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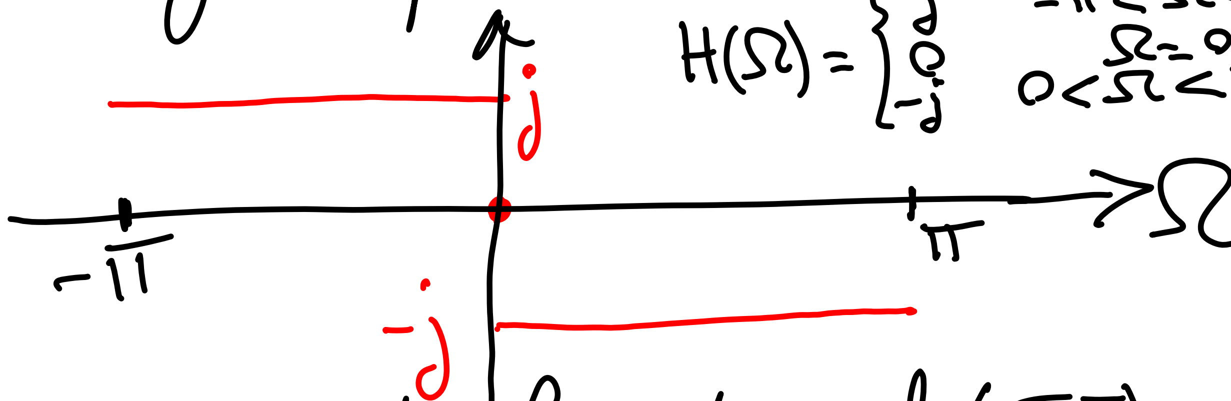
(Thanks to Prof. Victor Zue for inspiring this recitation)

Hilbert Transform & Amplitude Modulation

Consider the DT-LTI filter whose frequency response is shown below:

$$H(\Omega) = \begin{cases} j & -\pi < \Omega < 0 \\ 0 & \Omega = 0 \\ -j & 0 < \Omega < \pi \end{cases}$$



Replicates outside the interval $(-\pi, \pi)$ with period 2π.

a) What can you infer about the structure of h, the impulse response of the filter?

$H(-\Omega) = -H(\Omega)$ and $H(\Omega)$ is imaginary

$\Rightarrow$ h is an odd fctn $\left(h[-n] = -h[n]\right)$

Right away we know that $h[0] = 0$

Alternative View:

We know that we can decompose any signal into even and odd components, as follows:

$$h[n] = h_e[n] + h_o[n], \text{ where}$$

$$h_e[n] = \frac{h[n] + h[-n]}{2} \quad \text{even}$$

and $$h_o[n] = \frac{h[n] - h[-n]}{2} \quad \text{odd}$$

We learned in an earlier recitation that

$$h_e[n] \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{Re}\{H(\Omega)\}$$

and $$h_o[n] \overset{\mathcal{F}}{\longleftrightarrow} j\operatorname{Im}\{H(\Omega)\}$$

For this Hilbert transform filter

$$\operatorname{Re}\{H(\Omega)\} = 0 \implies h_e[n] = 0 \implies$$

$$h[n] = h_o[n]$$ That is, h is an odd function.

(b) Can this filter be stable?

No! $H(\Omega)$ is discontinuous at $\Omega = 0$.

In a previous recitation we explained, without proof, that this indicates two things:

(i) h is not absolutely summable (and hence not stable)

$$\sum_n |h[n]| \text{ doesn't converge}$$

and (ii) h is square-summable (i.e., it has finite energy)

$$\sum_n |h[n]|^2 < \infty$$

We'll look to verify these two aspects once we determine $h[n]$ in the next part.

(c) Determine, and provide a well-labeled plot of, $h[n]$.

$$h[n] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\Omega)e^{j\Omega n}\,d\Omega$$

Case I: $n=0$

$$h[0] = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} H(\Omega)\,d\Omega}_{=0 \text{ b/c } H(\Omega) \text{ is odd.}}$$

$\Rightarrow h[0] = 0$, as we had predicted earlier.

