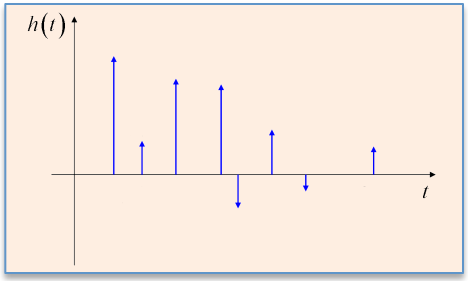
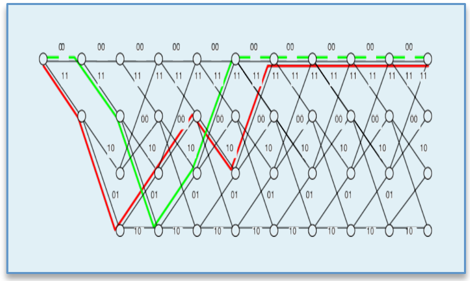
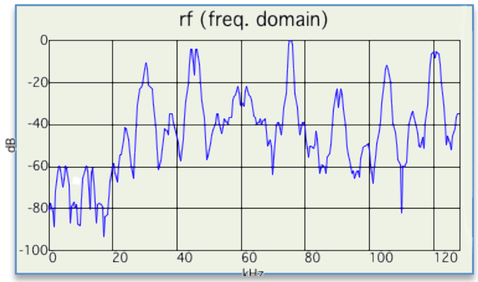
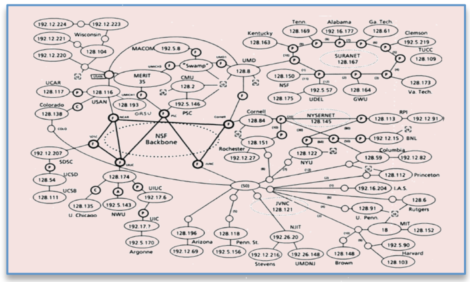








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2013
Lecture #15

- The FFT
- Modulation
- Demodulation

DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

where

$$X(\Omega) = \sum_m x[m] e^{-j\Omega m}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$.

$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

$$X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_k m}, \qquad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n}$$

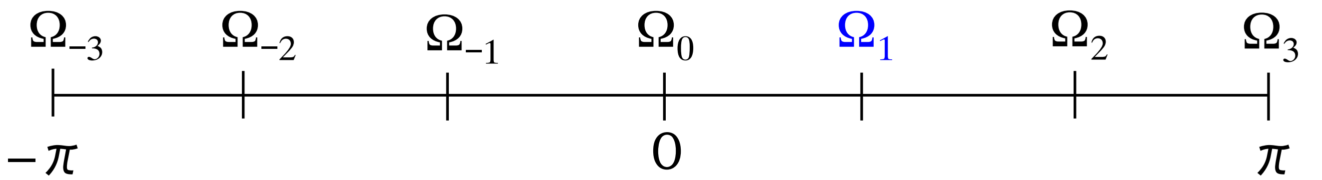
where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that $P \geq L$, where [0,L-1] is the time interval outside of which x[n] is zero, and k ranges from –P/2 to (P/2)–1 (for even P).

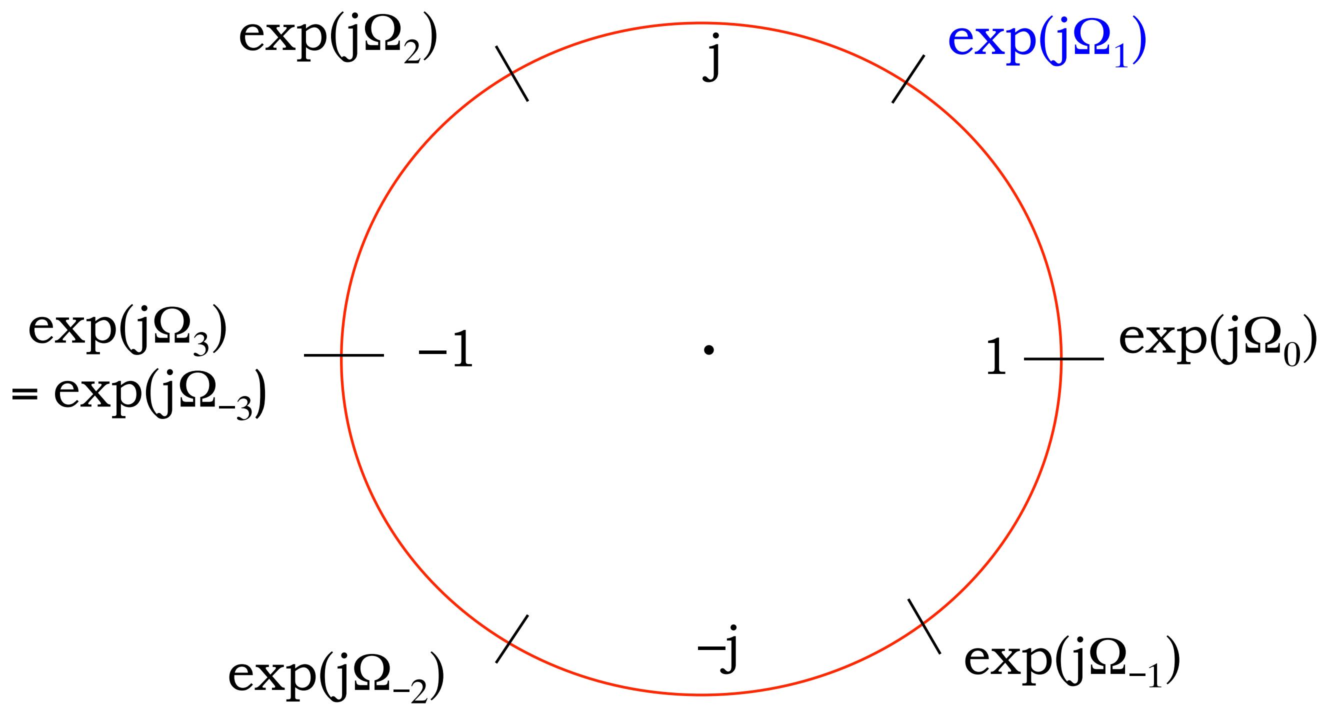
Computing these sums directly involves $O(P^2)$ operations – when P gets large, the computations get very s l o w….

Happily, in 1965 Cooley and Tukey published a fast (divide-and-conquer) method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

$P = 1024$, $P^2 = 1{,}048{,}576$, $P \log P \approx 10{,}240$

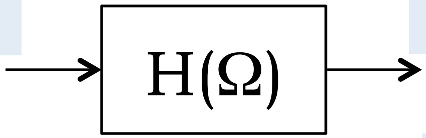
Where do the Ω_k live?
e.g., for P=6 (even)





Input/Output Behavior of LTI System in Frequency Domain

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$



$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

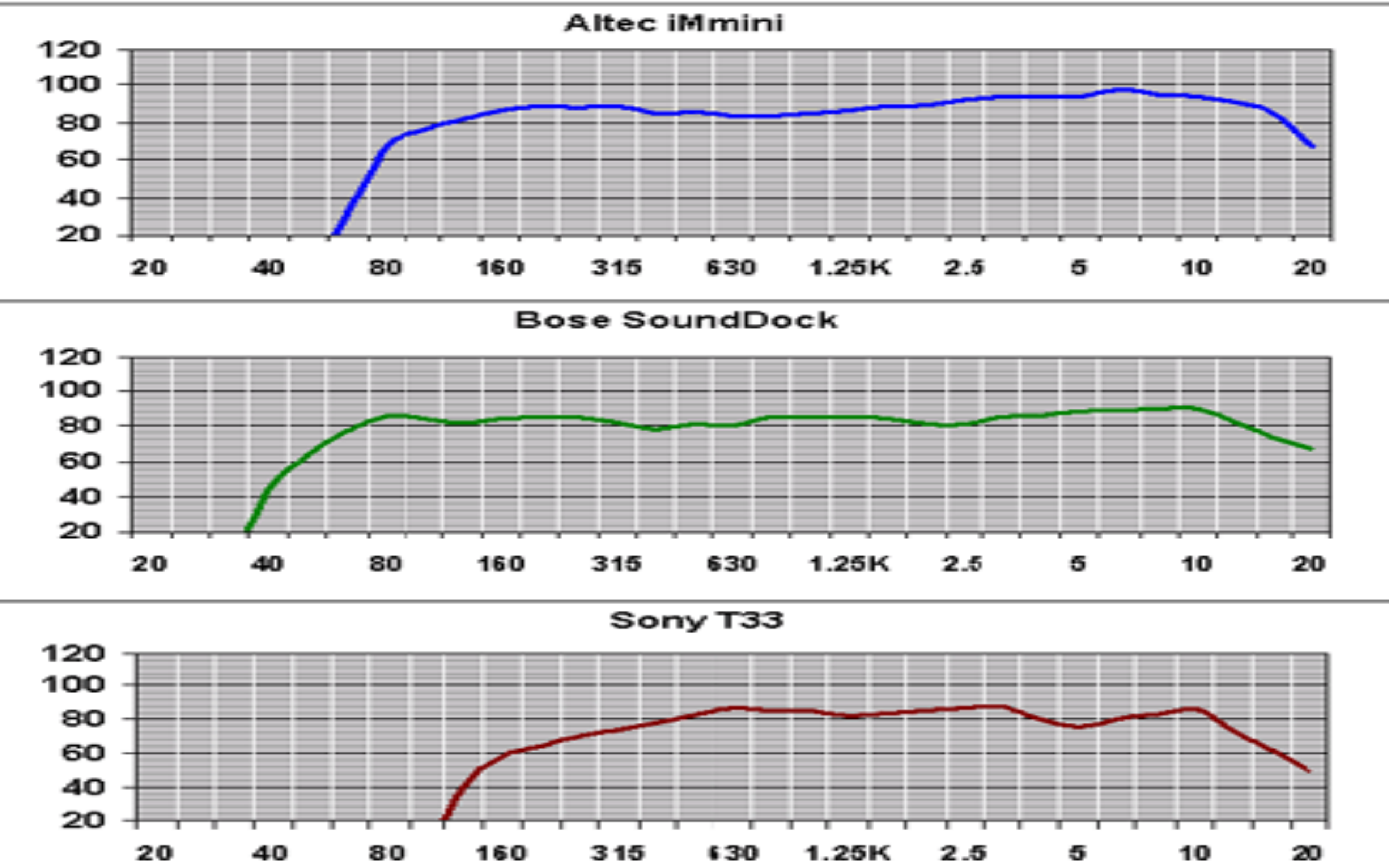
$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Spectral content of output

Frequency response of system

Spectral content of input

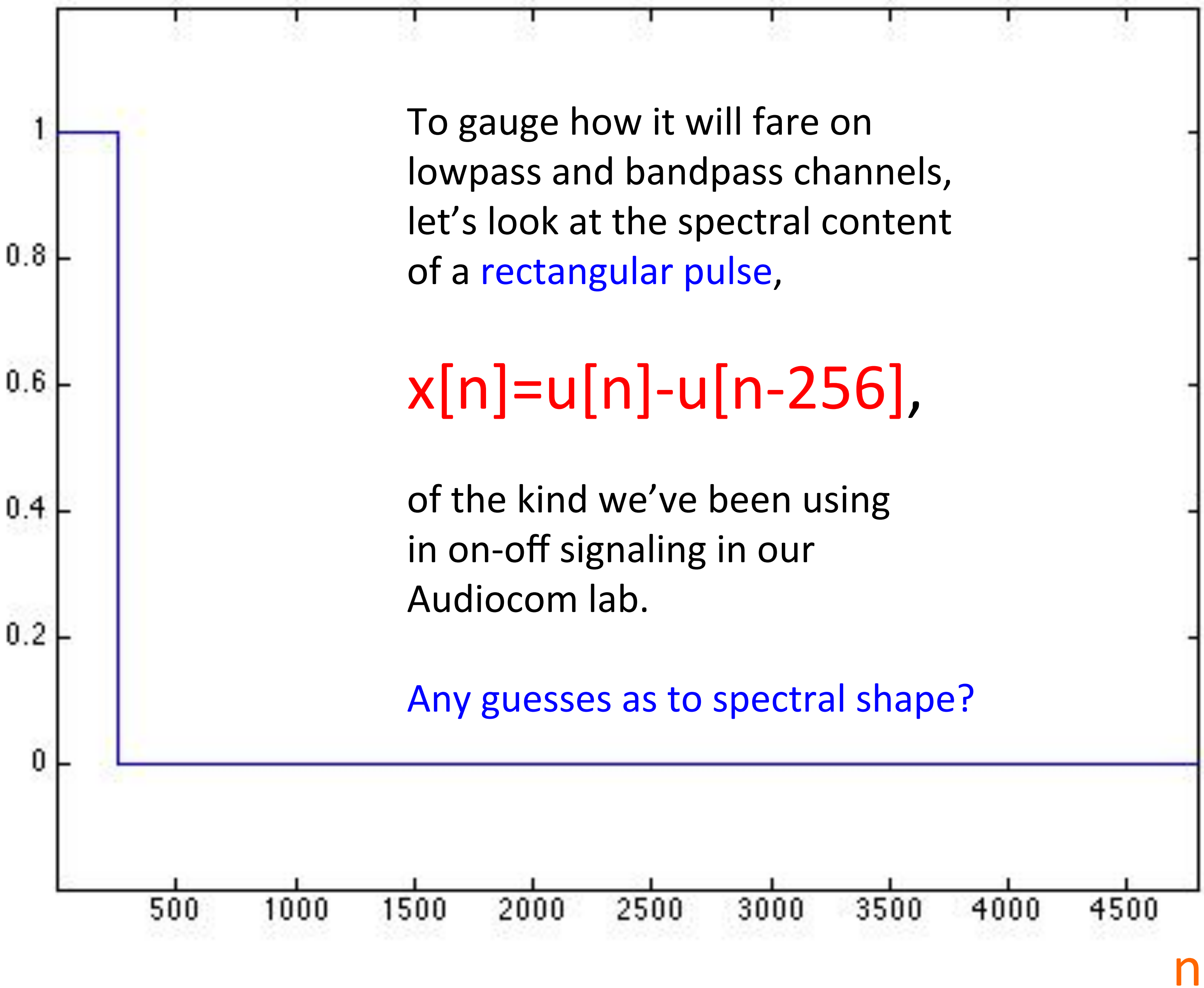


http://www.pcmag.com/article2/0,2817,1769243,00.asp

Phase of the frequency response is important too!

