• Modulation
  – to match the transmitted signal to the physical medium
• Demodulation
Single Link Communication Model

Original source

Digitize (if needed)

Source binary digits ("message bits")

Source coding

Bit stream

End-host computers

Receiving app/user

Render/display, etc.

Source decoding

Bit stream

Channel Coding (bit error correction)

Mapper + Xmit samples

Signals (Voltages) over physical link

Recv samples + Demapper

Channel Decoding (reducing or removing bit errors)

End-host computers
DT Fourier Transform (DTFT) for Spectral Representation of General $x[n]$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega$$

where

$$X(\Omega) = \sum_{m} x[m]e^{-j\Omega m}$$

This Fourier representation expresses $x[n]$ as a weighted combination of $e^{j\Omega n}$ for all $\Omega$ in $[-\pi, \pi]$. $X(\Omega_o)d\Omega$ is the spectral content of $x[n]$ in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$.
The input/output behavior of an LTI system in the frequency domain is described by the following equations:

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega)X(\Omega)e^{j\Omega n} d\Omega \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\Omega)e^{j\Omega n} d\Omega \]

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

Spectral content of output

Spectral content of input

Frequency response of system
http://www.pcmag.com/article2/0,2817,1769243,00.asp
Phase of the frequency response is important too!

• Maybe not if we are only interested in audio, because the ear is not so sensitive to phase distortions

• But it’s certainly important if we are using an audio channel to transmit non-audio signals such as digital signals representing 1’s and 0’s, not intended for the ear
To gauge how it will fare on lowpass and bandpass channels, let's look at the spectral content of a rectangular pulse,

\[ x[n] = u[n] - u[n-256], \]

of the kind we've been using in on-off signaling in our Audiocom lab.

Any guesses as to spectral shape?
Derivation of DTFT for rectangular pulse

\( x[m] = u[m] - u[m-N] \)

\[
X(\Omega) = \sum_{m=0}^{N-1} x[m] e^{-j\Omega m}
\]

\[
= 1 + e^{-j\Omega} + e^{-j2\Omega} + \ldots + e^{-j\Omega(N-1)}
\]

\[
= \frac{(1 - e^{-j\Omega N})}{(1 - e^{-j\Omega})}
\]

\[
= e^{-j\Omega(N-1)/2} \frac{\sin(\Omega N / 2)}{\sin(\Omega / 2)}
\]

Height \( N \) at the origin, first zero-crossing at \( 2\pi /N \)

Shifting in time only changes the phase term in front. If the rectangular pulse is centered at 0, this term is 1.
Simpler case: DTFT of \( x[n] = u[n+5] - u[n-6] \)
(centered rectangular pulse of length 11)

\[ N \]

A periodic sinc
(or “Dirichlet kernel”) – not the sinc we’ve seen before!

\[ 2\pi / N \]

https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html
Magnitude of preceding DTFT

https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html
DTFT of $x[n] = u[n] - u[n-10]$, rectangular pulse of length 10 starting at time 0

http://cnx.org/content/m0524/latest/
Back to our Audiocom lab example
\[ x[n] = u[n] - u[n-256] \]
| DTFT | of $x[n]=u[n]-u[n-256]$, rectangular pulse of length 256:

48000 samples of $|\text{DTFT}|$ spread evenly between $[-\pi, \pi]$, computed using FFT (around 3000 times faster than direct computation in this case!)

If sampling rate is 48 kHz, then this is 24,000 Hz

$\Omega = \pi$

$f = f_s / 2$

0 rads/sample

0 Hz
Zooming in:

\[ 256 = N \]

Too much of the signal’s energy misses the loudspeaker’s passband!