Case II: $n \neq 0$

$$h[n] = \frac{1}{2\pi}\left(j\int_{-\pi}^{0} e^{j\Omega n}\,d\Omega - j\int_{0}^{\pi} e^{j\Omega n}\,d\Omega\right)$$

$$= \frac{\cancel{j}}{2\pi}\left(\frac{e^{j\Omega n}}{\cancel{j}n}\Bigg|_{-\pi}^{0} - \frac{e^{j\Omega n}}{\cancel{j}n}\Bigg|_{0}^{\pi}\right)$$

$$= \frac{1}{2\pi n}\left[\left(1-e^{-j\pi n}\right) - \left(e^{j\pi n}-1\right)\right]$$

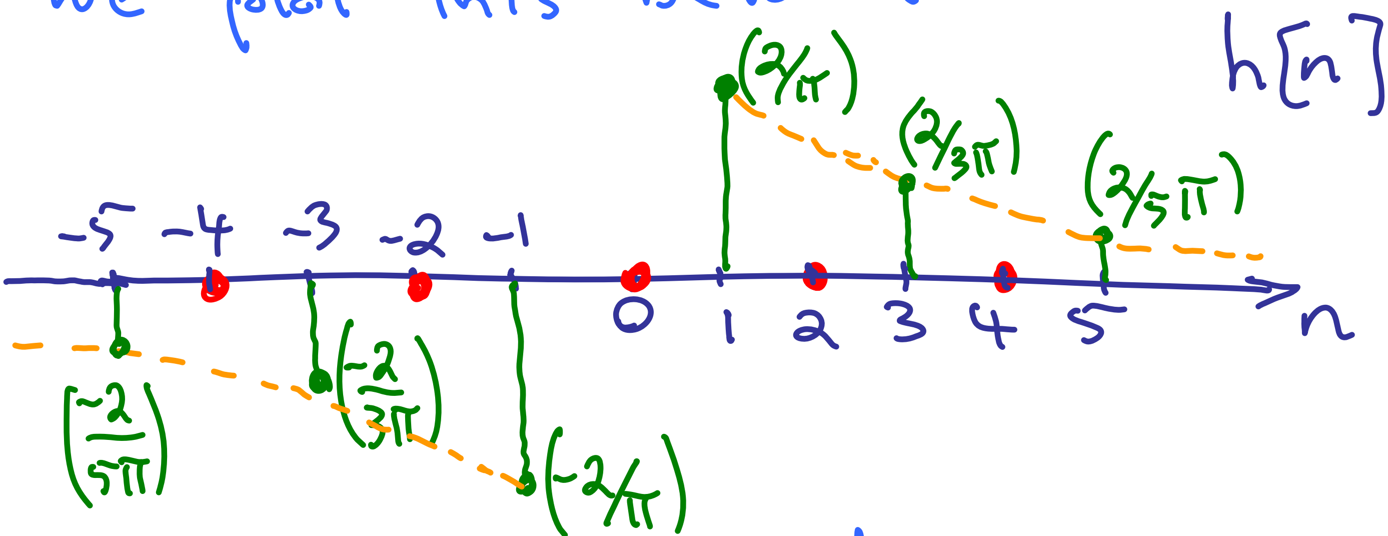
$$= \frac{2-\left(e^{j\pi n}+e^{-j\pi n}\right)}{2\pi n} = \frac{2-2\cos(\pi n)}{2\pi n}$$

$$= \frac{1-\cos\pi n}{\pi n} = \frac{1-(-1)^n}{\pi n} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$$

Summarizing the two cases, we have:

$$h[n] = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd} \end{cases}$$

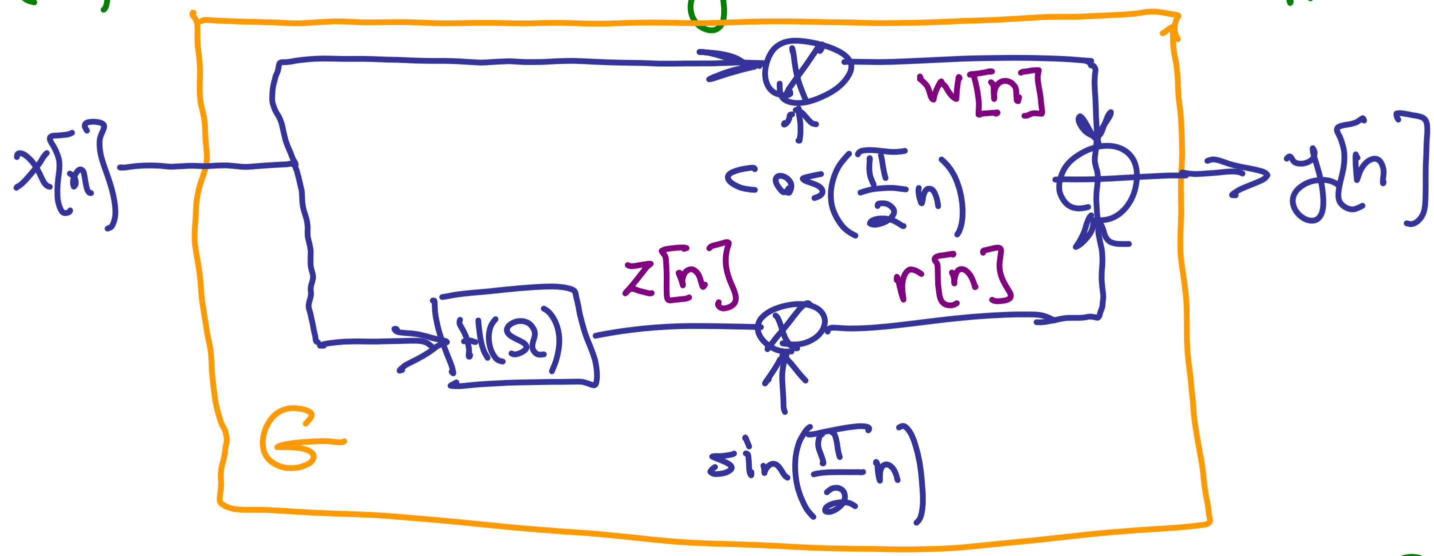
We plot this below:



Note that the dotted orange lines indicate the decay on each side. As the expression for $h[n]$ shows, the decay is on the order of $\frac{1}{n}$, which is not fast enough for absolute convergence.

However, $|h[n]|^2$ decays on the order of $\frac{1}{n^2}$, which is fast enough for convergence $\Rightarrow$ h has finite energy, as discussed earlier.

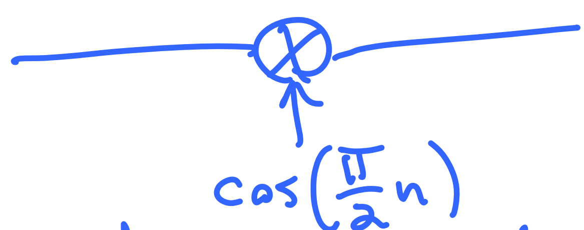
(d) Consider the system shown below:



(i) Can the system G be LTI?

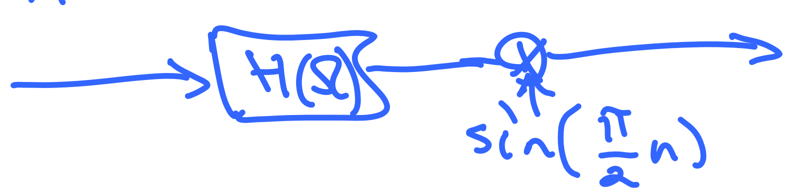
G is linear, but not time invariant.

G is a parallel interconnection of two branches. The upper branch



is modulation, so it's linear, but not time invariant.

The lower branch is a cascade of an LTI filter and modulation

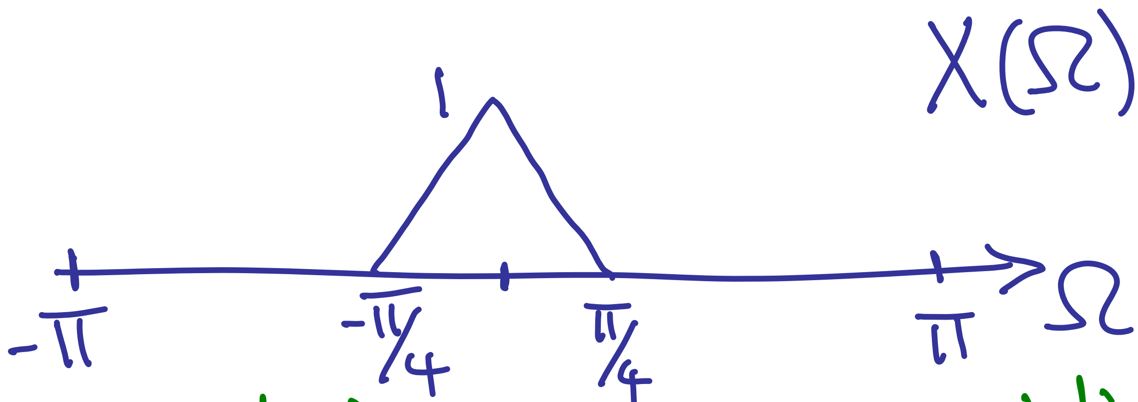


So, the lower branch is linear, but not time invariant.