- Maybe not if we are only interested in audio, because the ear is not so sensitive to phase distortions

- But it's certainly important if we are using an audio channel to transmit non-audio signals such as digital signals representing 1's and 0's, not intended for the ear



To gauge how it will fare on lowpass and bandpass channels, let's look at the spectral content of a rectangular pulse,

$$x[n]=u[n]-u[n-256],$$

of the kind we've been using in on-off signaling in our Audiocom lab.

Any guesses as to spectral shape?

Derivation of DTFT for rectangular pulse x[m]=u[m]-u[m-N]

$$X(\Omega) = \sum_{m=0}^{N-1} x[m] e^{-j\Omega m}$$

$$= 1 + e^{-j\Omega} + e^{-j2\Omega} + \ldots + e^{-j\Omega(N-1)}$$

$$= (1 - e^{-j\Omega N}) / (1 - e^{-j\Omega})$$

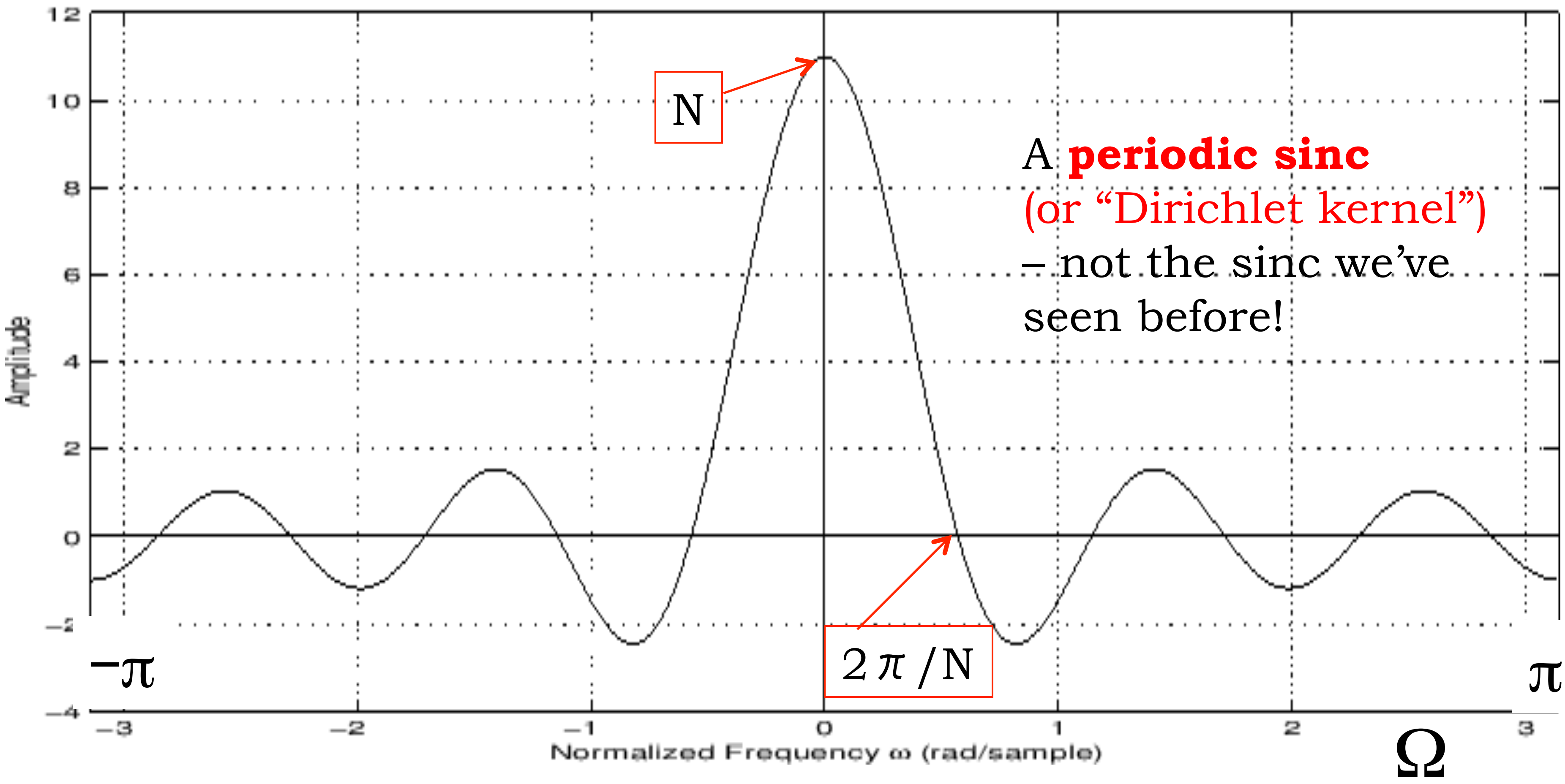
$$= e^{-j\Omega(N-1)/2} \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

Height N at the origin, first zero-crossing at 2π/N

Shifting in time only changes the phase term in front.
If the rectangular pulse is centered at 0, this term is 1.

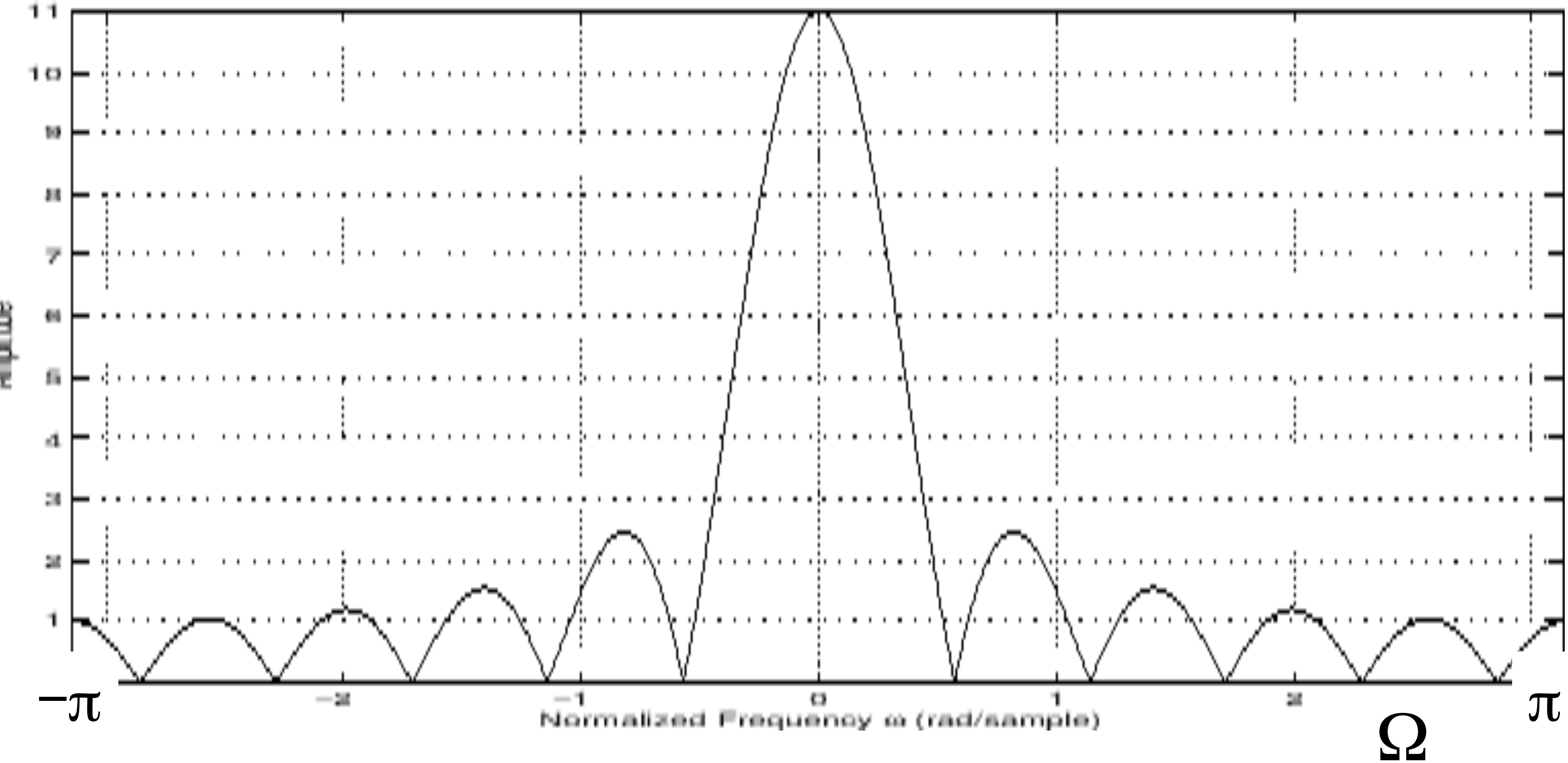
DTFT of x[n] = u[n+5] – u[n-6]

(centered rectangular pulse of length 11)



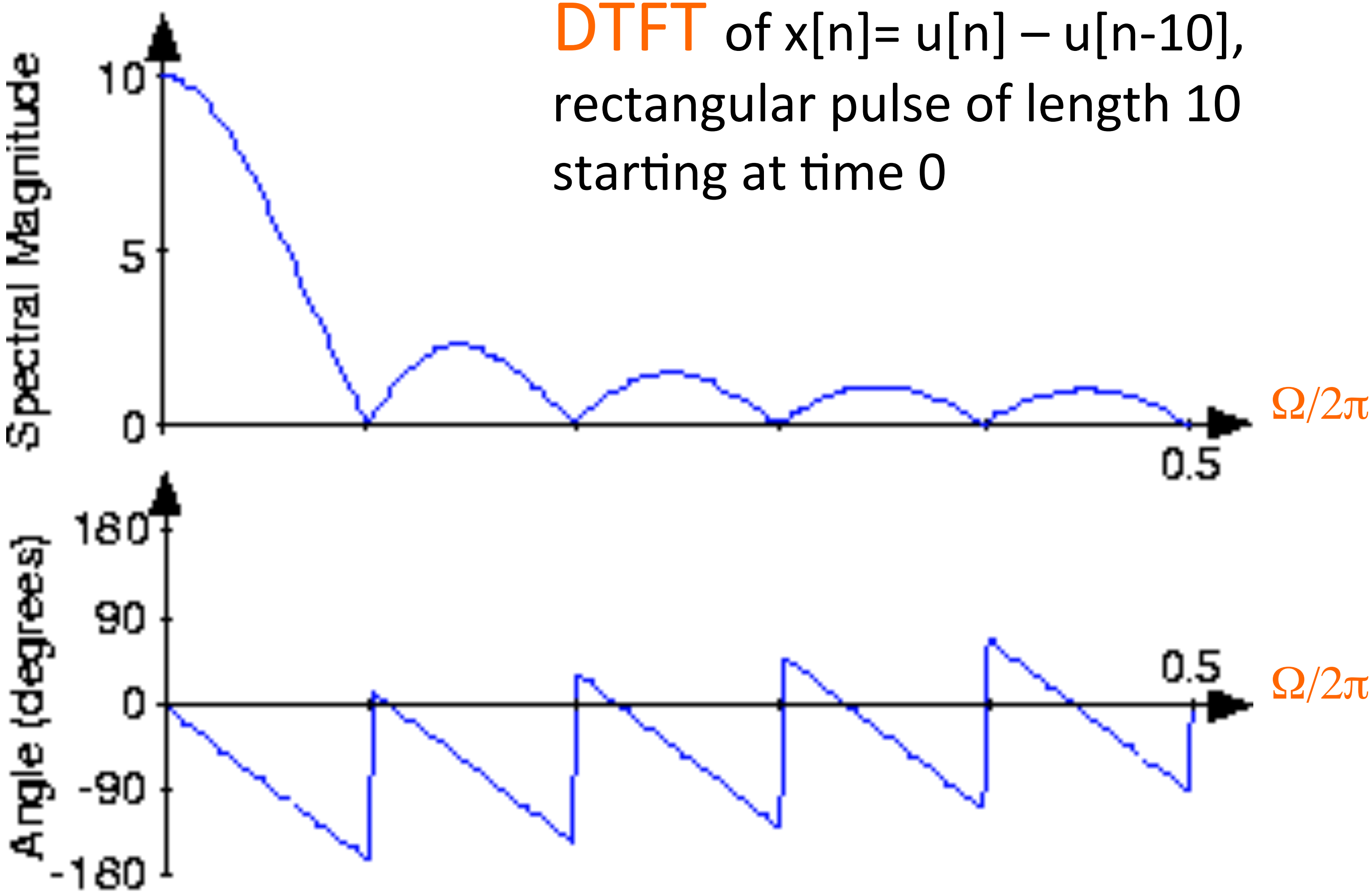
https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html

