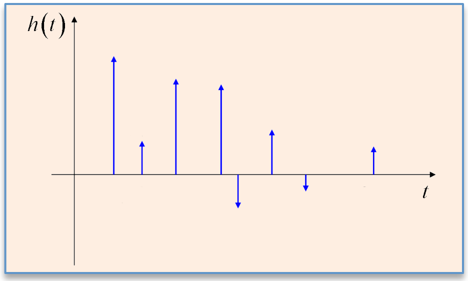
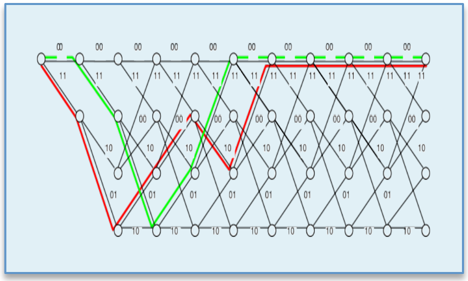
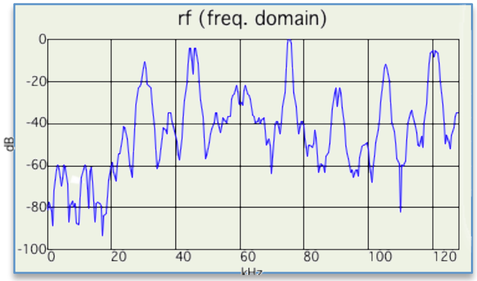
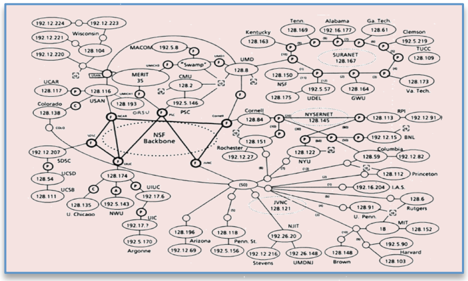








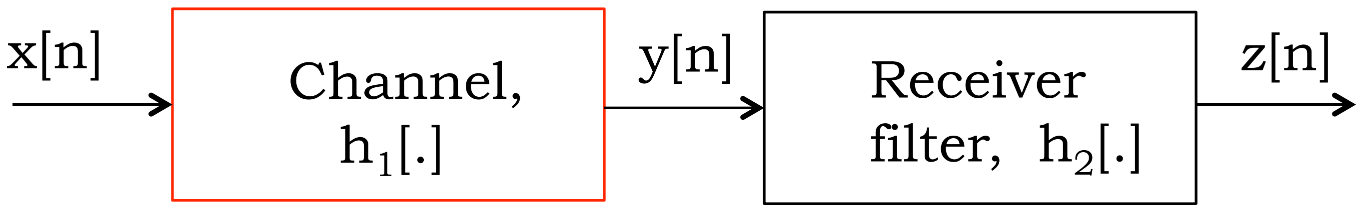
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2012
Lecture #14

- Spectral content via the DTFT

Demo: "Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

$$h_1[n]=\delta[n]+0.8\delta[n-1]$$

$$H_1(\Omega) = ?? = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

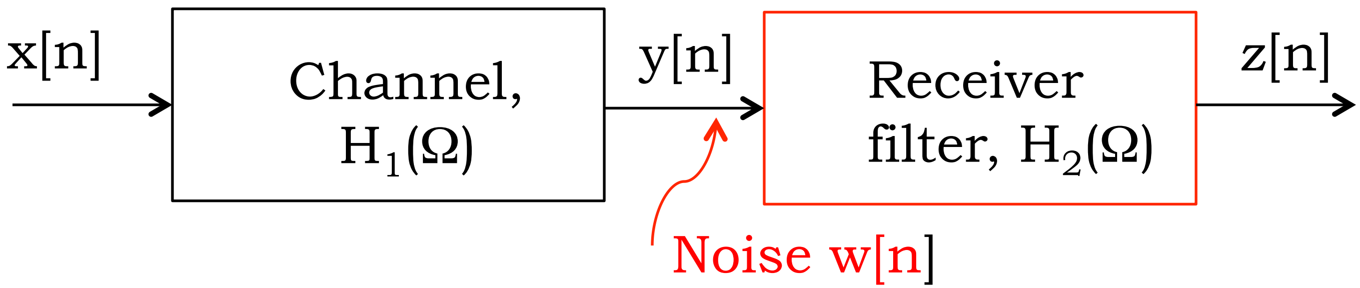
$$= 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$$ *EVEN function of* Ω;

$$<H_1(\Omega) = \arctan[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]$$ *ODD*.

A Frequency-Domain view of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get $z[n]=x[n]$?

⇒ $H_2(\Omega) = 1/H_1(\Omega)$ "Inverse filter"

$$= (1/|H_1(\Omega)|).\ \exp\{-j\angle H_1(\Omega)\}$$

Inverse filter at receiver does very badly in the presence of noise that adds to $y[n]$:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad H(\Omega) = \sum_m h[m] e^{-j\Omega m}$$

Any contiguous interval of length 2π

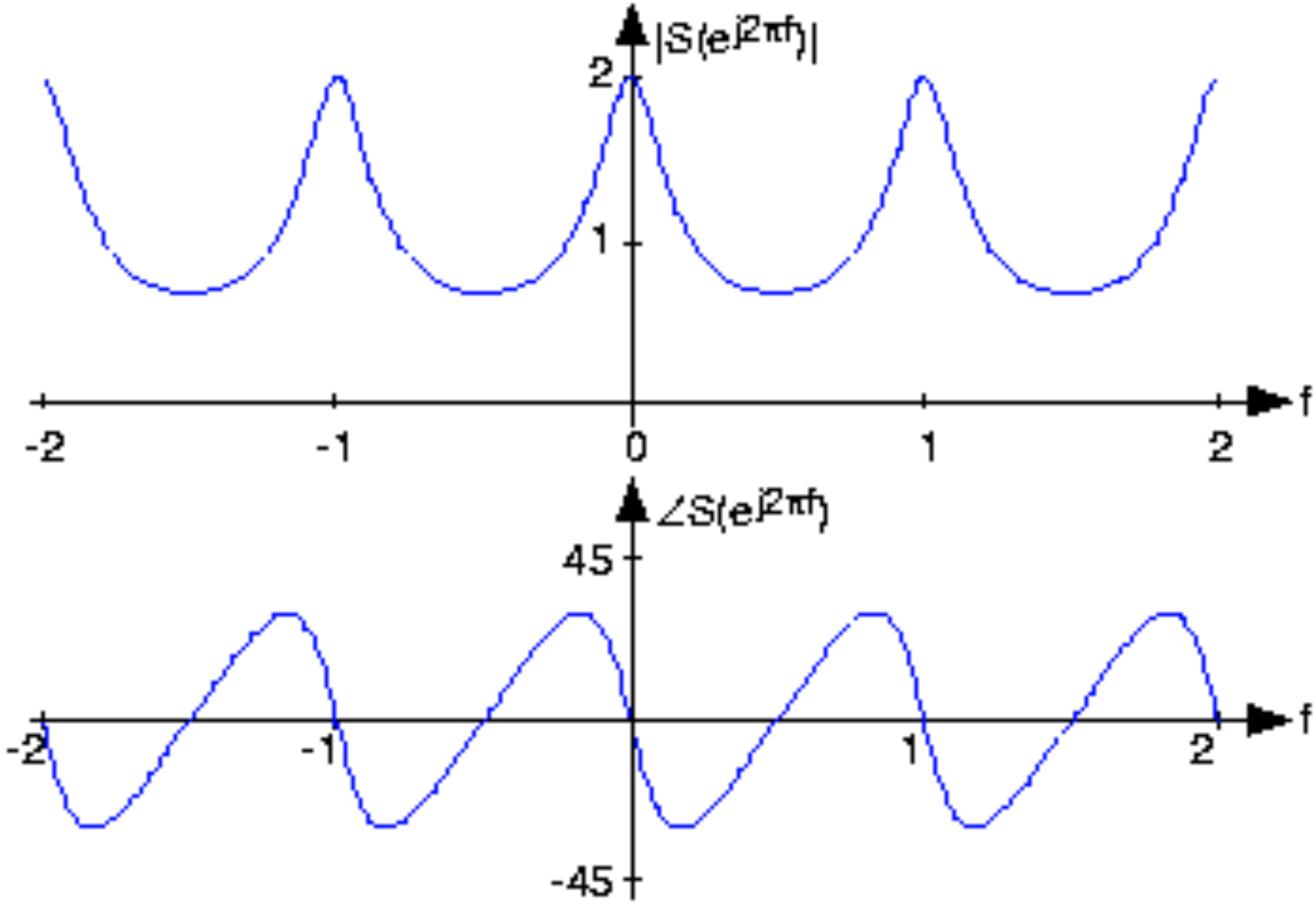
then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_m x[m] e^{-j\Omega m}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$.