The overall structure is linear, but not time invariant.

We expect the composite system G to create new frequencies that are not present in the input x. In fact, this is the case, as we'll see below.

(e) Consider the input signal x whose spectrum is shown below:



It's understood that $X(\Omega)$ is 2π-periodic in Ω.

(i) Say as much as you can about the time-domain structure of the input signal x.

x is real and even because $X(\Omega)$ is real and even.

(ii) Provide well-labeled plots of $W(\Omega)$, $Z(\Omega)$, $R(\Omega)$, and $Y(\Omega)$, the spectra of the signals w, z, r, and y, respectively.

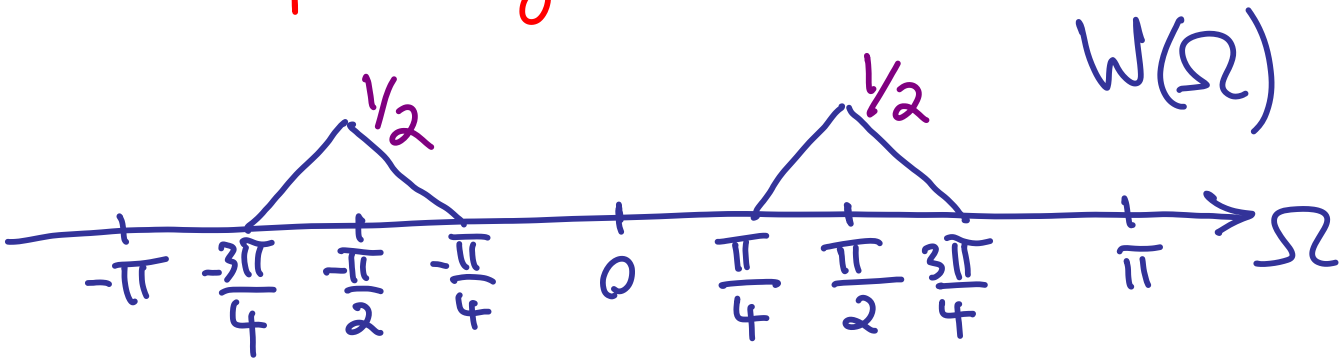
$$w[n] = x[n]\cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}x[n]\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$$

First, notice that x is real & even, as is $\cos\left(\frac{\pi}{2}n\right)$. So w is a real and even signal $\Rightarrow W(\Omega)$ is real and even. In fact, this is the case.

$$W(\Omega) = \frac{1}{2}\left[X\left(\Omega - \frac{\pi}{2}\right) + X\left(\Omega + \frac{\pi}{2}\right)\right]$$

due to multiplication by $e^{j\frac{\pi}{2}n}$

due to multiplication by $e^{-j\frac{\pi}{2}n}$

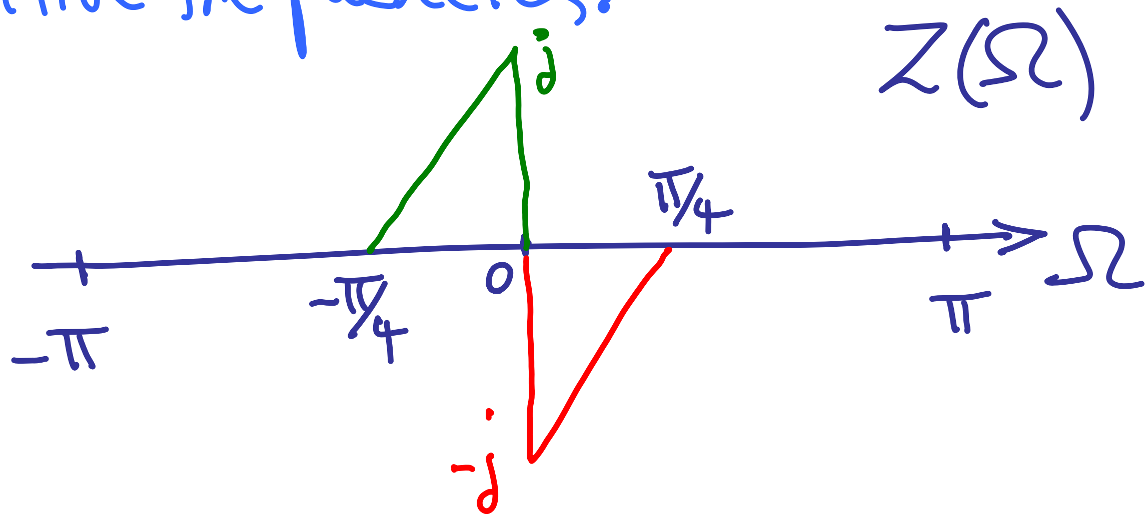


$$Z(\Omega) = X(\Omega)H(\Omega)$$

$\left.\begin{array}{l} x \text{ is real \& even} \\ h \text{ is real \& odd} \end{array}\right\} \Rightarrow z = x * h$ is real & odd

$\Rightarrow Z(\Omega)$ is imaginary & odd.