Magnitude of preceding DTFT



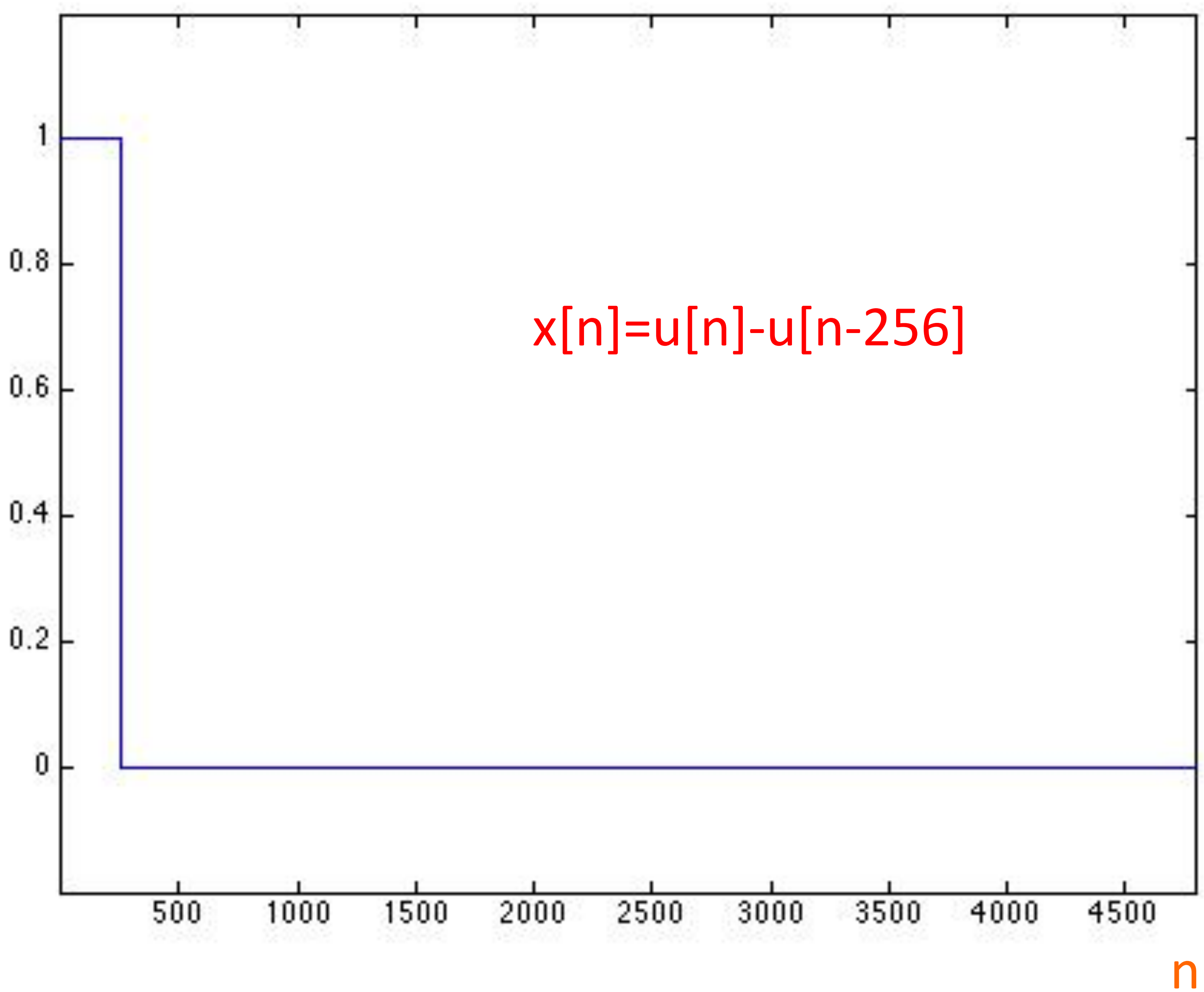
https://ccrma.stanford.edu/~jos/sasp/Rectangular_Window.html

DTFT of x[n]= u[n] – u[n-10], rectangular pulse of length 10 starting at time 0

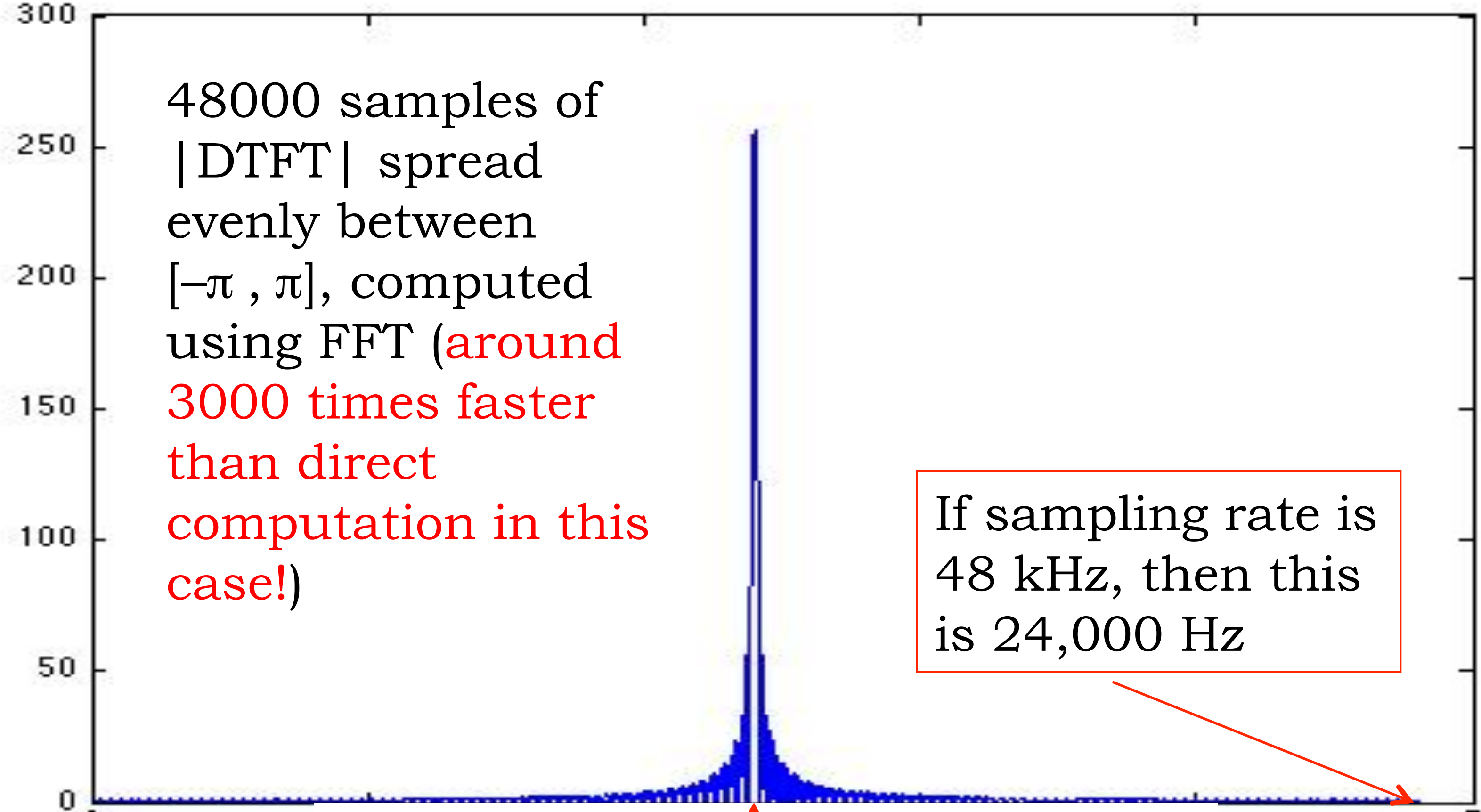


http://cnx.org/content/m0524/latest/

Back to our Audiocom lab example


1
0.8
0.6
0.4
0.2
0
500
1000
1500
2000
2500
3000
3500
4000
4500
x[n]=u[n]-u[n-256]
n

|DTFT| of x[n]=u[n]-u[n-256], rectangular pulse of length 256:



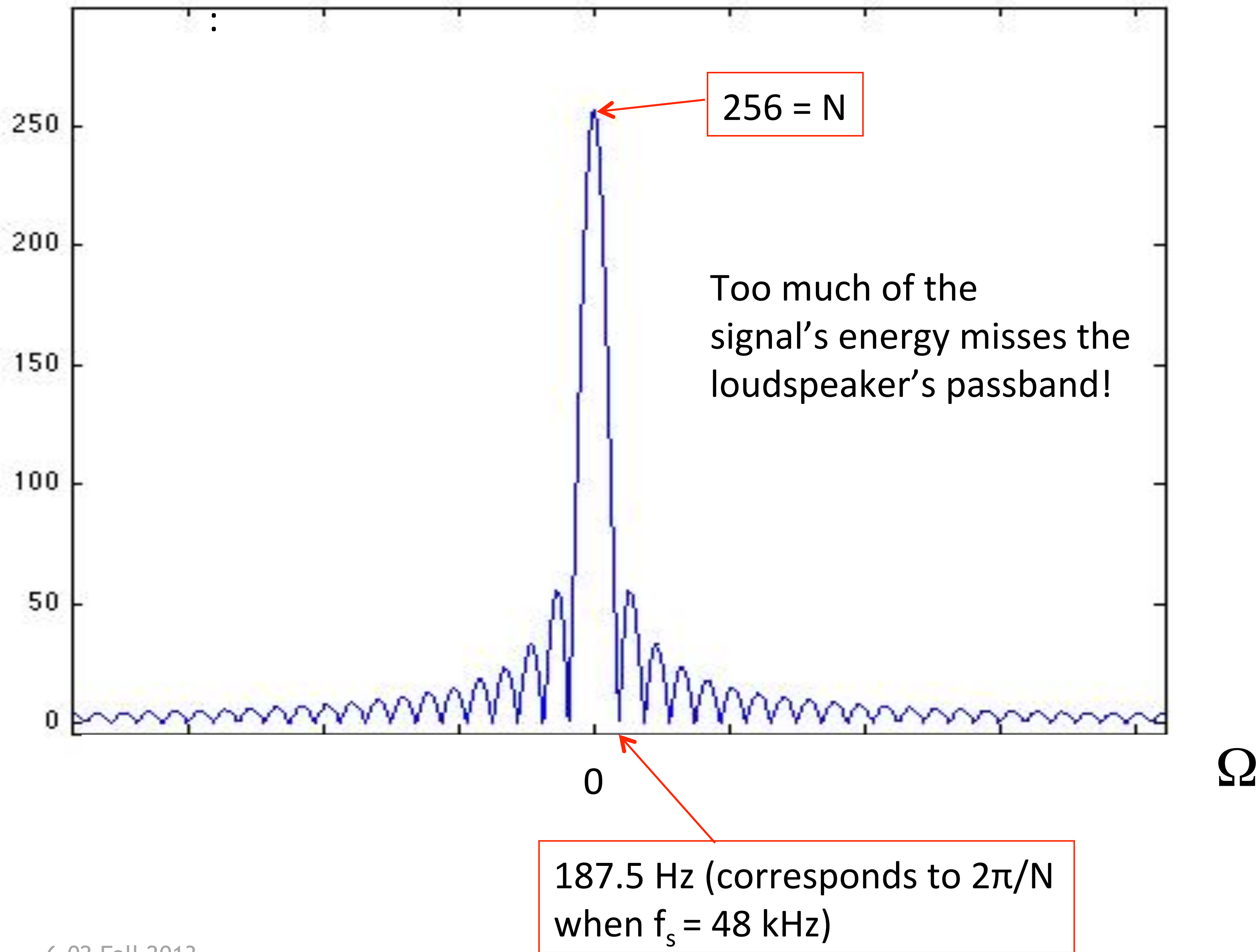
$-\pi$

$\Omega = \pi$

$\mathbf{f = f_s / 2}$

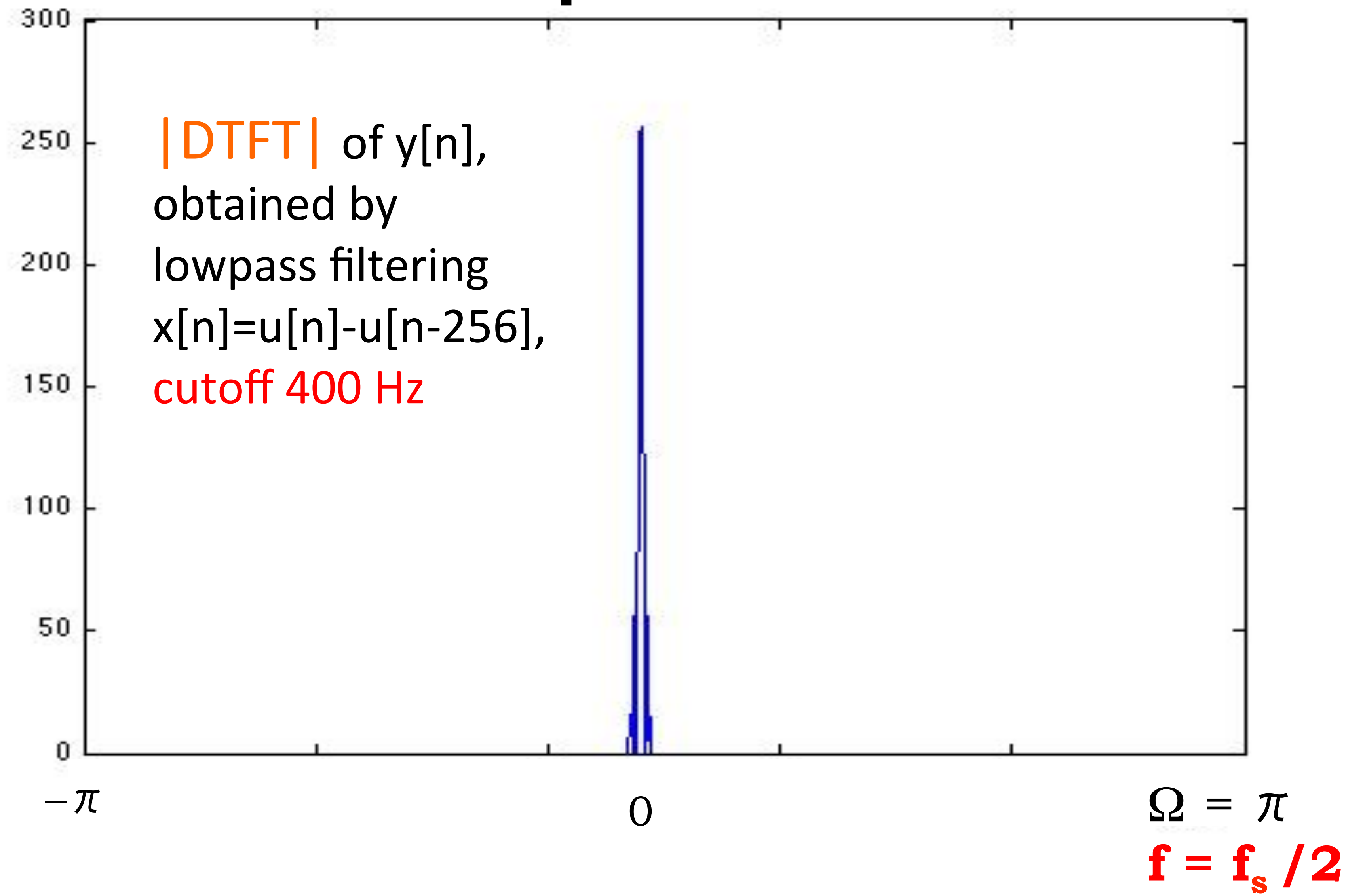
0 rads/sample
0 Hz

Zooming in on |DTFT| of x[n]=u[n]-u[n-256]

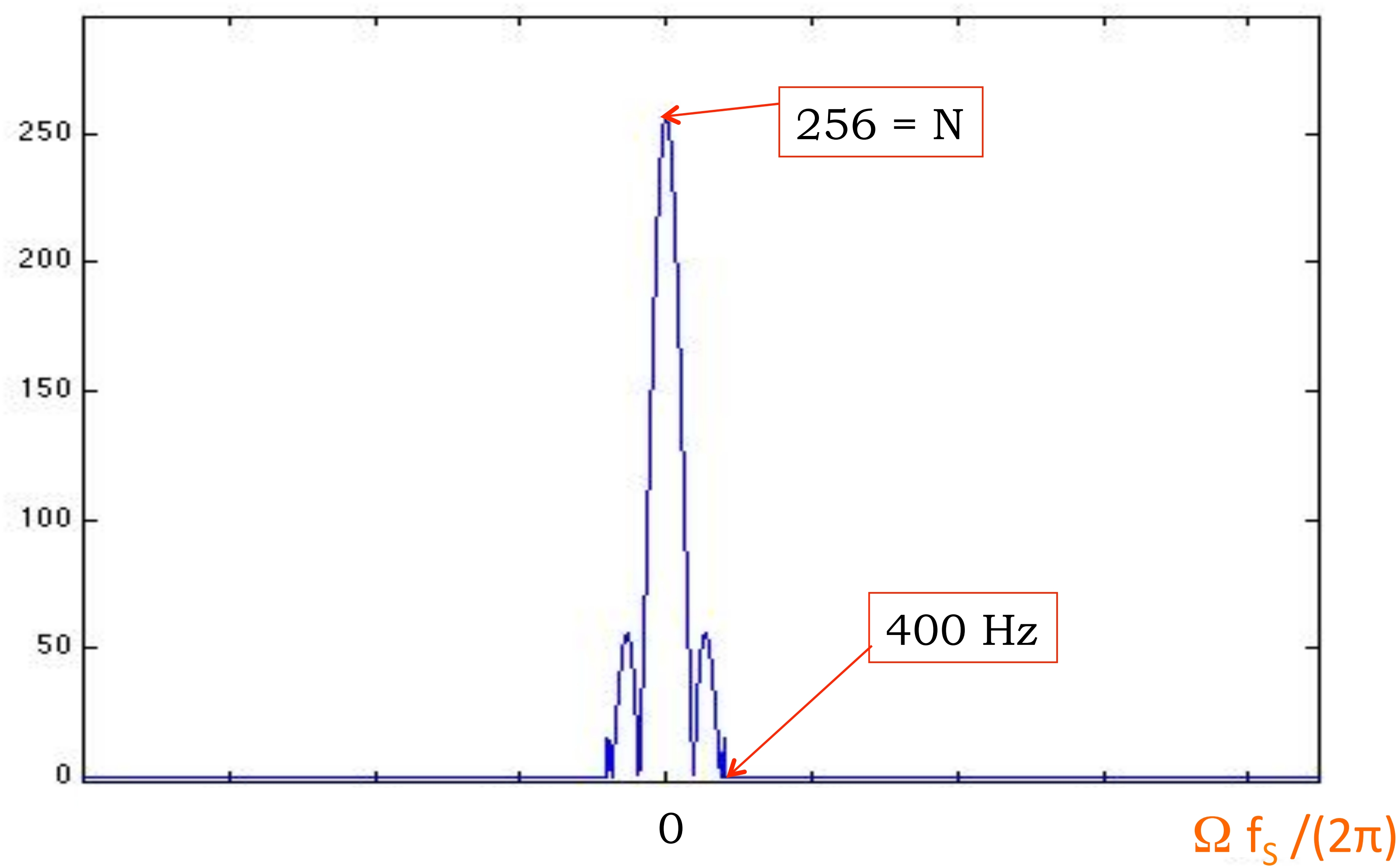


Too much of the signal's energy misses the loudspeaker's passband!

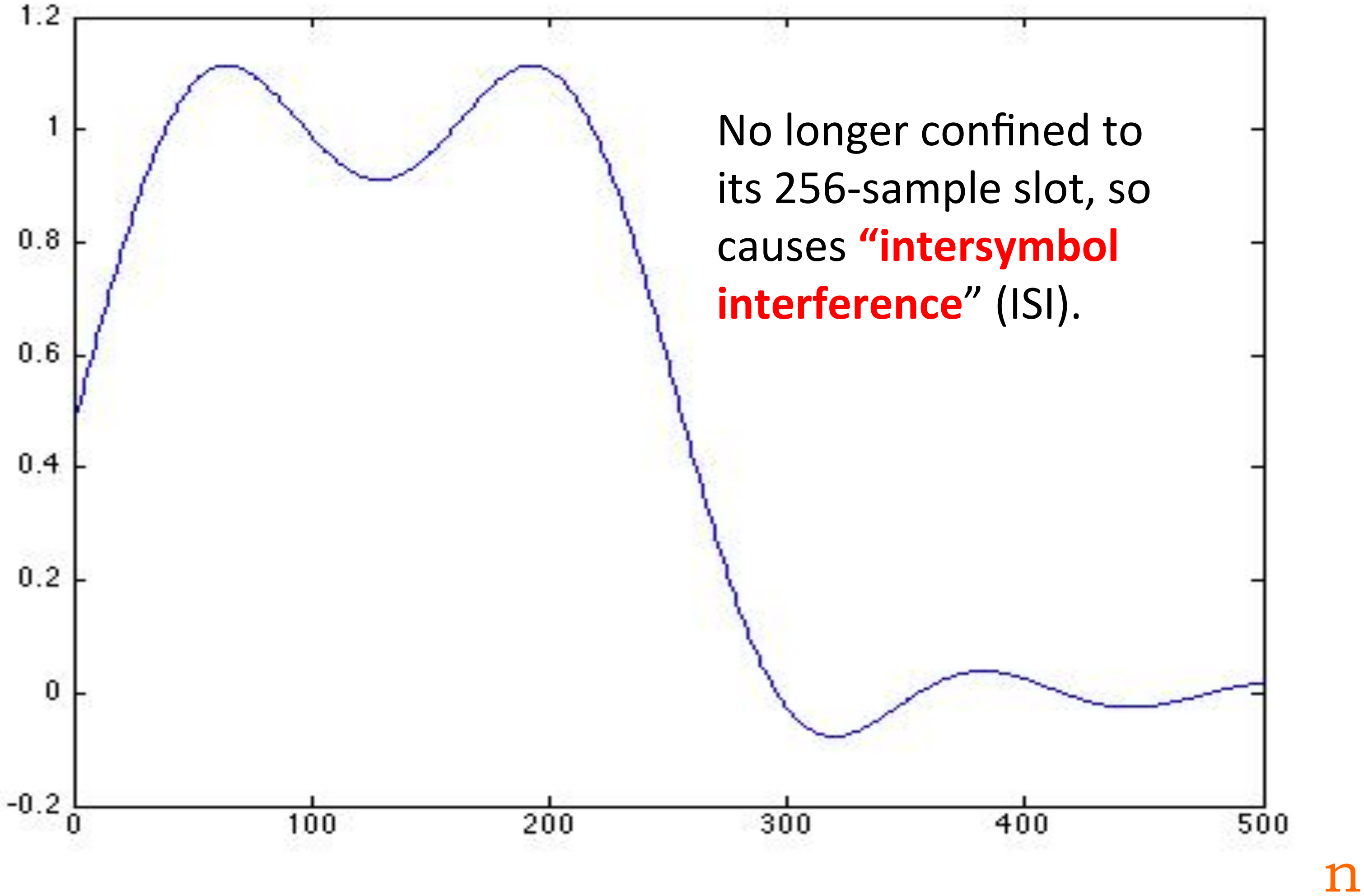
What if we sent this pulse through an ideal lowpass channel?



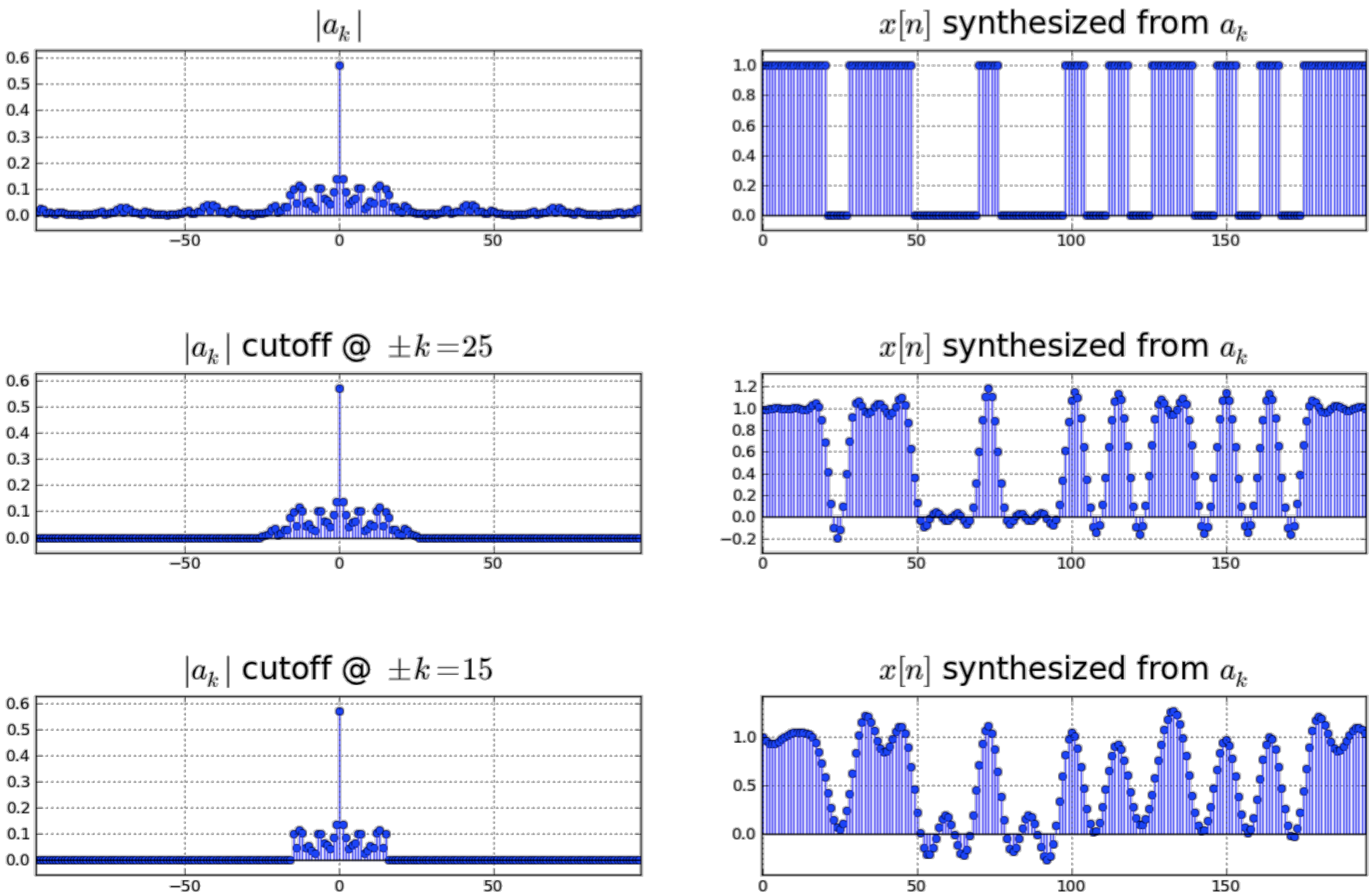
Zooming in on |DTFT| of y[n], obtained by lowpass filtering of x[n]=u[n]-u[n-256], cutoff 400 Hz