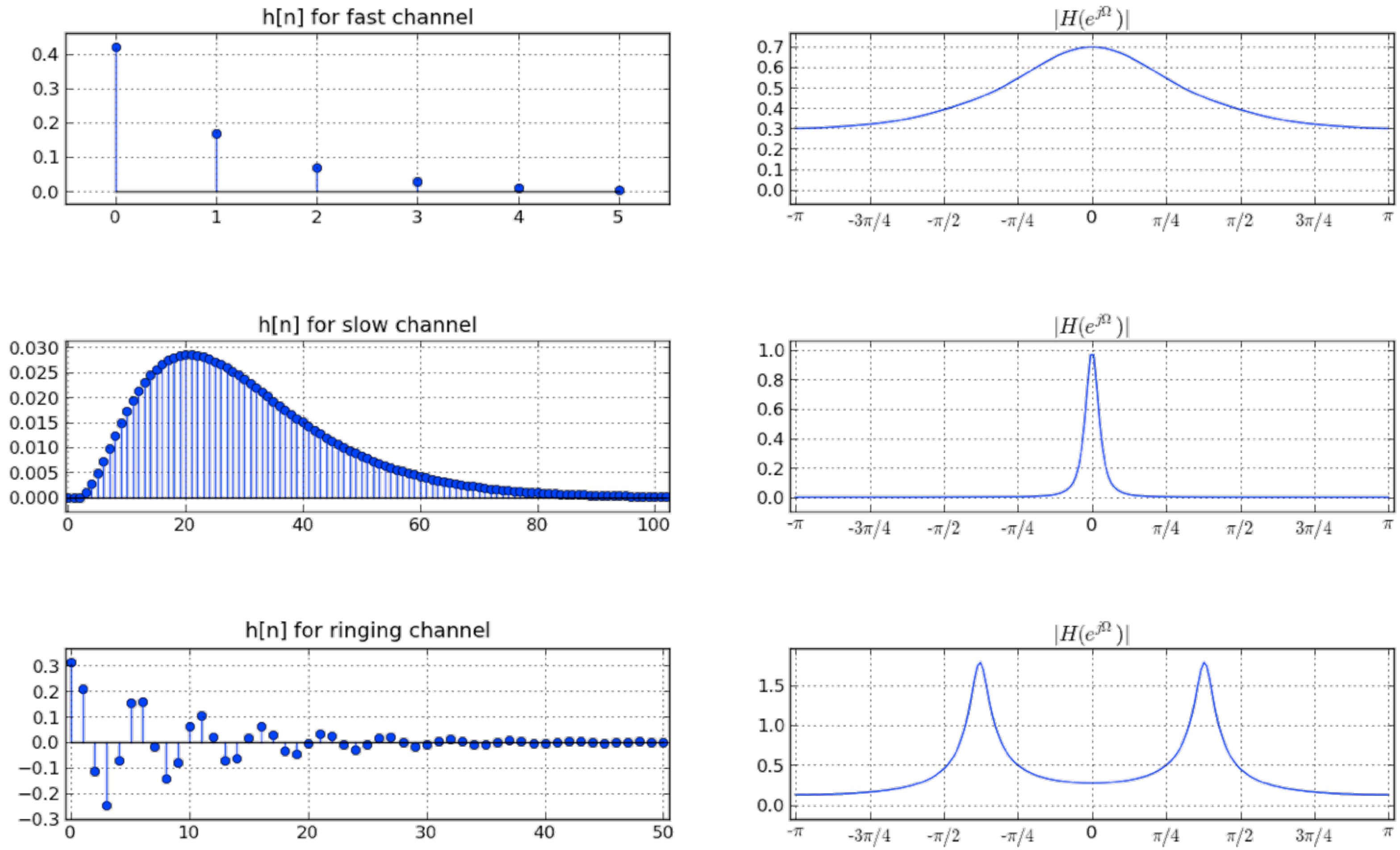
$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

The spectrum of the exponential signal $(0.5)^n u[n]$ is shown over the frequency range $\Omega = 2\pi f$ in $[-4\pi, 4\pi]$, The angle has units of degrees.



http://cnx.org/content/m0524/latest/

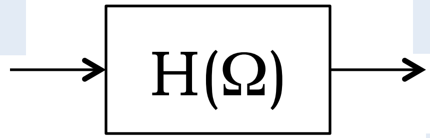
x[n] and X(Ω)



h[n] for fast channel
|H(e^{jΩ})|
h[n] for slow channel
|H(e^{jΩ})|
h[n] for ringing channel
|H(e^{jΩ})|
-π
-3π/4
-π/2
-π/4
0
π/4
π/2
3π/4
π

Input/Output Behavior of LTI System in Frequency Domain

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$



$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Compare with y[n]=(h*x)[n]

