• Revisit/expand story on bit detection in noise
• Introduction to modulation and demodulation
• Modeling the channel
Bit Detection in Noise

We’ve decided it’s easier to tell you the full story rather than a partial one!
So:

Recall that the receiver samples the value

\[ y = x + w \]

in a particular bit slot (one sample per bit slot, for now),
where \( x \) is the transmitted value

- \( = V_0 \) if the sender’s codeword bit \( B=0 \), probability \( p_0 \)
- \( = V_1 \) if the sender’s codeword bit \( B=1 \), probability \( p_1 \)

\( V_0 = 0 \) and \( V_1 = V_p \) for on-off signaling

\( V_0 = -V_p \) and \( V_1 = V_p \) for bipolar signaling

and

\( w \) is the value of the additive channel noise in this bit slot.
Conditional PDFs of received sample $Y$

- Think in terms of random variables,
  
  $$Y = X + W$$
with each of these taking specific values $y$, $x$, $w$ in the particular bit slot.

$(X$ and $W$ are assumed independent, i.e., knowing one tells us nothing about the other --- if the channel noise is signal-dependent, things get harder)

- How is $Y$ distributed if $B=0$ ? $\implies$ Described by conditional PDF $f_{Y|B}(y|0)$

- How is $Y$ distributed if $B=1$ ? $\implies$ Described by conditional PDF $f_{Y|B}(y|1)$

- What is $f_{Y|B}(y|0)$ if $W$ is Gaussian, mean $0$, variance $\sigma^2$ ?
  Gaussian, mean $V_0$, variance $\sigma^2$

- What is $f_{Y|B}(y|1)$ if $W$ is Gaussian, mean $0$, variance $\sigma^2$ ?
  Gaussian, mean $V_1$, variance $\sigma^2$
Bit Detection with Min Probability of Error

What is the probability that the received sample falls in this interval of length $dy$?

$$f_{Y|B}(y|0) \, dy \quad \text{if} \quad B=0$$
$$f_{Y|B}(y|1) \, dy \quad \text{if} \quad B=1$$

What is the probability of error if receiver decides “0” when $y$ lies here?

$$p_1 \cdot f_{Y|B}(y|1) \, dy$$

What is the probability of error if receiver decides “1” when $y$ lies here?

$$p_0 \cdot f_{Y|B}(y|0) \, dy$$
So, for min $P(\text{error})$ ...

- Decide “1” for all $y$ where $p_1 \cdot f_{Y|B}(y|1) > p_0 \cdot f_{Y|B}(y|0)$

- Decide “0” for all $y$ where $p_1 \cdot f_{Y|B}(y|1) < p_0 \cdot f_{Y|B}(y|0)$

- And the associated probability of error is

  $$\int p_1 \cdot f_{Y|B}(y|1) \, dy \quad \text{over the “0” region}$$

  $$+ \int p_0 \cdot f_{Y|B}(y|0) \, dy \quad \text{over the “1” region}$$

- This is bit-by-bit detection, i.e., **hard detection** (not soft)
The Q(.) Function --- Area in the Tail of a Standard Gaussian

\[ Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\nu^2/2} \, d\nu \]

\[ Q(-t) = 1 - Q(t) \]

\[ \frac{t}{(1 + t^2)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} < Q(t) < \frac{1}{t} \frac{e^{-t^2/2}}{\sqrt{2\pi}}, \quad t > 0 \]
Tail probability of a general Gaussian in terms of the Q(.) function

\[
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{t}^{\infty} e^{-(v-\mu)^2/(2\sigma^2)} \, dv
= Q\left(\frac{t - \mu}{\sigma}\right)
\]
Connecting the “SNR” and BER for Bipolar Signaling

\[ p_0 = 0.5 \quad \text{and} \quad p_1 = 0.5 \]

\[ \frac{\mu}{\sigma} = -V_p \]

\[ \sigma = \sigma_{\text{noise}} \]

\[ V_p = \sqrt{E_s} \]

\[ 2\sigma^2 = N_0 \]