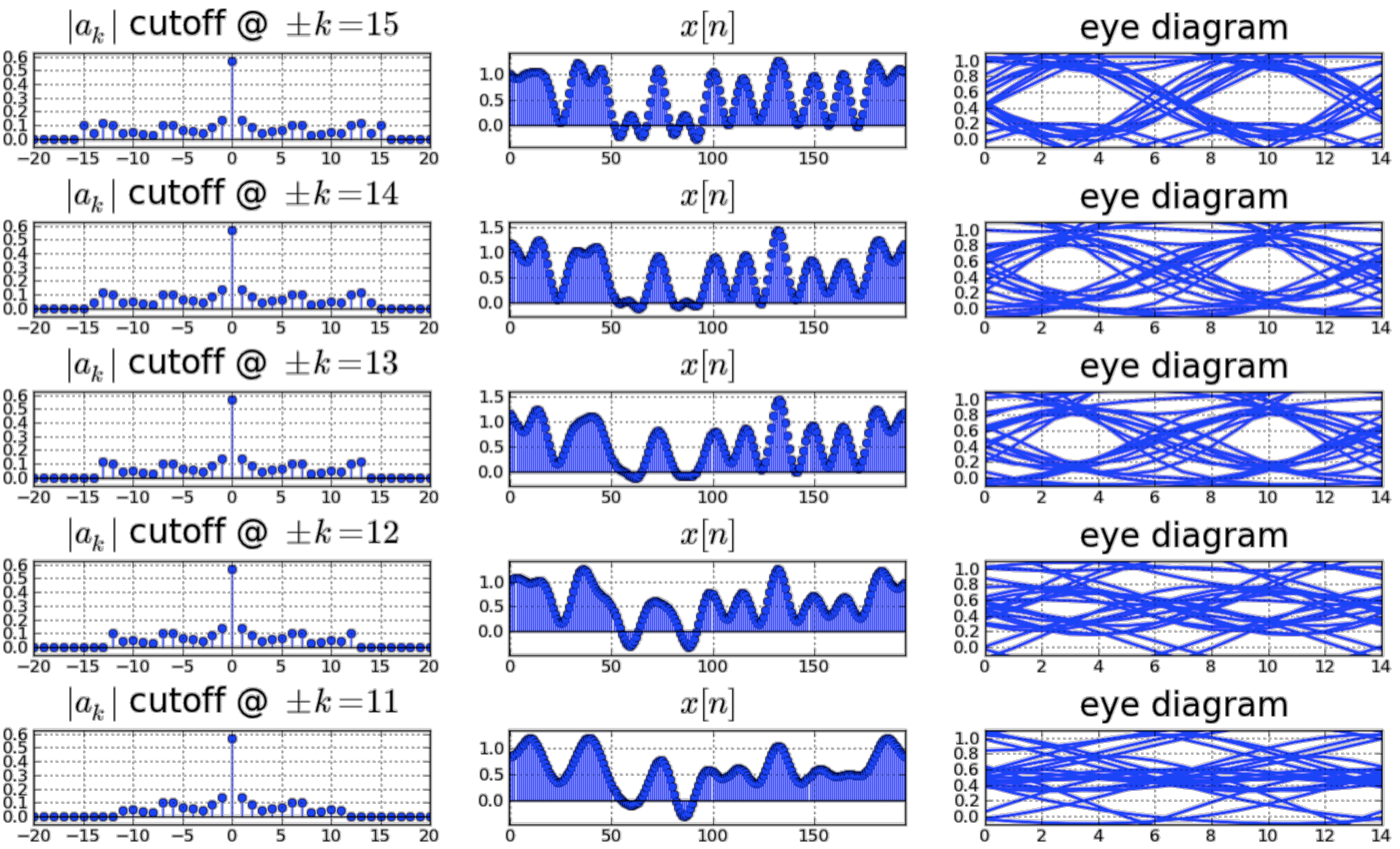
Corresponding output pulse y[n], obtained by lowpass filtering the rectangular pulse x[n]



Effect of Low-Pass Channel



How Low Can We Go?



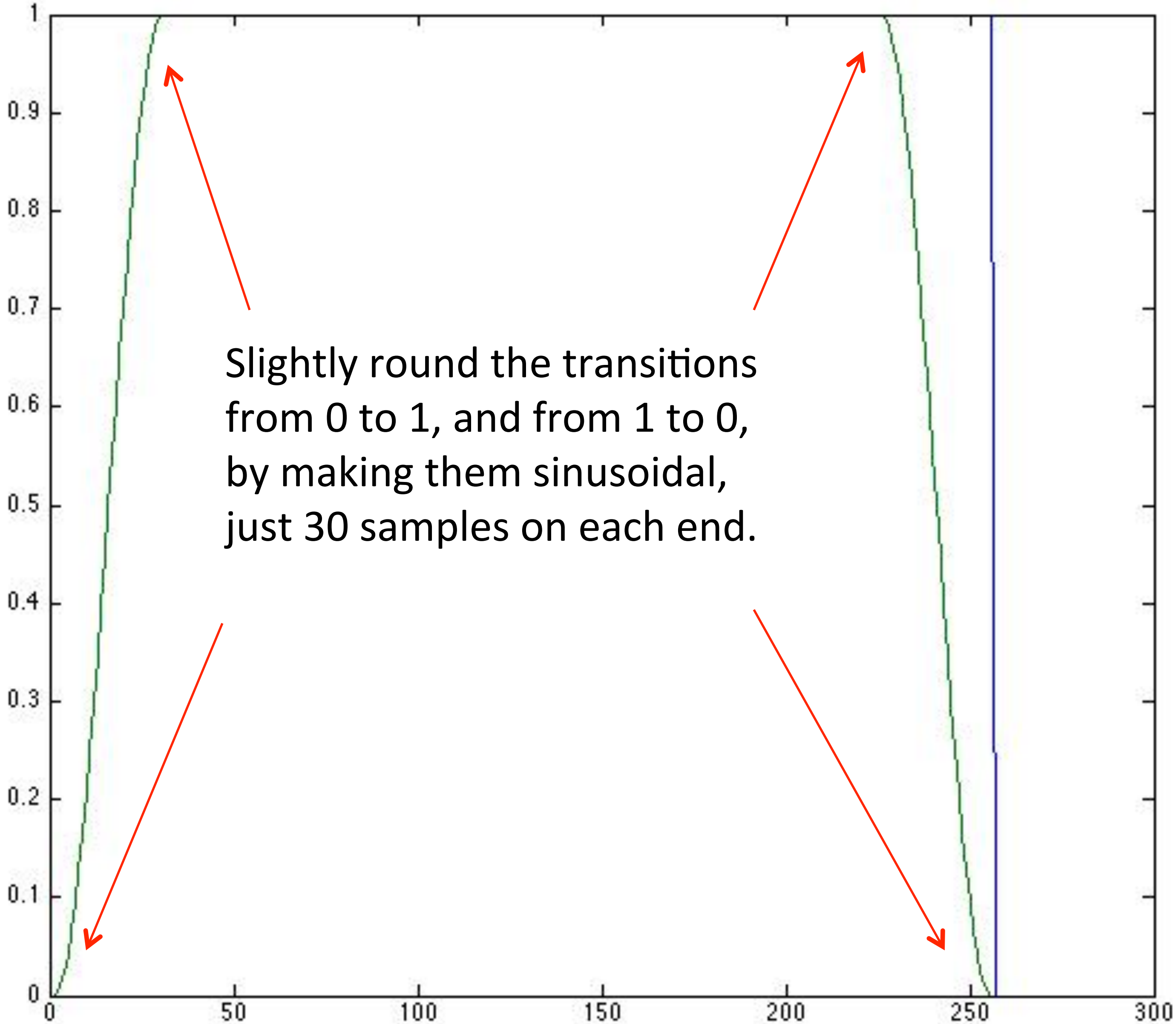
Complementary/dual behavior in time and frequency domains

- Wider in time, narrower in frequency; and vice versa.
  - This is actually the basis of the uncertainty principle in physics!

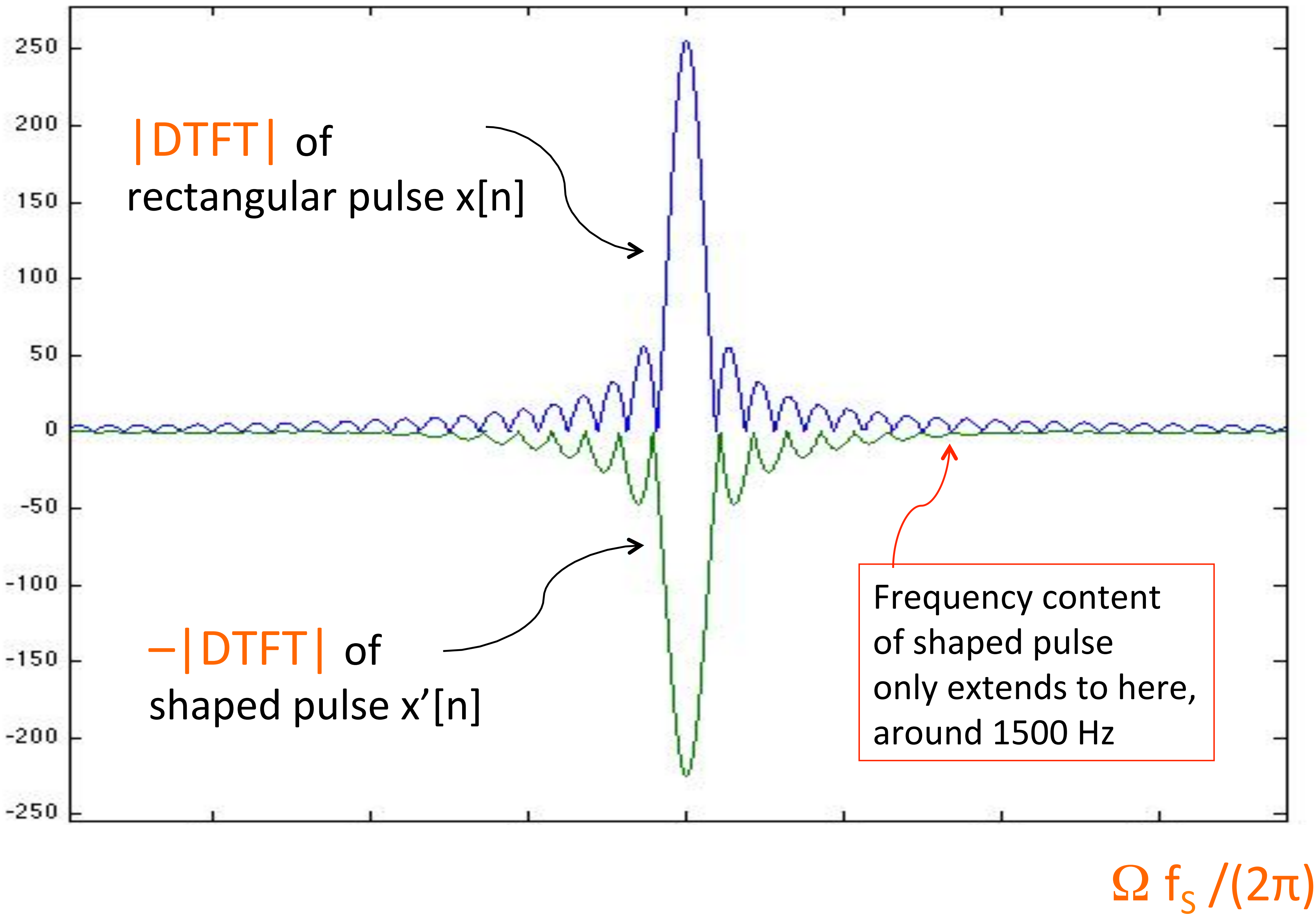
- Smoother in time, sharper in frequency; and vice versa

- Rectangular pulse in time is a (periodic) sinc in frequency, while rectangular pulse in frequency is a sinc in time; etc.