Again, convolution in time has mapped to multiplication in frequency

Magnitude and Angle

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$\Updownarrow$$

$$|Y(\Omega)| = |H(\Omega)|.|X(\Omega)|$$

and

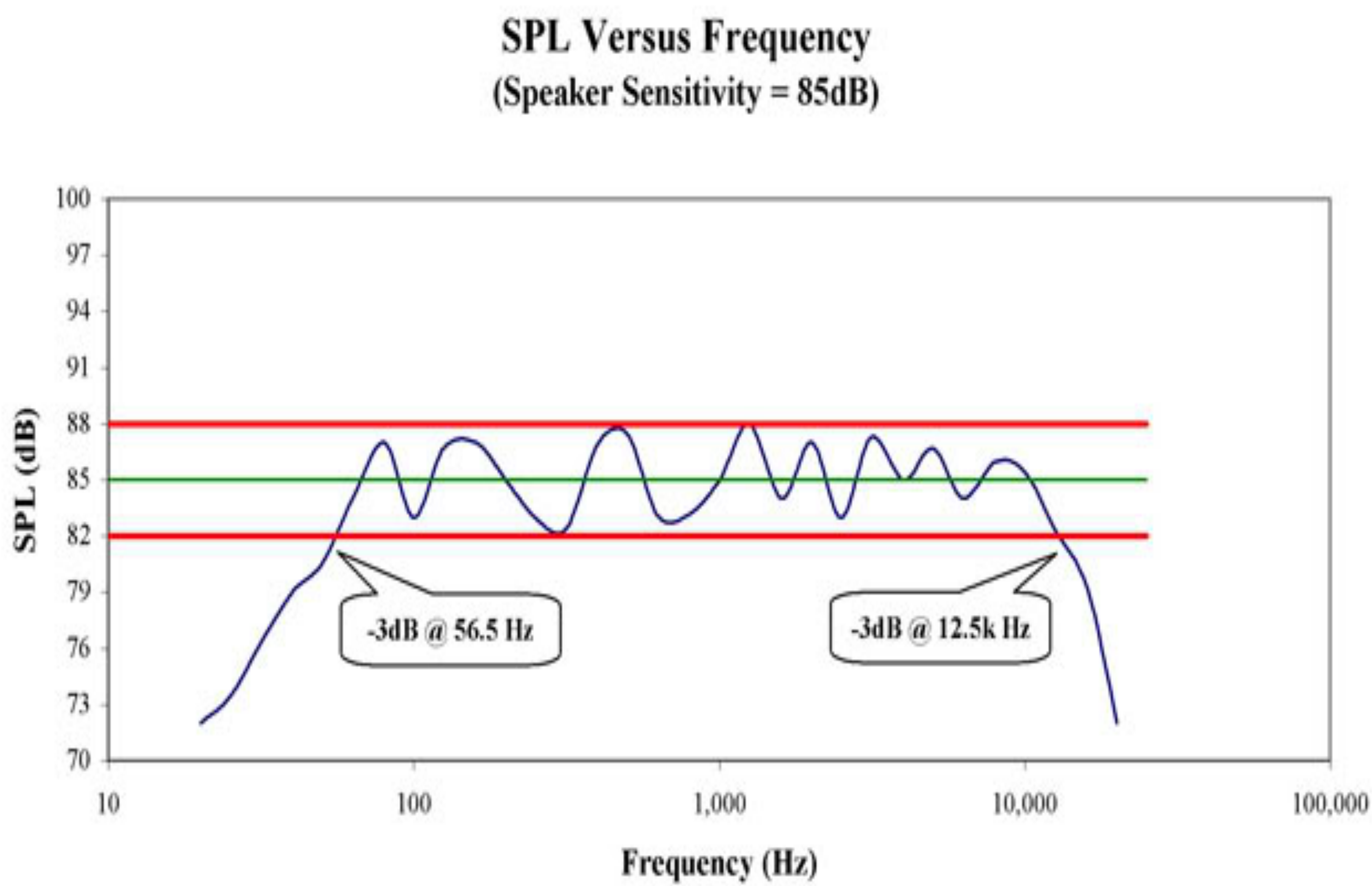
$$< Y(\Omega) = < H(\Omega) + < X(\Omega)$$

Core of the Story

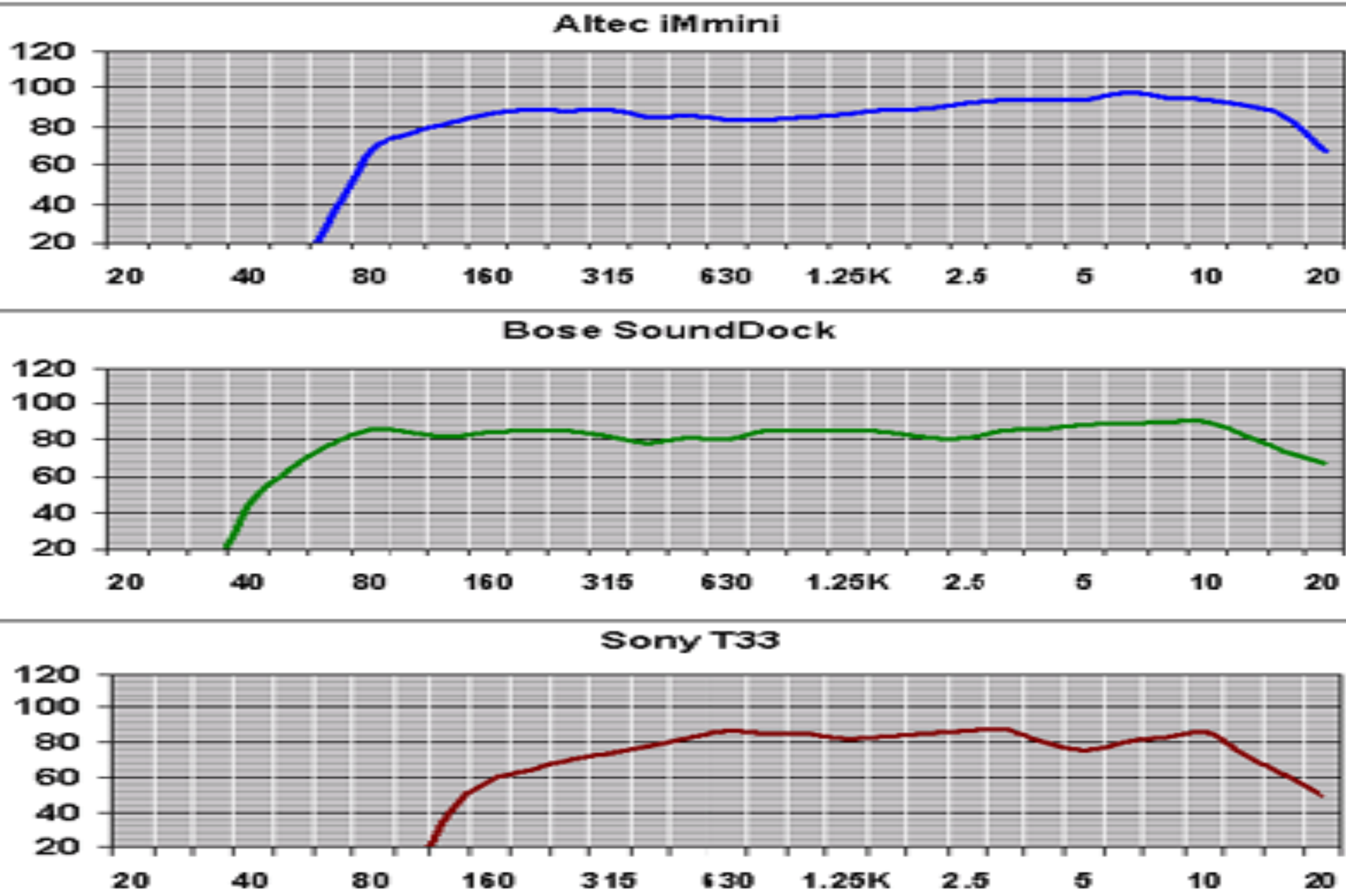
1. A huge class of DT and CT signals
can be written --- using **Fourier** transforms --- as a
weighted sums of sinusoids (ranging from very slow to very fast)
or (equivalently, but more compactly) complex exponentials.
The sums can be discrete $\sum$ or continuous $\int$ (or both).