187.5 Hz (corresponds to \( 2 \pi / N \) when \( f_s = 48 \text{ kHz} \))
What if we sent this pulse through an ideal lowpass channel?

| DTFT | of lowpass filtered version of x[n]=u[n]-u[n-256],
cutoff 400 Hz

\[
\Omega = \pi
\]

\[
f = \frac{f_s}{2}
\]
Zooming in:

- $256 = N$
- $400 \text{ Hz}$
Corresponding pulse in time, i.e., lowpass filtered version of rectangular pulse

No longer confined to its 256-sample slot, so causes “intersymbol interference” (ISI).
Effect of Low-Pass Channel

$|a_k|$ cutoff @ $\pm k = 25$

$x[n]$ synthesized from $a_k$

$|a_k|$ cutoff @ $\pm k = 15$

$x[n]$ synthesized from $a_k$
How Low Can We Go?
Complementary/dual behavior in time and frequency domains

• Wider in time, narrower in frequency; and vice versa.
  – This is actually the basis of the uncertainty principle in physics!

• Smoother in time, sharper in frequency; and vice versa

• Rectangular pulse in time is a (periodic) sinc in frequency, while rectangular pulse in frequency is a sinc in time; etc.
A shaped pulse versus a rectangular pulse:

Slightly round the transitions from 0 to 1, and from 1 to 0, by making them sinusoidal, just 30 samples on each end.
In the spectral domain:

- |DTFT| of rectangular pulse
- Negative |DTFT| of shaped pulse
- Frequency content of shaped pulse only extends to here, around 1500 Hz
After passing the two pulses through a 400 Hz cutoff lowpass filter:

The lowpass filtered shaped pulse conforms more tightly to the 256-sample slot, and settles a little quicker.
But loudspeakers are bandpass, not lowpass
Spectrum of rectangular pulse after ideal bandpass filtering, 100 Hz to 10,000 Hz
Zooming in:

- 0 Hz
- 100 Hz
- 10,000 Hz
Corresponding pulse in time, i.e., **bandpass filtered** version of rectangular pulse

Won’t do at all!!
The Solution: Modulation

- Shift the spectrum of the signal $x[n]$ into the loudspeaker’s passband by **modulation**!

$$x[n] \cos(\Omega_c n) = 0.5x[n](e^{j\Omega_c n} + e^{-j\Omega_c n})$$

$$= \frac{0.5}{2\pi} \left[ \int_{-\pi}^{\pi} X(\Omega')e^{j(\Omega' + \Omega_c)n} d\Omega' + \int_{-\pi}^{\pi} X(\Omega'')e^{j(\Omega'' - \Omega_c)n} d\Omega'' \right]$$

$$= \frac{0.5}{2\pi} \left[ \int_{-\pi}^{\pi} X(\Omega - \Omega_c)e^{j\Omega n} d\Omega + \int_{-\pi}^{\pi} X(\Omega + \Omega_c)e^{j\Omega n} d\Omega \right]$$

Spectrum of modulated signal comprises **half-height** replications of $X(\Omega)$ centered as $\pm\Omega_c$ (i.e., plus and minus the carrier frequency). So choose carrier frequency comfortably in the passband, leaving room around it for the spectrum of $x[n]$. 
Is Modulation Linear? Time-Invariant? ...

\[ x[n] \times \cos(\Omega_c n) \rightarrow t[n] \]

... as a system that takes input \( x[n] \) and produces output \( t[n] \) for transmission?

Yes, linear!

No, not time-invariant!
So for our rectangular pulse example:

Time domain:
Pulse modulated onto 1000 Hz carrier
Corresponding spectrum of signal modulated onto carrier
Zooming in:

- 100 Hz, lower cutoff of bandpass filter
- 10,000 Hz, upper cutoff of bandpass filter
- 128, i.e. half height of original

-1000 Hz

0 Hz

1000 Hz

100 Hz, lower cutoff of bandpass filter
Pulse modulated onto 1000 Hz carrier makes it through the bandpass channel with very little distortion.
SCARY GOOD!!
At the Receiver: Demodulation

• In principle, this is (as easy as) modulation again:

If the received signal is
\[ r[n] = x[n]\cos(\Omega_c n), \]
then simply compute
\[ d[n] = r[n]\cos(\Omega_c n) = x[n]\cos^2(\Omega_c n) = 0.5 \{x[n] + x[n]\cos(2\Omega_c n)\} \]

• What does the spectrum of \( d[n] \) look like?
• What constraint on the bandwidth of \( x[n] \) is needed for perfect recovery of \( x[n] \)?