To plot $Z(\Omega)$ we multiply by j the portion of $X(\Omega)$ corresponding to negative frequencies, and we multiply by $-j$ the portion corresponding to positive frequencies.



$r[n] = z[n] \sin\left(\frac{\pi}{2} n\right)$

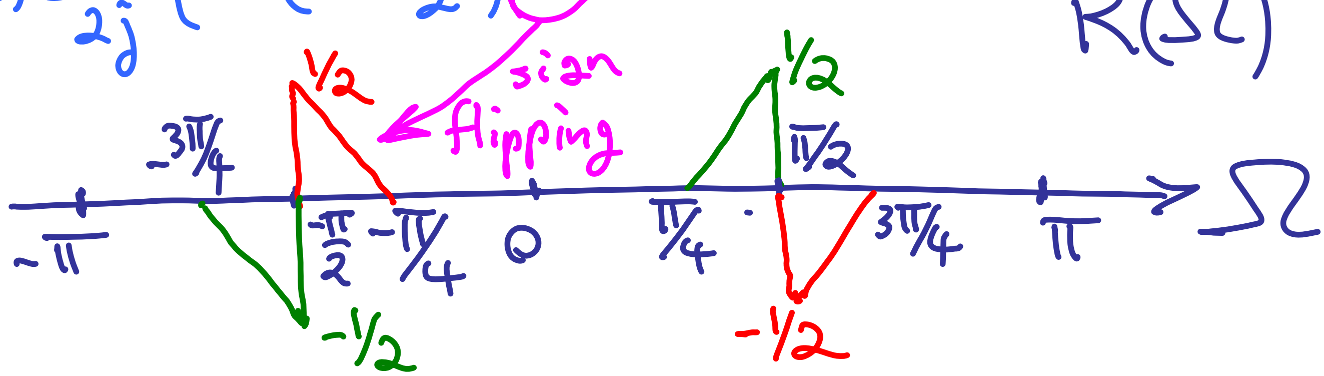
$\left.\begin{array}{l} z \text{ is real \& odd} \\ \sin(\cdot) \text{ is real \& odd} \end{array}\right\} \Rightarrow r \text{ is real \& even}$

$\Rightarrow$ we expect R to be real and even. In fact, this is the case.

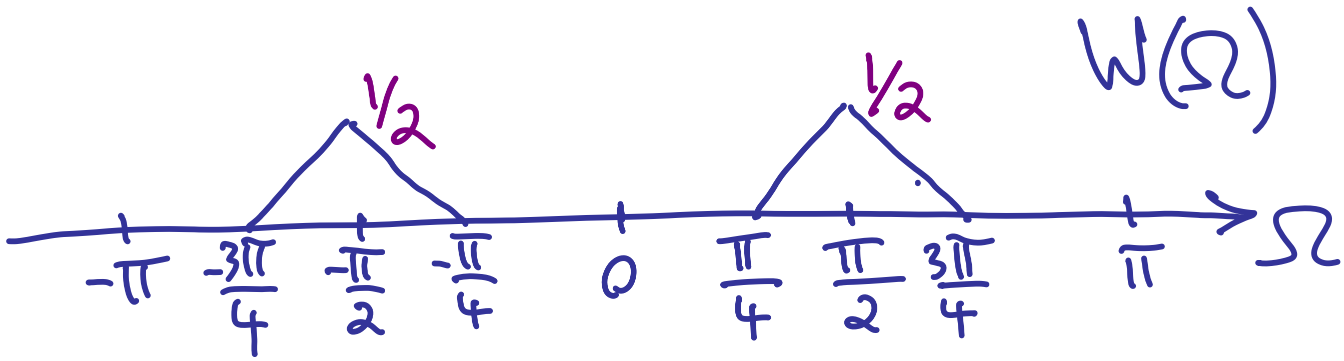
$$r[n] = \frac{1}{2j} z[n] e^{j\frac{\pi}{2}n} - \frac{1}{2j} z[n] e^{-j\frac{\pi}{2}n} \Longrightarrow$$