2. **LTI** systems act very simply on sums of sinusoids:
superposition of responses to each sinusoid, with the
frequency response determining the frequency-dependent
scaling of magnitude, shifting in phase.

Loudspeaker Bandpass Frequency Response

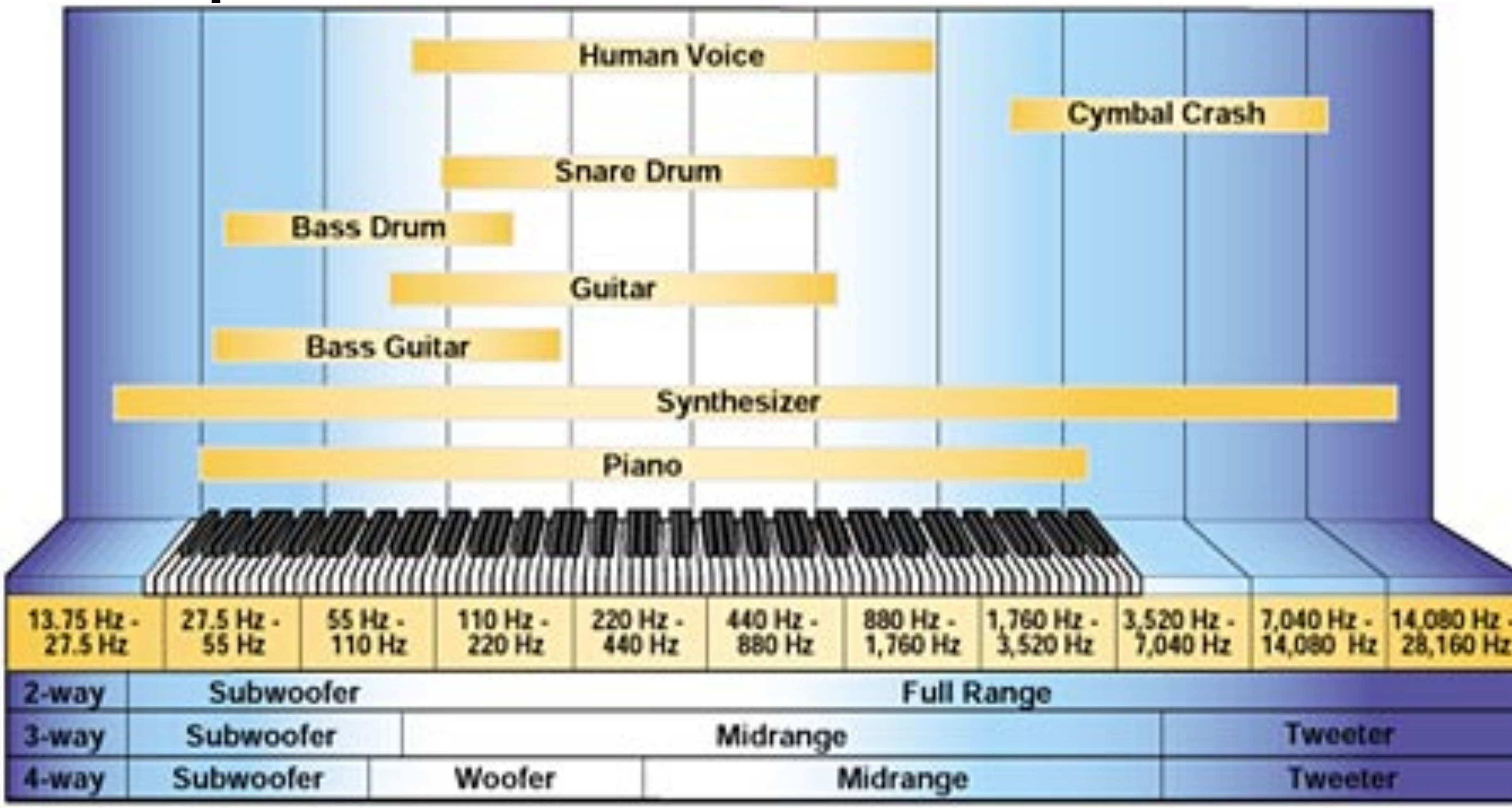


http://forum.blu-ray.com/showthread.php?t=150915



http://www.pcmag.com/article2/0,2817,1769243,00.asp

Spectral Content of Various Sounds



http://forum.blu-ray.com/showthread.php?t=150915

Connection between CT and DT

The continuous-time (CT) signal

$$x(t) = \cos(\omega t) = \cos(2\pi f t)$$

sampled every T seconds, i.e., at a sampling frequency of $f_s = 1/T$, gives rise to the discrete-time (DT) signal

$$x[n] = x(nT) = \cos(\omega n T) = \cos(\Omega n)$$

So $\Omega = \omega T$

and $\Omega = \pi$ corresponds to $\omega = \pi/T$ or $f = 1/(2T) = f_s/2$

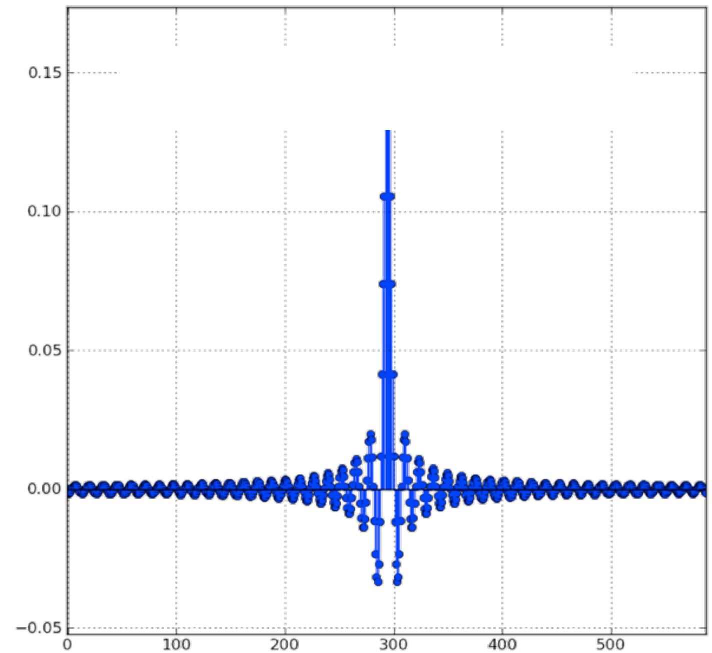
Signal x[n] that has its frequency content uniformly distributed in [$-\Omega_c$, Ω_c]