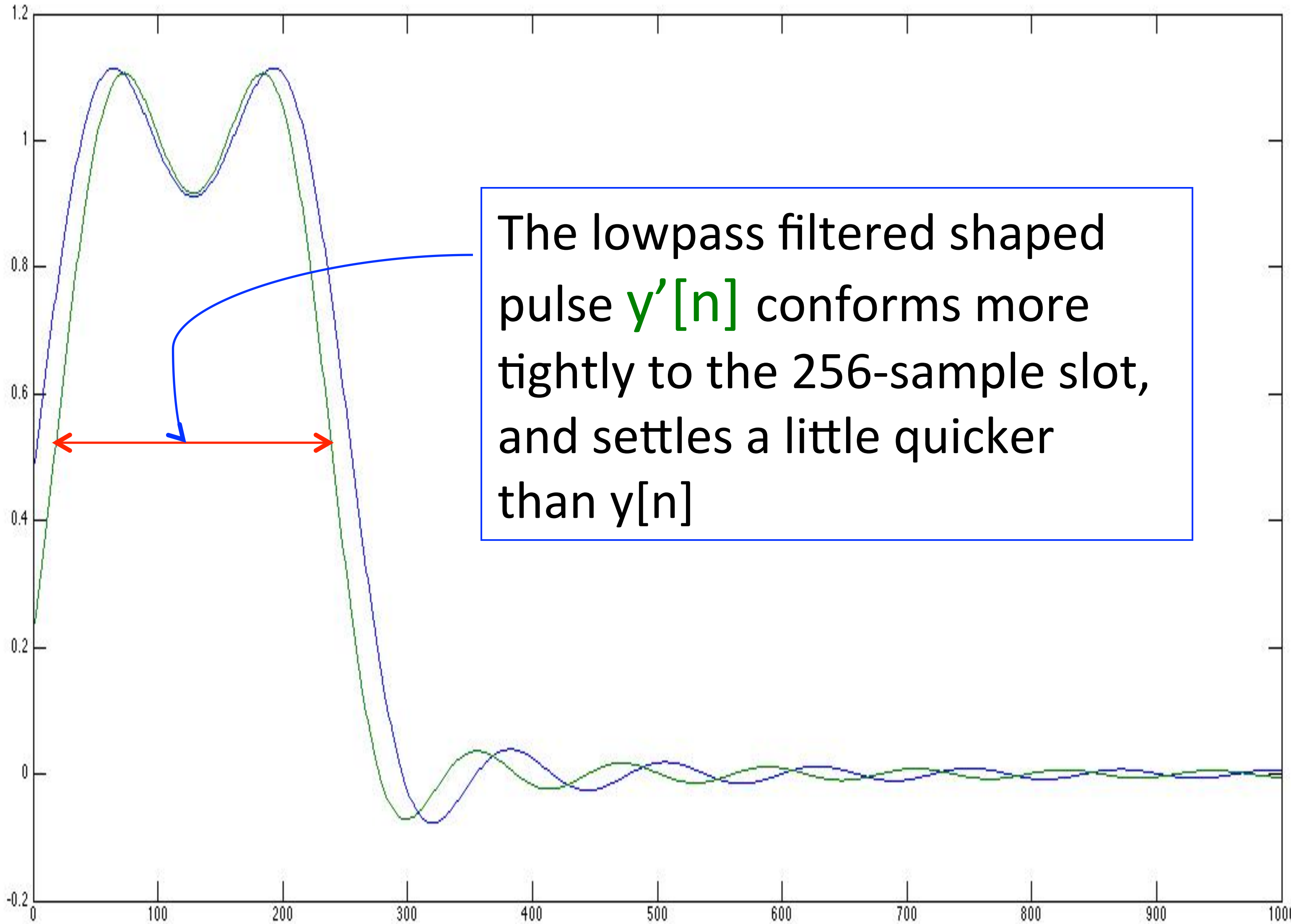
A shaped pulse x’[n] versus a rectangular pulse x[n]



In the spectral domain:



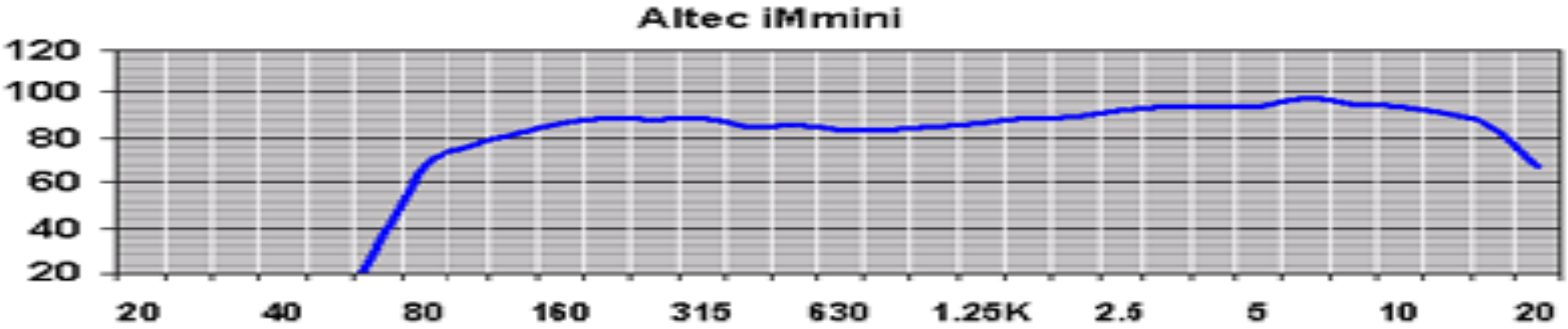
After passing the two pulses through a 400 Hz cutoff lowpass filter:

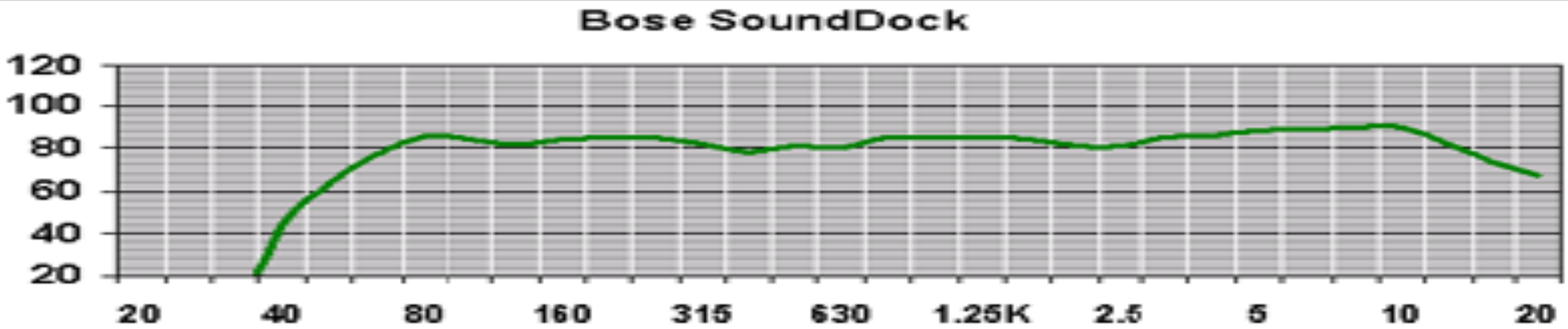


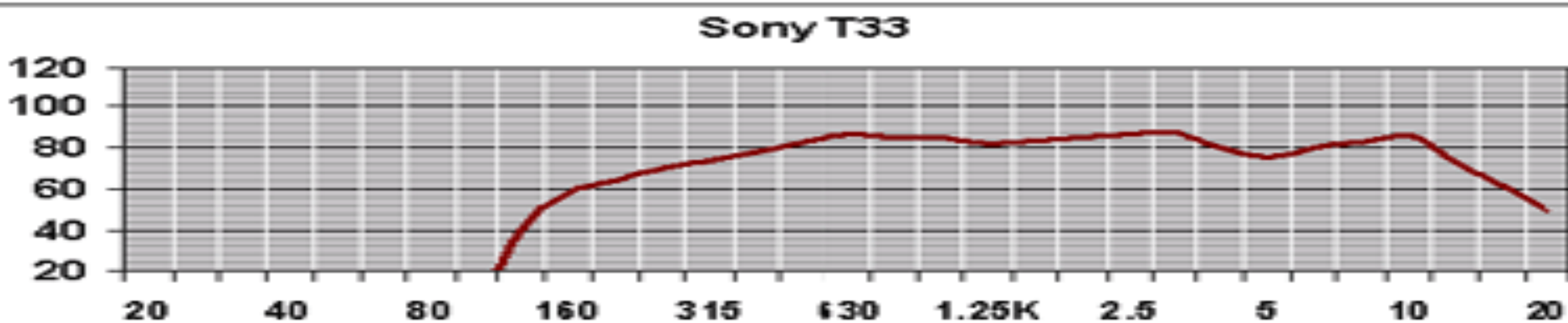
The lowpass filtered shaped pulse y’[n] conforms more tightly to the 256-sample slot, and settles a little quicker than y[n]