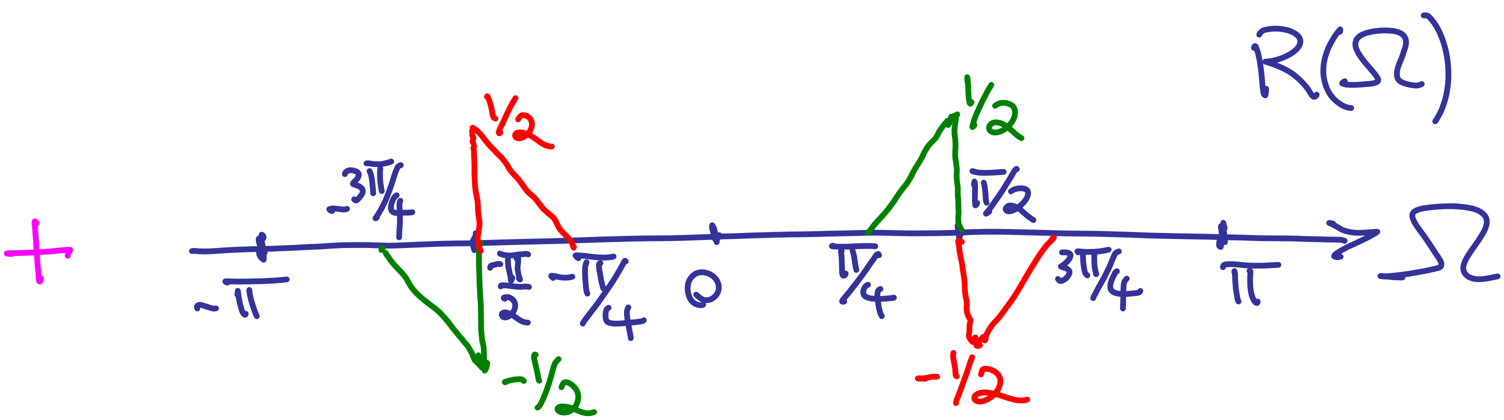
$$R(\Omega) = \frac{1}{2j}\left[Z\left(\Omega - \frac{\pi}{2}\right) - Z\left(\Omega + \frac{\pi}{2}\right)\right]$$

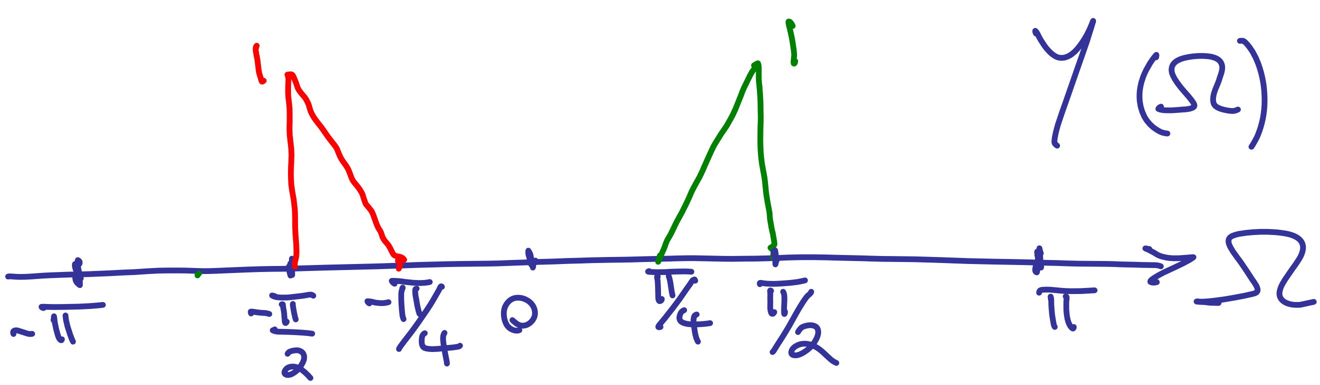
sign flipping



The plot for $Y(\Omega)$ is now a straightforward addition of the plots for $W(\Omega)$ and $R(\Omega)$. Portions of the spectrum cancel.







$y[n] = w[n] + r[n]$ $\Longrightarrow$ y real & even

real & even   real & even $\Downarrow$

$Y(\Omega)$ real & even

Notice: New frequencies have been created, as expected!