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

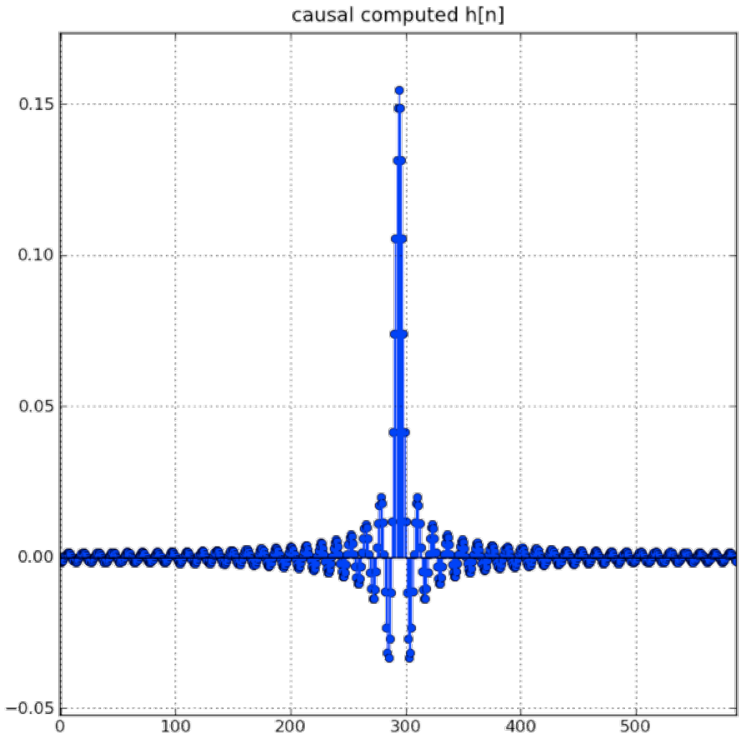
$$= \frac{\sin(\Omega_C n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_C / \pi \quad , \quad n = 0$$

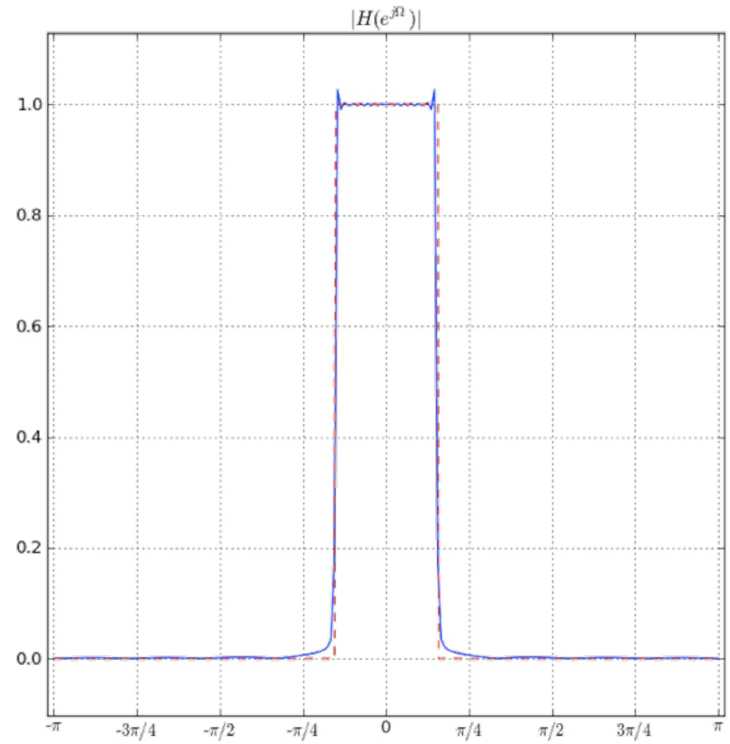


DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n)

x[n] and X(Ω)

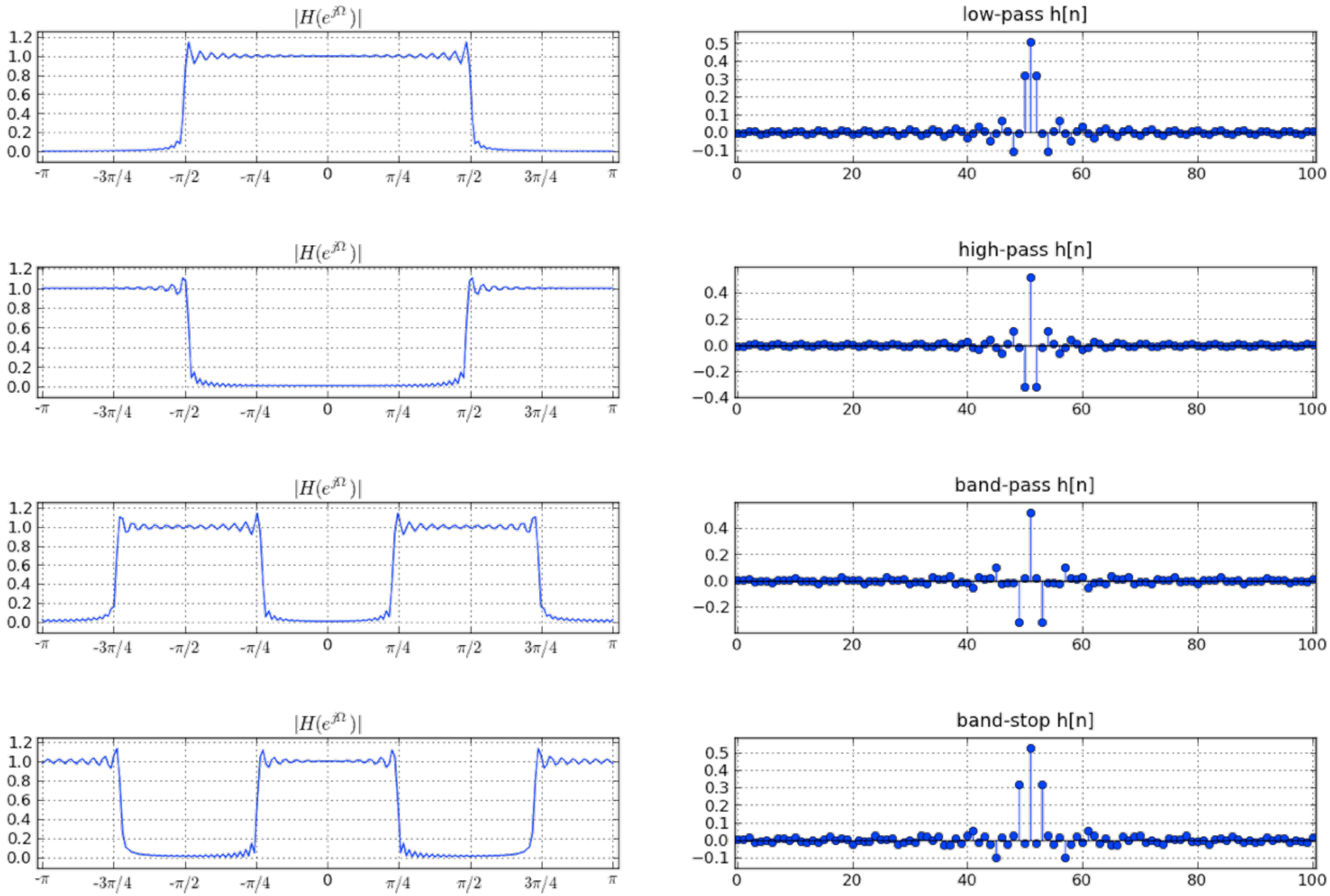


causal computed h[n]
0.15
0.10
0.05
0.00
−0.05
0
100
200
300
400
500



$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
0
$\pi/4$
$\pi/2$
$3\pi/4$
π