"SNR" = \( \frac{E_s}{N_0} \)

\[ P(\text{error}) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-V_p}^{+V_p} e^{-\frac{w^2}{2\sigma^2}} \, dw \]

\[ = Q\left(\frac{V_p}{\sigma}\right) \]

\[ = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \]

How does this relate to \( p \) of BSC?
Bit Error Rate for Bipolar Signaling Scheme with Single-Sample Decision

\[ Q(\sqrt{2E_s/N_0}) \]
Noise Model for iid Process \( w[n_j] \)

- Assume each \( w[n_j] \) is distributed as a Gaussian random variable \( W \), with mean 0 and variance \( \sigma^2 \), and independently of \( w[.\] at sample times in all other bit slots.

- Under the iid assumption

\[
P(w_a < w[n_1] < w_b \text{ and } w_c < w[n_2] < w_d)
\]

\[
= \int_{w_a}^{w_b} f_W(\alpha) \, d\alpha \cdot \int_{w_c}^{w_d} f_W(\beta) \, d\beta
\]

etc. --- i.e., individual PDFs multiply for independent continuous random variables (just as probabilities multiply for independent discrete events) --- so in Gaussian case exponents add.

→ Justifies soft-decision decoding with sum-of-squares metric
Soft Decoding Beats Hard Decoding

2 dB improvement
But we can do better!

• Why just take a single sample from a bit interval?
• Instead, average M (≤L) samples with independent noise components:

\[ y[n] = \pm V_p + w[n] \text{ so } \text{avg} \{y[n]\} = \pm V_p + \text{avg} \{w[n]\} \]

• **Claim:** \text{avg} \{w[n]\} is still Gaussian, still has mean 0, but its variance is now \( \sigma^2/M \) instead of \( \sigma^2 \), so

“SNR” is increased by a factor of M

• Same analysis and formulas as before, but now with

\[ \sigma^2 \rightarrow \sigma^2/M \text{, or equivalently} \]

sample energy \( E_s \rightarrow \) bit (or symbol) energy \( E_b = M.E_s \)
Implications for Signaling Rate

• As the noise intensity increases, we need to slow down the signaling rate, i.e., increase the number of samples per bit (L), to get higher energy in the (M≤L) samples extracted from a bit interval, if we wish to maintain the same error performance.

– e.g. Voyager 2 was transmitting at 115 kilobits/s when it was near Jupiter in 1979. When it was over 9 billion miles away, 13 light hours away from the sun, twice as far away from the sun as Pluto, it was transmitting at only 160 bits/s. The received power at the Deep Space Network antennas on earth when Voyager was near Neptune was on the order of 10^(-16) watts!! --- 20 billion times smaller than an ordinary digital watch consumes. The power now is estimated at less than 10^(-19) watts.
Flipped bits can have serious consequences!

- “On November 30, 2006, a telemetered command to Voyager 2 was incorrectly decoded by its on-board computer—in a random error—as a command to turn on the electrical heaters of the spacecraft's magnetometer. These heaters remained turned on until December 4, 2006, and during that time, there was a resulting high temperature above 130 °C (266 °F), significantly higher than the magnetometers were designed to endure, and a sensor rotated away from the correct orientation. It has not been possible to fully diagnose and correct for the damage caused to the Voyager 2's magnetometer, although efforts to do so are proceeding.”

- “On April 22, 2010, Voyager 2 encountered scientific data format problems as reported by the Associated Press on May 6, 2010. On May 17, 2010, JPL engineers revealed that a flipped bit in an on-board computer had caused the issue, and scheduled a bit reset for May 19. On May 23, 2010, Voyager 2 has resumed sending science data from deep space after engineers fixed the flipped bit.”

http://en.wikipedia.org/wiki/Voyager_2
The moral of the story is …

… if you’re doing appropriate/optimal processing at the receiver, your effective SNR (and therefore your error performance) in the case of iid Gaussian noise is determined — through the $Q(.)$ function — by the ratio of bit (or symbol) energy (not sample energy) to noise variance.