• Spectral content of signals via the DTFT
Determining $h[n]$ from $H(\Omega)$

$$H(\Omega) = \sum_{m} h[m] e^{-j\Omega m}$$

Multiply both sides by $e^{j\Omega n}$ and integrate over a (contiguous) $2\pi$ interval. Only one term survives!

$$\int_{<2\pi>} H(\Omega)e^{j\Omega n} \, d\Omega = \int_{<2\pi>} \sum_{m} h[m] e^{-j\Omega (m-n)} \, d\Omega$$

$$= 2\pi \cdot h[n]$$

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega)e^{j\Omega n} \, d\Omega$$
Design ideal lowpass filter with cutoff frequency $\Omega_c$ and $H(\Omega)=1$ in passband

$$h[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} H(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_c n)}{\pi n}, \quad n \neq 0$$

$$= \frac{\Omega_c}{\pi}, \quad n = 0$$

DT “sinc” function (extends to $\pm\infty$ in time, falls off only as $1/n$)
Approximating an ideal lowpass filter

Idea: Delay \( h[n] \) to get causal LTI system (after truncation of tails). Will the result still be a lowpass filter?
Causal approximation to ideal lowpass filter

\[ h_C[n] = h[n-300] \]

Determine \(|H_C(\Omega)|\)
Exercise: Frequency response of $h[n-D]$

Given an LTI system with unit sample response $h[n]$ and associated frequency response $H(\Omega)$,

determine the frequency response $H_D(\Omega)$ of an LTI system whose unit sample response is

$$h_D[n] = h[n-D].$$

**Answer:**

$$H_D(\Omega) = \exp\{-j\Omega D\}.H(\Omega)$$

so:

$$|H_D(\Omega)| = |H(\Omega)|, \quad \text{i.e., magnitude unchanged}$$

$$<H_D(\Omega) = -\Omega D + <H(\Omega), \quad \text{i.e., linear phase term added}$$
Useful Filters
Lowpass filtering (10 Hz cutoff) of blood flow velocity in middle cerebral artery, measured using transcranial Doppler ultrasound.
Frequency Response of Channels

- $h[n]$ for fast channel
- $|H(e^{j\omega})|$ for fast channel

- $h[n]$ for slow channel
- $|H(e^{j\omega})|$ for slow channel

- $h[n]$ for ringing channel
- $|H(e^{j\omega})|$ for ringing channel
Loudspeaker Bandpass Frequency Response

Connection between CT and DT

The continuous-time (CT) signal

\[ x(t) = \cos(\omega t) = \cos(2\pi ft) \]

sampled every \( T \) seconds, i.e., at a sampling frequency of \( f_s = 1/T \), gives rise to the discrete-time (DT) signal

\[ x[n] = x(nT) = \cos(\omega nT) = \cos(\Omega n) \]

So \( \Omega = \omega T \)

and \( \Omega = \pi \) corresponds to \( \omega = \pi/T \) or \( f = 1/(2T) = f_s/2 \)
A Deeper Reason for Interest in Sinusoidal Inputs

- General inputs $x[.]$ can we written as “sums” of sinusoids

- Each input sinusoidal component is mapped via the frequency response $H(\Omega)$ to its corresponding sinusoidal output component

- Superposition of these output components yields the general response $y[.]$
DT Fourier Transform (DTFT) for Spectral Representation of General $x[n]$

If we can write

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega)e^{j\Omega n} d\Omega$$

where

$$H(\Omega) = \sum_{m} h[m]e^{-j\Omega m}$$

then we can write

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega$$

where

$$X(\Omega) = \sum_{m} x[m]e^{-j\Omega m}$$

This Fourier representation expresses $x[n]$ as a weighted combination of $e^{j\Omega n}$ for all $\Omega$ in $[-\pi, \pi]$. $X(\Omega_o)d\Omega$ indicates the spectral content of $x[n]$ in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$. 
$x[n]$ and $X(\Omega)$

Rapidly decaying $x[n]$

Slowly decaying $x[n]$

Oscillatory $x[n]$
Signal $x[n]$ that has its frequency content uniformly distributed in $[-\Omega_c, \Omega_c]$:

$$x[n] = \frac{1}{2\pi} \left< \int_{-\Omega_c}^{\Omega_c} X(\Omega) e^{j\Omega n} d\Omega \right>$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_c n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_c / \pi, \quad n = 0$$

DT “sinc” function (extends to $\pm \infty$ in time, falls off only as $1/n$)
Spectral Content of Various Sounds

Ask Prof. Zue** to read this speech spectrogram

http://en.wikipedia.org/wiki/Spectrogram

**Victor was the first person to be able to read these! Get a sense of his achievements at http://www.okawa-foundation.or.jp/en/activities/prize/data/2012_evi.pdf
Dolphin sounds
Instantaneous Heart Rate

![Graph showing instantaneous HR signal over time](image)
Heart-Rate Power Spectral Density $|X(\Omega)|^2$ averaged over many time-windows

- Breathing frequency: 0.18Hz
Relating Output Spectral Content to Input Spectral Content for an LTI System

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} \, d\Omega \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega) X(\Omega) e^{j\Omega n} \, d\Omega \]

\[ Y(\Omega) = H(\Omega) X(\Omega) \]

Compare with \( y[n] = (h*x)[n] \)

Again, convolution in time has mapped to multiplication in frequency
Magnitude and Angle

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

\[ |Y(\Omega)| = |H(\Omega)| \cdot |X(\Omega)| \]

and

\[ < Y(\Omega) = < H(\Omega) + < X(\Omega) \]
Core of the Story

1. A huge class of DT and CT signals can be written --- using Fourier transforms --- as a weighted sums of sinusoids (ranging from very slow to very fast) or (equivalently, but more compactly) complex exponentials. The sums can be discrete $\sum$ or continuous $\int$ (or both).

2. LTI systems act very simply on sums of sinusoids: superposition of responses to each sinusoid, with the frequency response determining the frequency-dependent scaling of magnitude, shifting in phase.
Spectrum of Digital Transmissions

transmit @ 7 samples/bit

\[ |a_k| \text{ (scaled version of DTFT samples)} \]

\[ x[n] \text{ synthesized from } a_k \]
Observations on previous figure

• The waveform $x[n]$ cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (This is rough and qualitative, as $x[n]$ is not sinusoidal.)

• A period of 14 corresponds to a frequency of $2\pi / 14 = \pi / 7$, which is 1/7 of the way from 0 to the positive end of the frequency axis at $\pi$ (so $k$ approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

• There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

• And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.
Effect of Low-Pass Channel

\[ |a_k| \text{ cutoff } @ \pm k = 25 \]

\[ x[n] \text{ synthesized from } a_k \]

\[ |a_k| \text{ cutoff } @ \pm k = 15 \]

\[ x[n] \text{ synthesized from } a_k \]
How Low Can We Go?

7 samples/bit → 14 samples/period → \( k = \frac{N}{14} = \frac{196}{14} = 14 \)