X(Ω) and x[n]



$|H(e^{j\Omega})|$
low-pass h[n]
$|H(e^{j\Omega})|$
high-pass h[n]
$|H(e^{j\Omega})|$
band-pass h[n]
$|H(e^{j\Omega})|$
band-stop h[n]
$-\pi$ $-3\pi/4$ $-\pi/2$ $-\pi/4$ 0 $\pi/4$ $\pi/2$ $3\pi/4$ π
0 20 40 60 80 100

Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

$$X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_k m}, \qquad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n}$$

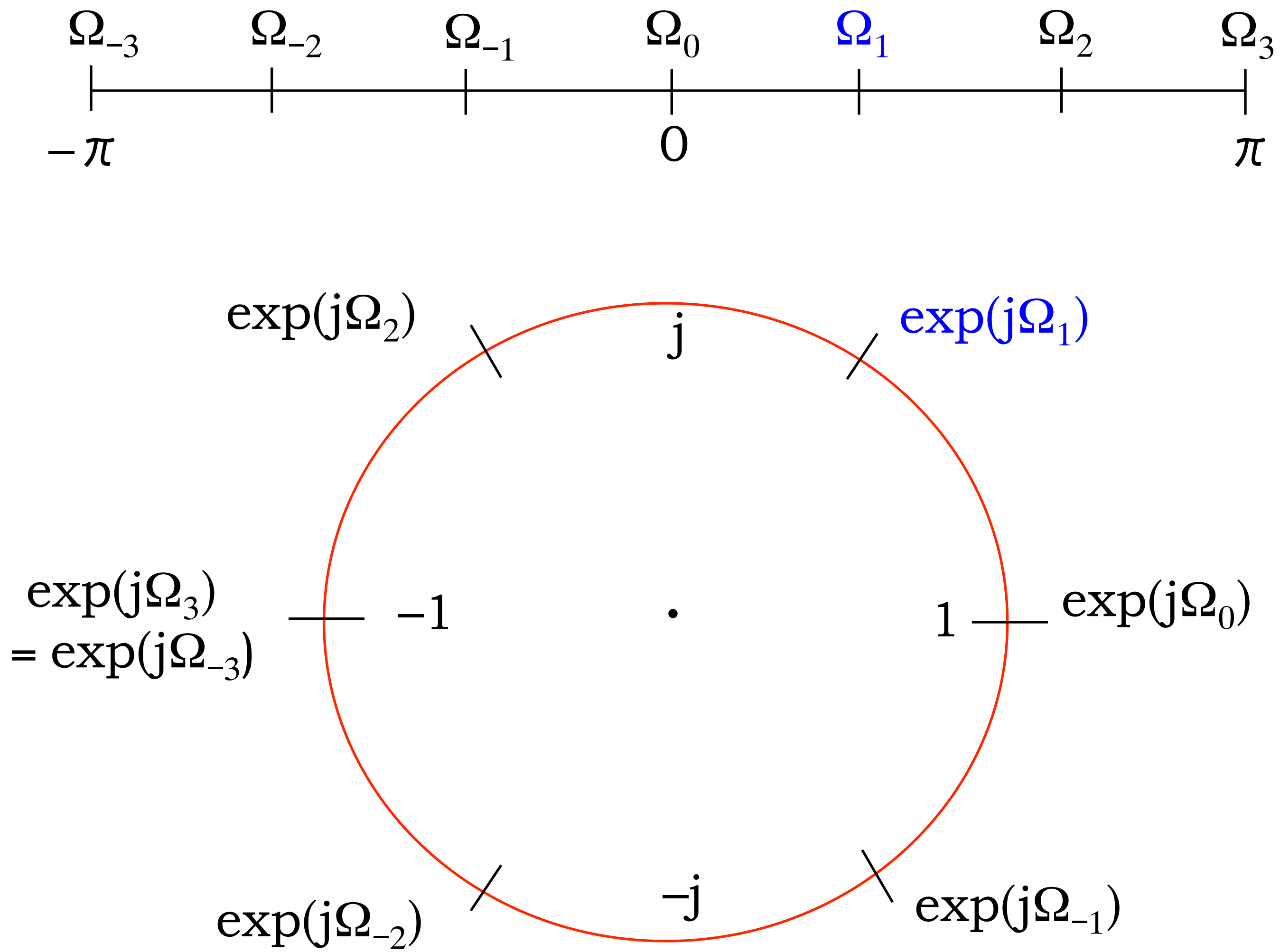
where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that P is longer than the time interval [0,L-1] over which x[n] is nonzero, and k ranges from –P/2 to (P/2)–1 (for even P).

Computing these series involves $O(P^2)$ operations – when P gets large, the computations get very s l o w....

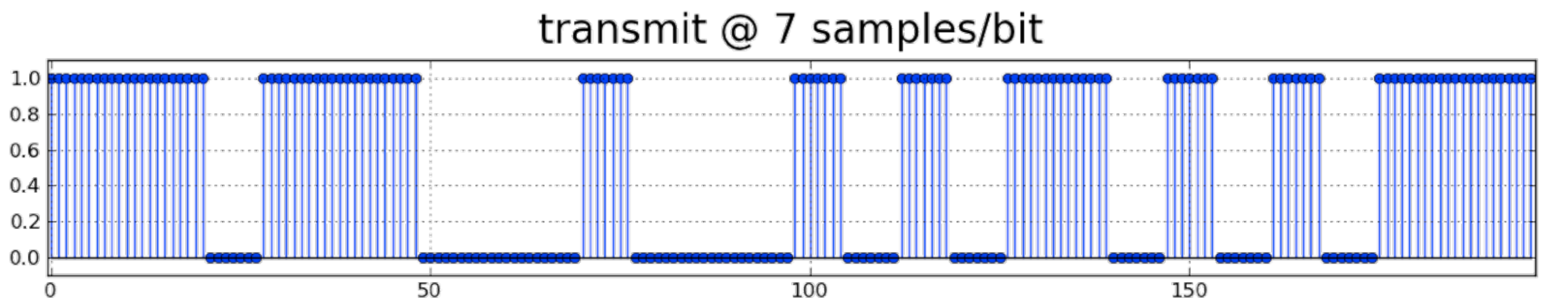
Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

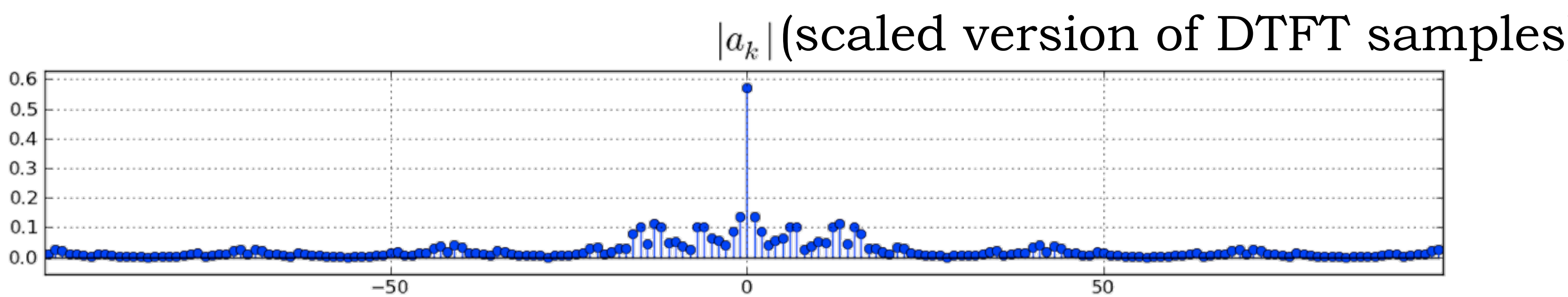
$P = 1024$, $P^2 = 1{,}048{,}576$, $P \log P \approx 10{,}240$