But loudspeakers are bandpass,

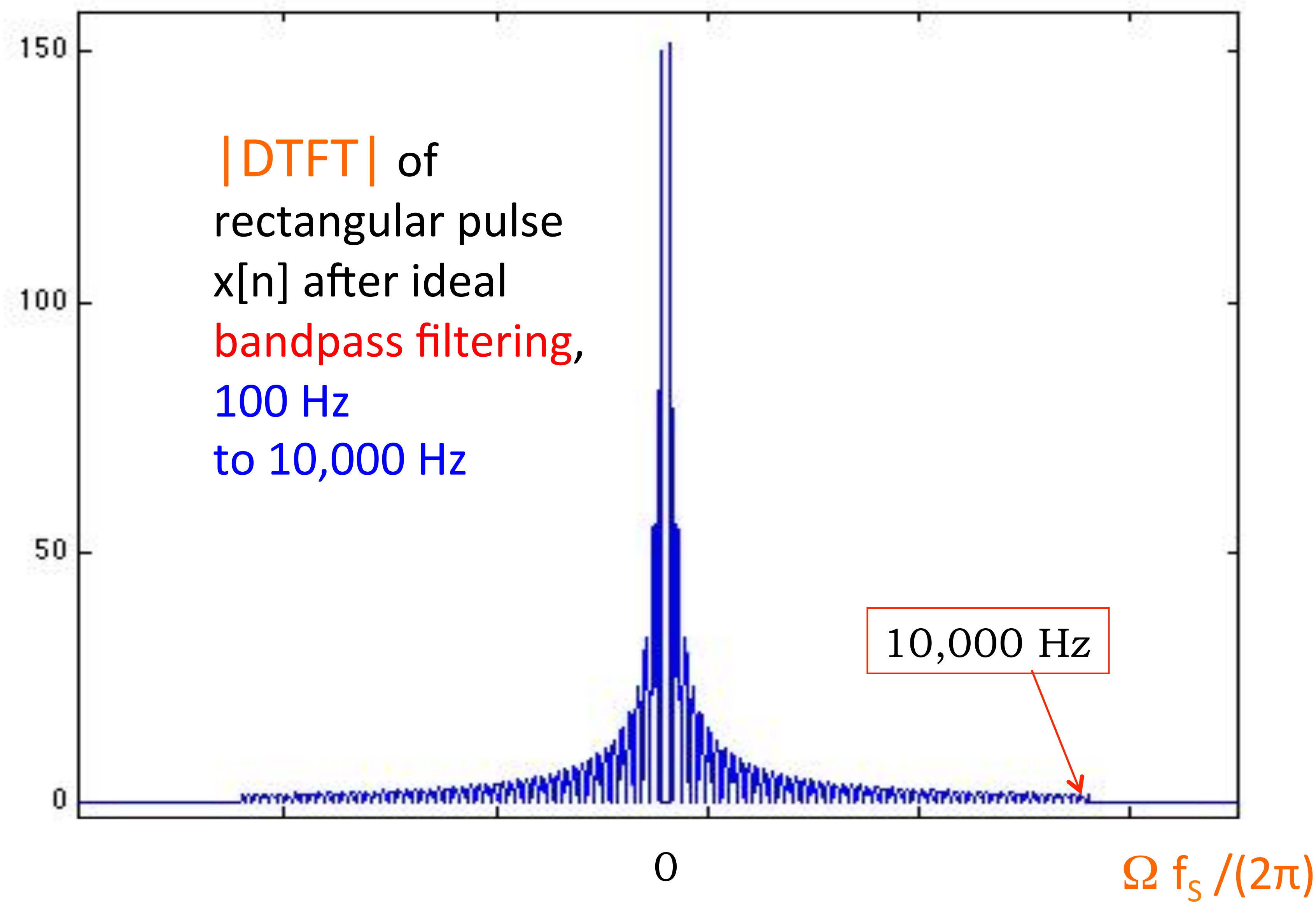
not lowpass



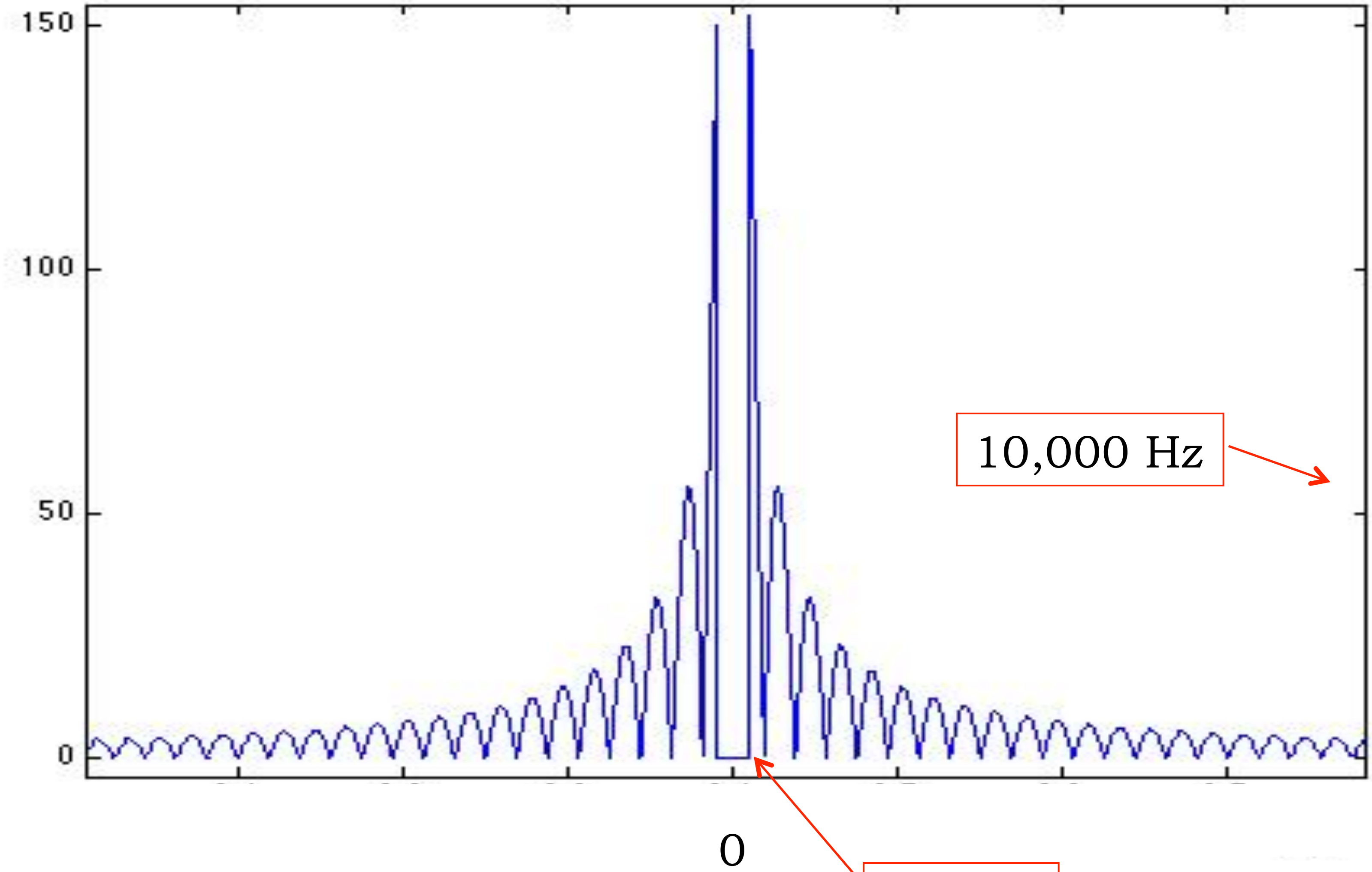




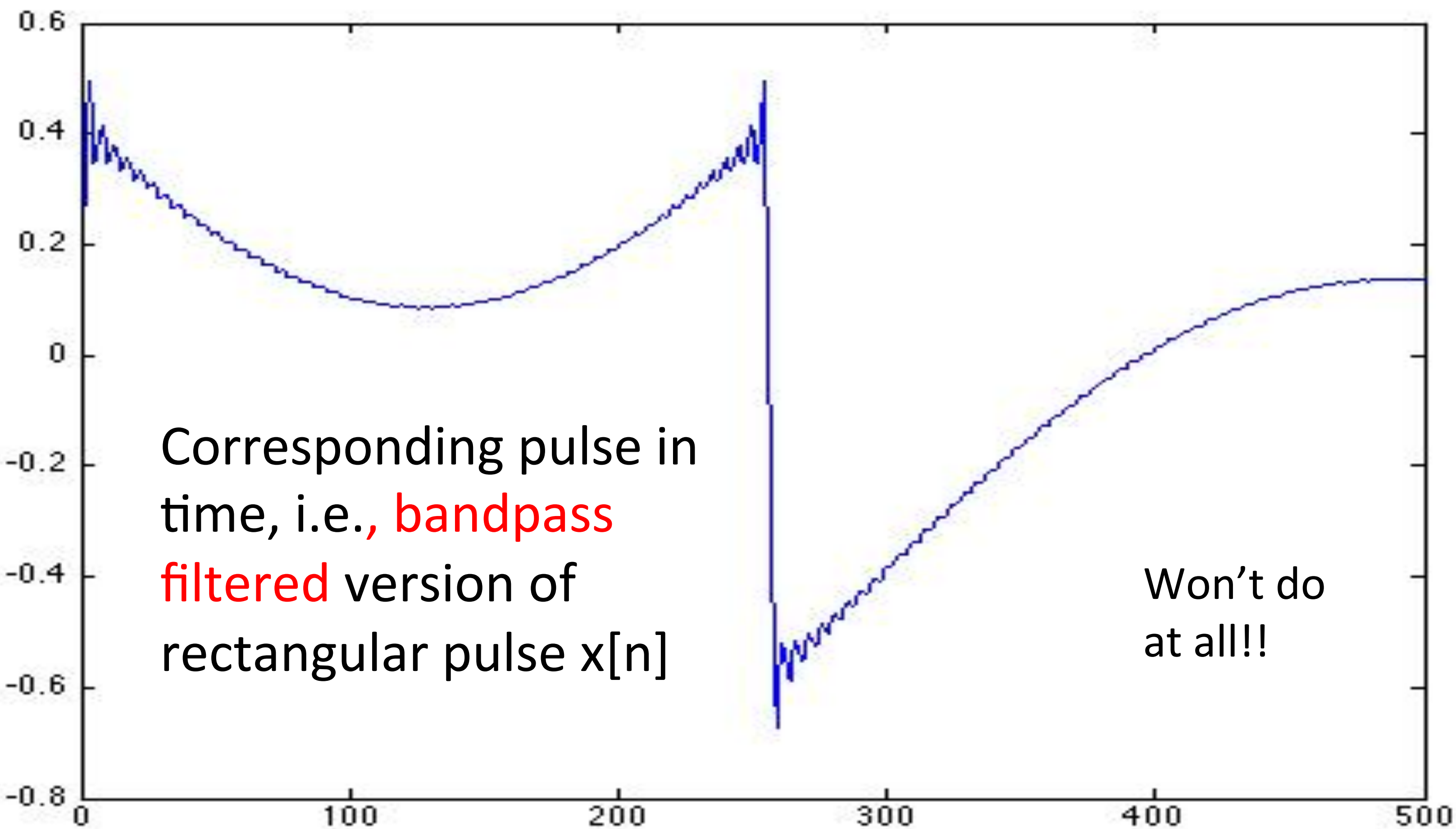
http://www.pcmag.com/article2/0,2817,1769243,00.asp


150
100
50
0
|DTFT| of
rectangular pulse
x[n] after ideal
bandpass filtering,
100 Hz
to 10,000 Hz
10,000 Hz
0
$\Omega f_s/(2\pi)$

Zooming in:



$\Omega\, f_s / (2\pi)$



Corresponding pulse in time, i.e., bandpass filtered version of rectangular pulse x[n]

Won't do at all!!

The Solution: Modulation

- Shift the spectrum of the signal x[n] into the loudspeaker's passband by **modulation!**

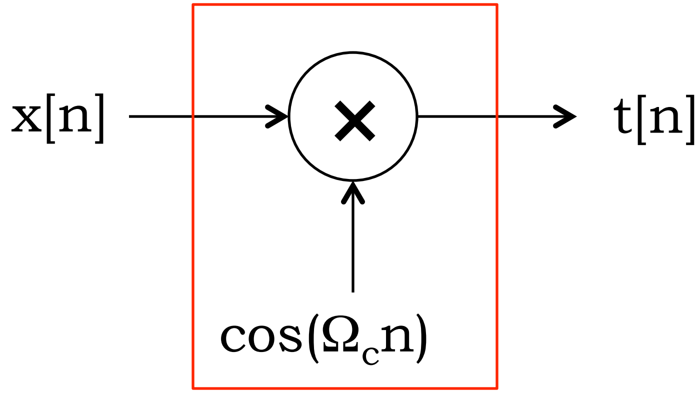
$$x[n]\cos(\Omega_c n) = 0.5x[n](e^{j\Omega_c n} + e^{-j\Omega_c n})$$

$$= \frac{0.5}{2\pi}[\int_{<2\pi>} X(\Omega')e^{j(\Omega'+\Omega_c)n}d\Omega' + \int_{<2\pi>} X(\Omega'')e^{j(\Omega''-\Omega_c)n}d\Omega'']$$

$$= \frac{0.5}{2\pi}[\int_{<2\pi>} X(\Omega-\Omega_c)e^{j\Omega n}d\Omega + \int_{<2\pi>} X(\Omega+\Omega_c)e^{j\Omega n}d\Omega]$$

Spectrum of modulated signal comprises half-height replications of X(Ω) centered as $\pm\Omega_c$ (i.e., plus and minus the carrier frequency). So choose carrier frequency comfortably in the passband, leaving room around it for the spectrum of x[n].

Is Modulation Linear? Time-Invariant?