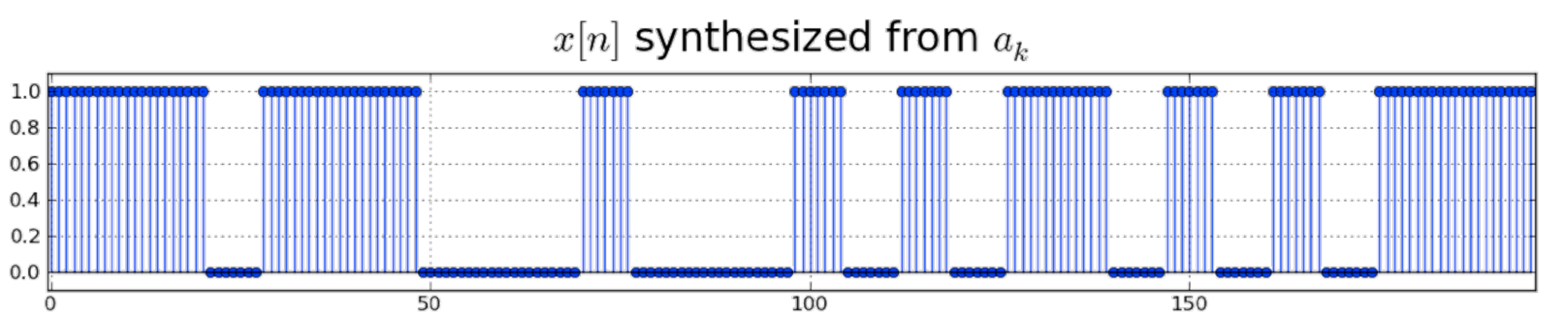
Where do the Ω_k live? e.g., for P=6 (even)



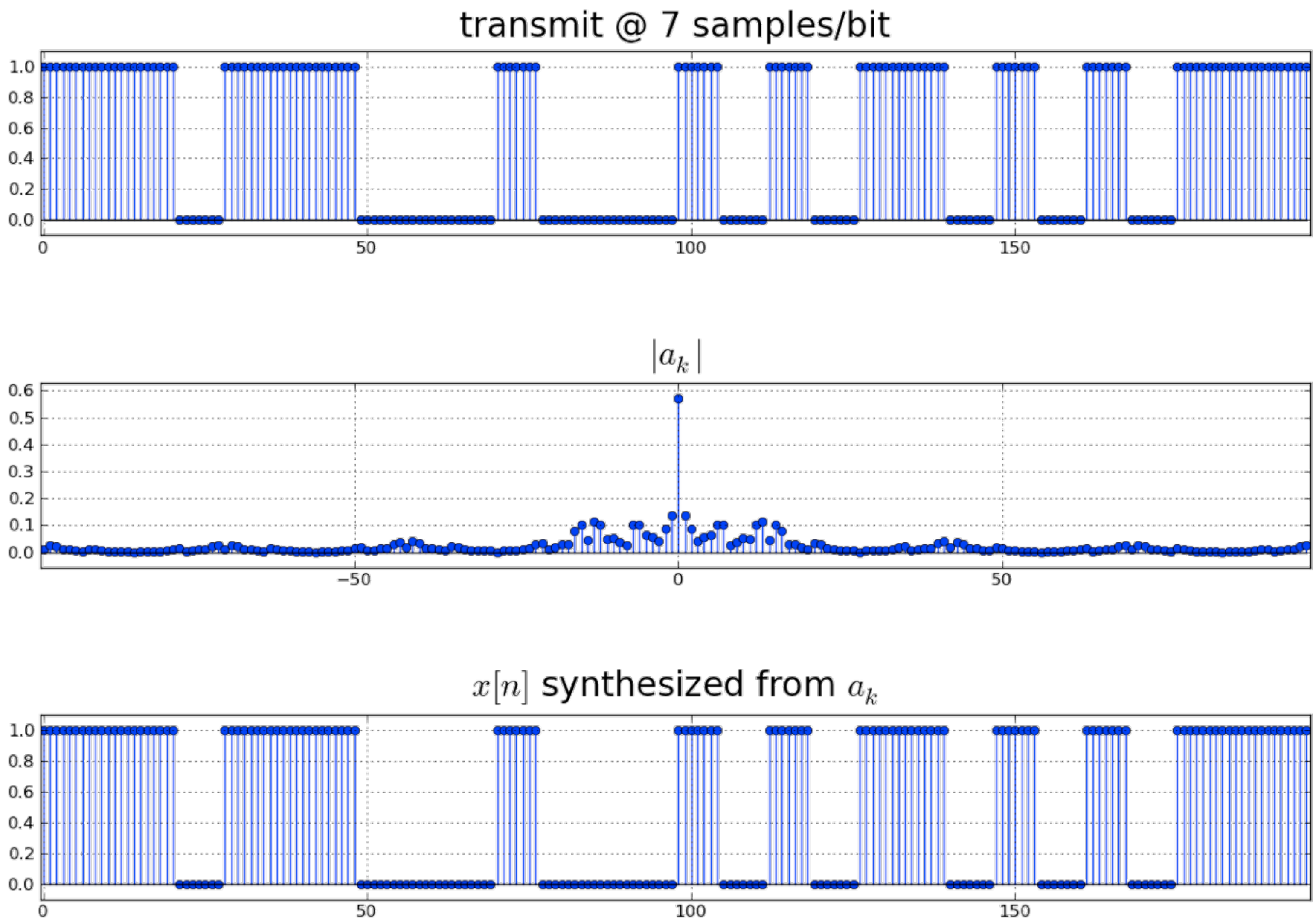
Spectrum of Digital Transmissions







Spectrum of Digital Transmissions



Observations on previous figure

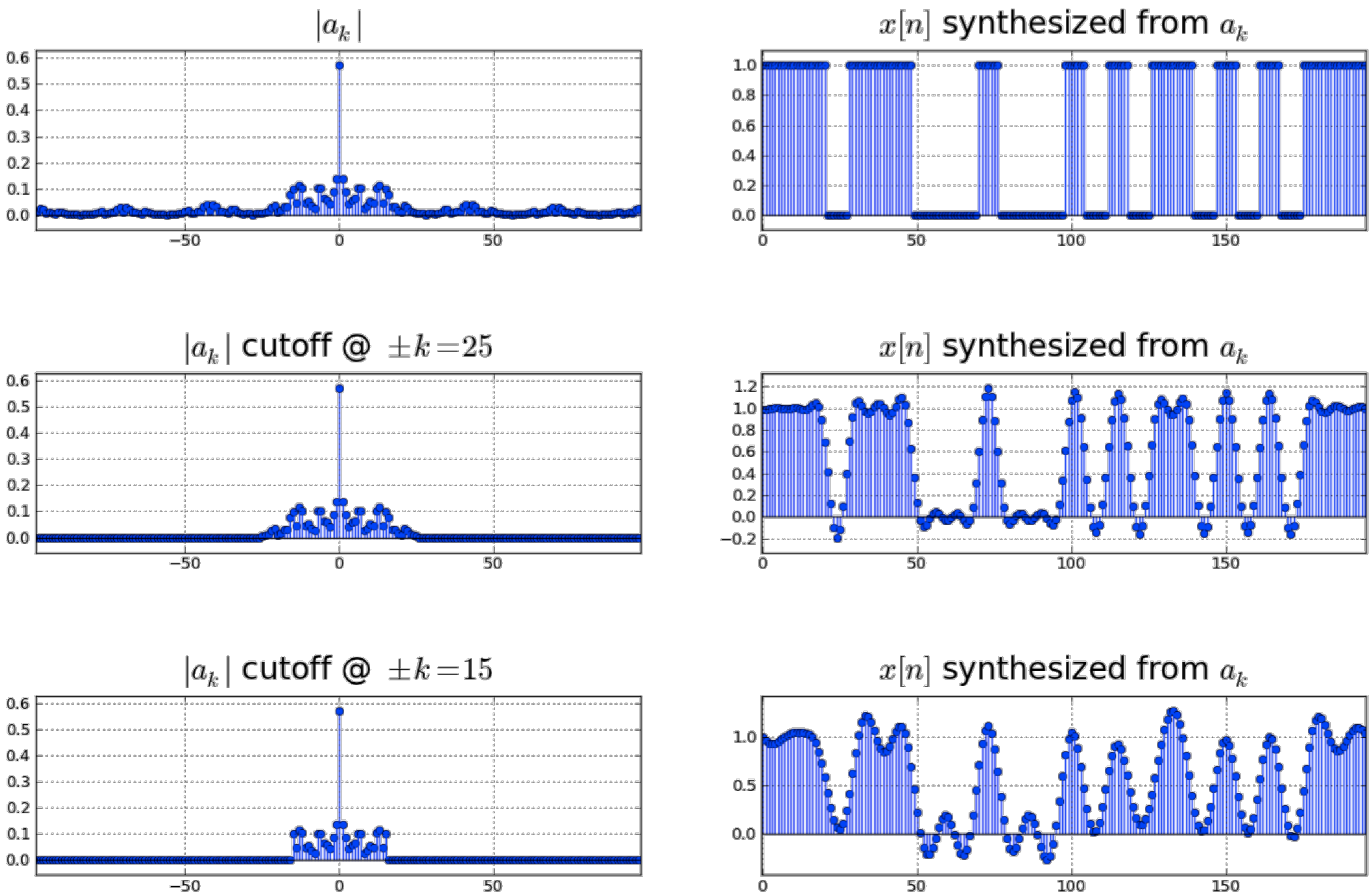
- The waveform x[n] cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (The is rough and qualitative, as x[n] is not sinusoidal.)

- A period of 14 corresponds to a frequency of $2\pi/14 = \pi/7$, which is 1/7 of the way from 0 to the positive end of the frequency axis at π (so k approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

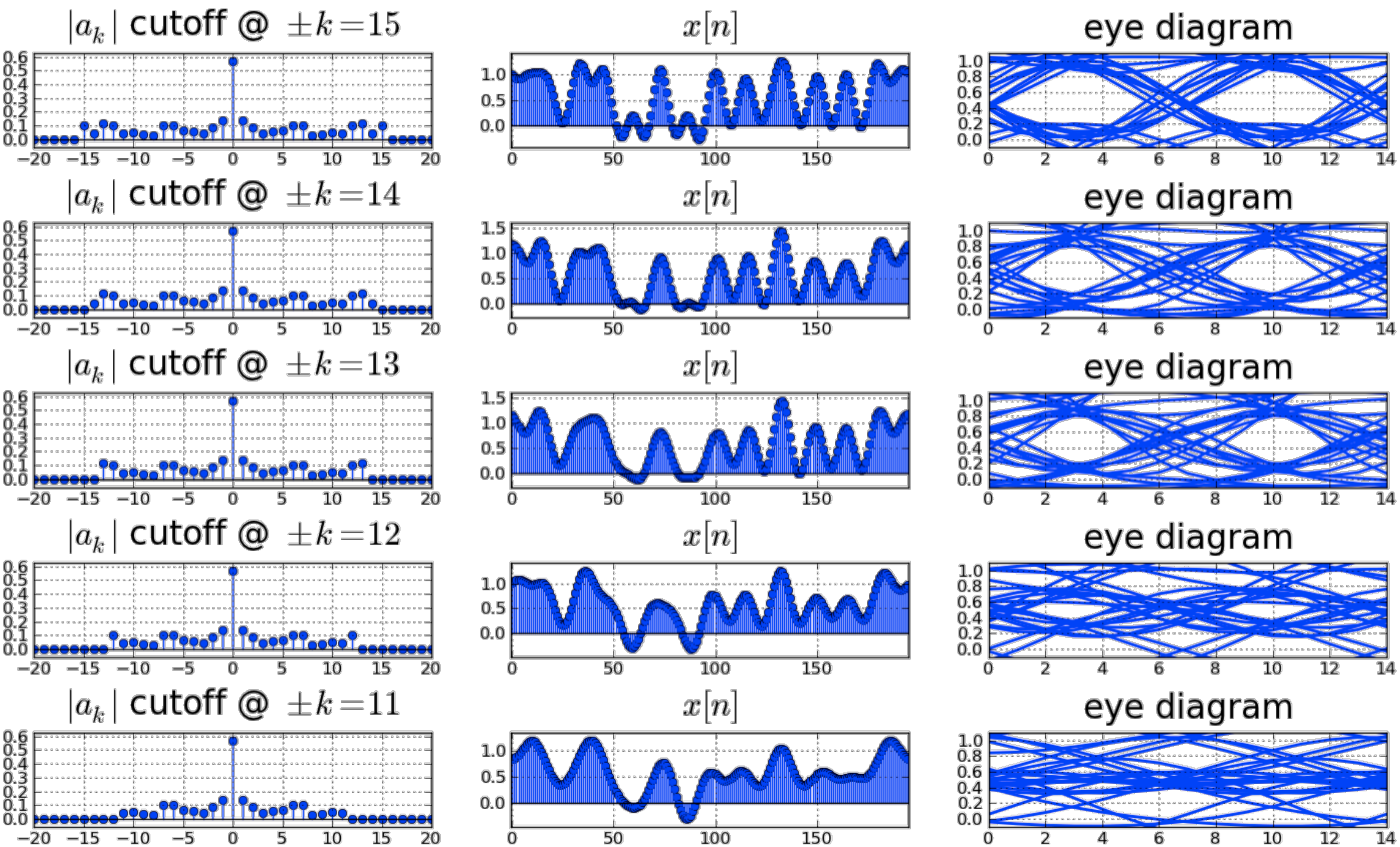
- There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

- And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.

Effect of Low-Pass Channel



How Low Can We Go?



7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14