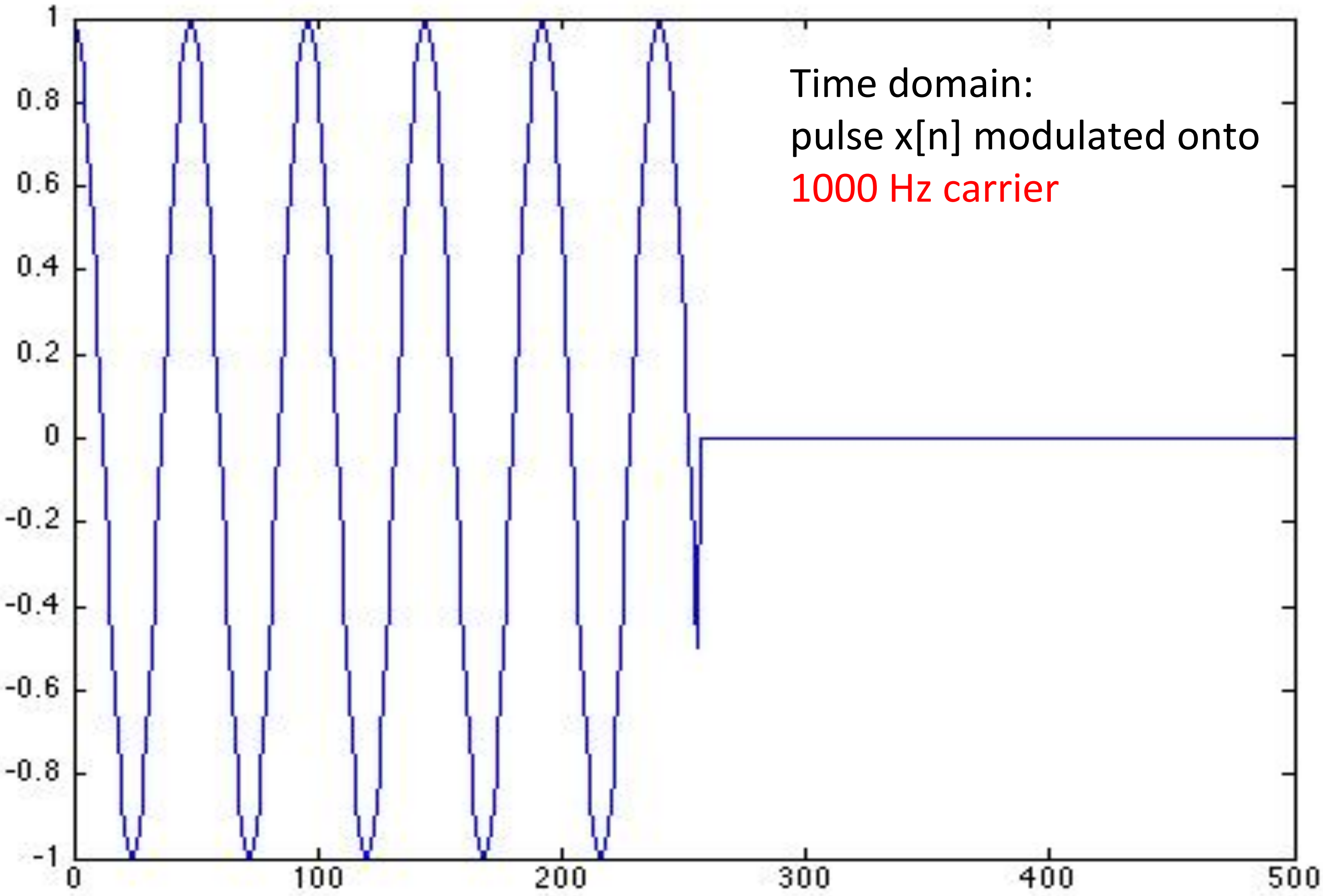


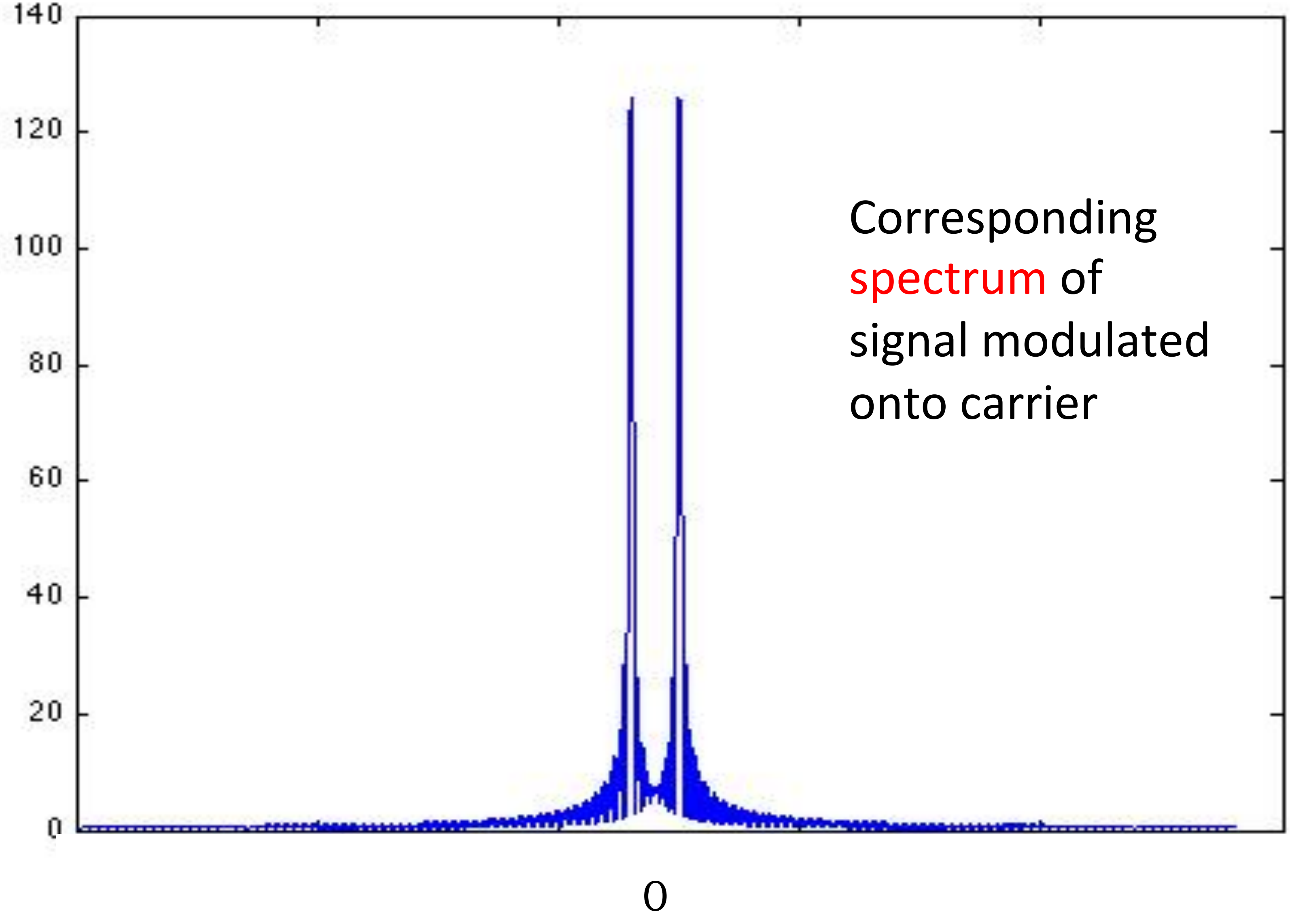
… as a system that takes input x[n] and produces output t[n] for transmission?

Yes, linear!

No, not time-invariant!

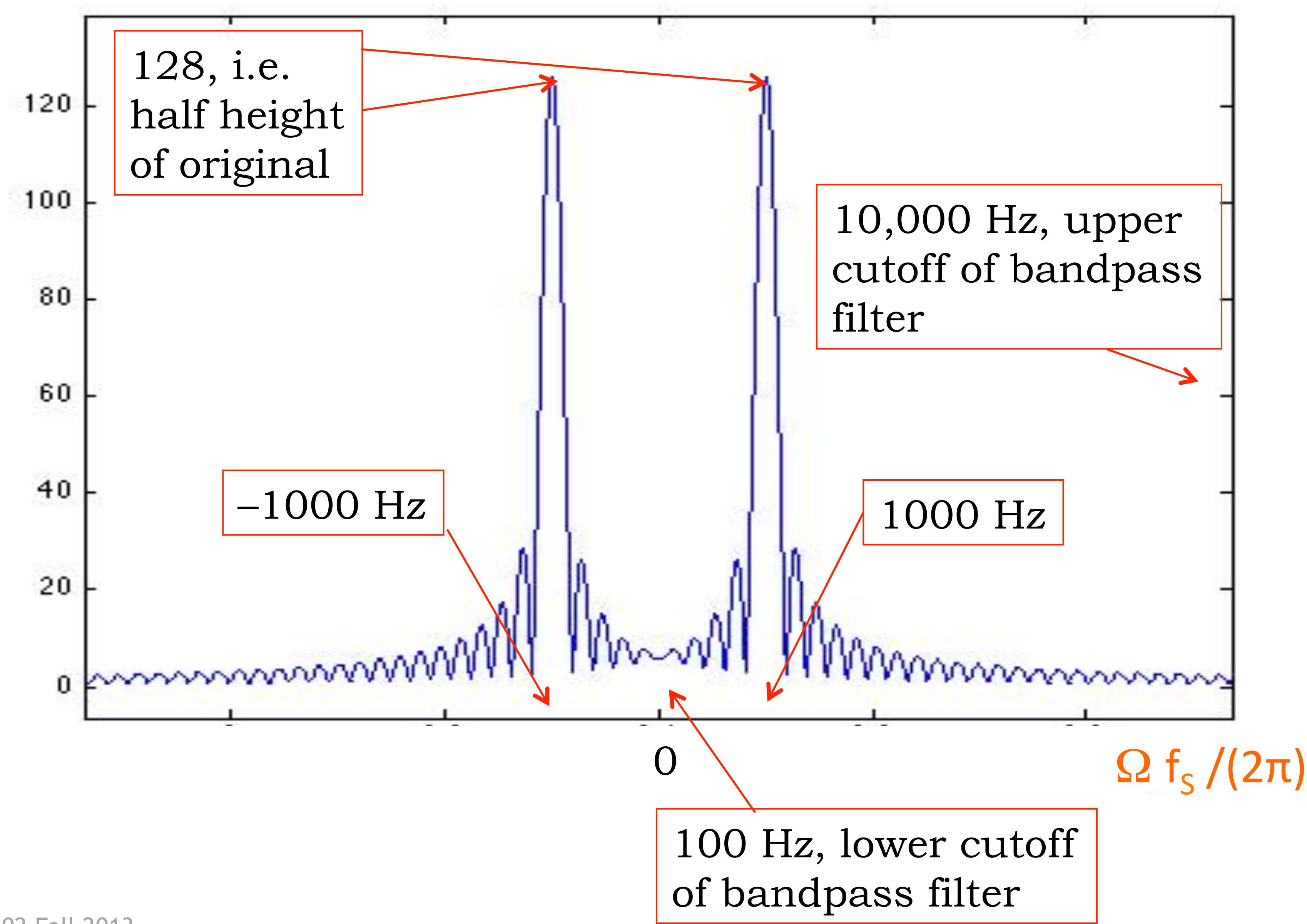
So for our rectangular pulse example:



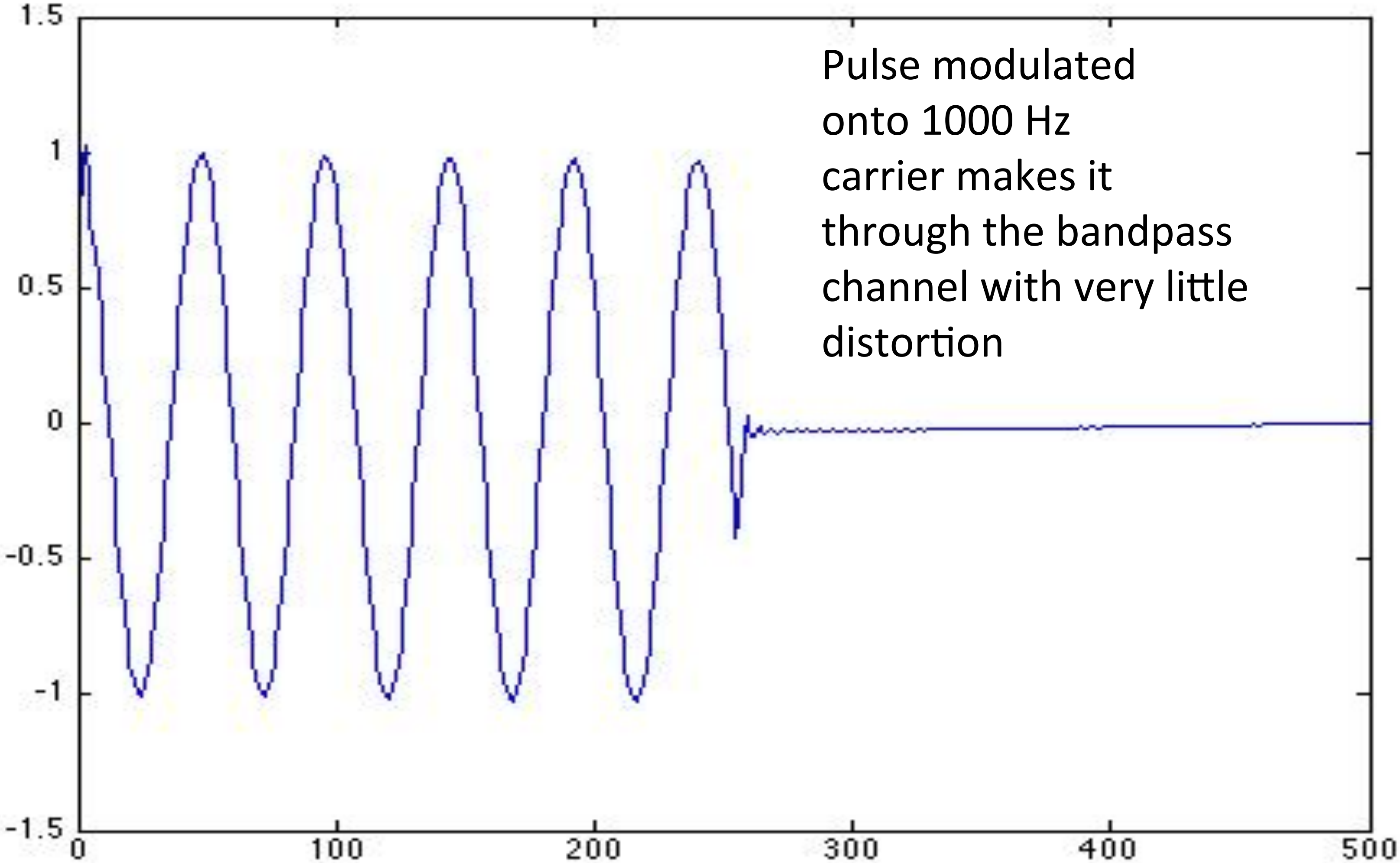


Corresponding spectrum of signal modulated onto carrier

Zooming in:



120
100
80
60
40
20
0
128, i.e. half height of original
10,000 Hz, upper cutoff of bandpass filter
–1000 Hz
1000 Hz
0
100 Hz, lower cutoff of bandpass filter
$\Omega f_s / (2\pi)$


1.5
1
0.5
0
-0.5
-1
-1.5
0
100
200
300
400
500
Pulse modulated onto 1000 Hz carrier makes it through the bandpass channel with very little distortion

At the Receiver: Demodulation

- In principle, this is (as easy as) modulation again:

If the received signal is

$$r[n] = x[n]\cos(\Omega_c n),$$

then simply compute

$$\begin{aligned} d[n] &= r[n]\cos(\Omega_c n) \\ &= x[n]\cos^2(\Omega_c n) \\ &= 0.5\,\{x[n] + x[n]\cos(2\Omega_c n)\} \end{aligned}$$

- What does the spectrum of d[n] look like?
- What constraint on the bandwidth of x[n] is needed for perfect recovery of x[n] by lowpass filtering of d[n]?