Name: ______________________________

Dept. of Electrical Engineering & Computer Science
Massachusetts Institute of Technology

6.02 Fall 2011

Quiz 2

November 17, 2011

"×" your section	Section	Time	Room	Recitation Instructor
☐	1	10-11	36-112	Paul Ampadu
☐	2	11-12	36-112	Sidhant Misra
☐	3	12-1	36-112	Sidhant Misra
☐	4	1-2	36-112	Paul Ampadu
☐	5	2-3	26-168	Karl Berggren
☐	6	3-4	26-168	Karl Berggren

"×" your TA

- ☐ Mukul Agarwal
- ☐ Jason Cloud
- ☐ Shuo Deng
- ☐ Lyla Fischer
- ☐ Rui Li
- ☐ Ruben Madrigal
- ☐ Surapap R.
- ☐ Xiawa Wang

Please write your name in the space at the top of this page.

There are **5 questions** (all with multiple parts) and **17 pages** in this quiz booklet. Answer each question according to the instructions given. You have **120 minutes** to answer the questions.

Please be neat and write legibly. If you find a question ambiguous, write down any assumptions you make, *and show your work for partial credit.*

Use the empty sides of this booklet if you need scratch space. You may also use them for answers, although you shouldn't need to. *If you use the blank sides for answers, make sure to say so!*

Two two-sided "crib sheets" and a calculator allowed. No other notes, books, computers, cell phones, PDAs, information appliances, carrier pigeons, etc.!

PLEASE DO NOT DISCUSS THIS QUIZ WITH OR AROUND ANYONE IN THE CLASS, UNLESS YOU ARE CERTAIN THEY HAVE TAKEN IT. MAKEUPS WILL BE DONE BY 5PM ON FRIDAY, NOVEMBER 18. THANKS!

Do not write in the boxes below

1 (x/15)	2 (x/20)	3 (x/25)	4 (x/25)	5 (x/15)	Total (x/100)

I System Properties

1. [15 points]: Each of the following equations describes the relationship that holds between the input signal $x[.]$ and output signal $y[.]$ of an associated discrete-time system, for all integers n. In each case, state whether or not the system is **(i) causal** , **(ii) linear**, **(iii) time-invariant**. We are looking for just a **Yes** or **No** here.

A. (3 points) $y[n] = 0.5x[n] + 0.5x[n-1]$

Causal?

Linear?

Time-invariant?

B. (3 points) $y[n] = x[n+1] + 7$

Causal?

Linear?

Time-invariant?

C. (3 points) $y[n] = \cos(3n)\, x[n-2]$

Causal?

Linear?

Time-invariant?

D. (3 points) $y[n] = x[n]\, x[n-1]$

Causal?

Linear?

Time-invariant?

E. (3 points) $y[n] = \sum_{k=13}^{n} x[k]$ for $n \geq 13$, otherwise $y[n] = 0$.

Causal?

Linear?

Time-invariant?

II LTI Systems in the Time Domain

2. [20 points]: Suppose the *unit* **step** *response* $s[n]$ of a particular linear, time-invariant (LTI) communication channel is given by

$$s[n] = \left(1 - \left(\frac{1}{2}\right)^n\right) u[n] \ ,$$

where $u[n]$ denotes the unit step function: $u[n] = 1$ for $n \geq 0$, and $u[n] = 0$ for $n < 0$. Show your reasoning and calculations in your answers below.

A. (3 points) Draw a labeled sketch of the above unit step response $s[n]$ for $0 \leq n \leq 4$.

B. (1 point) Suppose the input $x[n]$ to this channel is given by

$$\begin{aligned} x[n] &= 2 \quad \text{for } n = 0, 1, 2, \\ &= 0 \quad \text{for all other } n. \end{aligned}$$

Draw a labeled sketch of this $x[n]$ for n in the range -1 to 5 (just to fix this waveform in your mind).

C. (8 points) With the $x[.]$ from part **B**, determine the value of the output $y[n]$ at times $n = 1$ and $n = 4$. Explain your reasoning.

$y[1] =$

$y[4] =$

D. (6 points) Determine the unit *sample* response $h[n]$ of the channel, explaining how you arrived at it.

Also draw a labeled sketch of $h[n]$ for $0 \leq n \leq 4$.

The channel's unit sample response (shown in your sketch above) is given by the following expression:

$$h[n] =$$

E. (2 points) Is this channel bounded-input bounded-output stable? Explain your answer.

III LTI Systems in the Frequency Domain

3. [25 points]: Consider an LTI filter with input signal $x[n]$, output signal $y[n]$, and unit sample response

$$h[n] = a\delta[n] + b\delta[n-1] + b\delta[n-2] + a\delta[n-3] \ ,$$

where a and b are *positive* parameters, with $b > a > 0$. Thus $h[0] = h[3] = a$ and $h[1] = h[2] = b$, while $h[n]$ at all other times is 0. Your answers in this problem will be in terms of a and b. Show your reasoning throughout.

A. (4 points) Determine the frequency response $H(\Omega)$ of the filter.

$H(\Omega) =$

B. (4 points) Suppose $x[n] = (-1)^n$ for all integers n from $-\infty$ to ∞. Use your expression for $H(\Omega)$ in part **A** to determine $y[n]$ at *all* times n. Explain your reasoning.

$y[n] =$

C. (6 points) As a time-domain check on your answer from part **B**, use **convolution** to determine the values of $y[5]$ and $y[6]$ when $x[n] = (-1)^n$ for all integers n from $-\infty$ to ∞.

$y[5] =$

$y[6] =$

D. (4 points) The frequency response $H(\Omega) = |H(\Omega)|e^{j\angle H(\Omega)}$ that you found in part **A** for this filter can be written in the form

$$H(\Omega) = G(\Omega)e^{-j3\Omega/2} \ ,$$

where $G(\Omega)$ is a **real** function of Ω that can be positive or negative, depending on the values of a, b, and Ω. Determine $G(\Omega)$, writing it in a form that makes clear it is a real function of Ω.

$G(\Omega) =$

E. (7 points) Suppose the input to the filter is $x[n] = (-1)^n + \cos(\frac{\pi}{2}n + \theta_0)$ for all n from $-\infty$ to ∞, where θ_0 is some constant. Use the **frequency response** $H(\Omega)$ to determine the output $y[n]$ of the filter (writing it in terms of a, b, and θ_0).

(Depending on how you solve the problem, it may help you to recall that $\cos(\pi/4) = 1/\sqrt{2}$ and $\cos(3\pi/4) = -1/\sqrt{2}$. Also keep in mind our assumption that $b > a > 0$.)

$y[n] =$

IV Spectral Content and the DTFS

4. [25 points]: Consider a lowpass LTI communication channel with input $x[n]$, output $y[n]$, and frequency response $H(\Omega)$ given by

$$\begin{aligned} H(\Omega) &= e^{-j3\Omega} \quad \text{for } 0 \leq |\Omega| < \Omega_m \, , \\ &= 0 \qquad\quad \text{for } \Omega_m \leq |\Omega| \leq \pi \, . \end{aligned}$$

Here Ω_m denotes the cutoff frequency of the channel; the output $y[n]$ will contain no frequency components in the range $\Omega_m \leq |\Omega| \leq \pi$. The different parts of this problem involve different choices for Ω_m.

A. (4 points) Picking $\Omega_m = \pi/4$, provide separate and properly labeled sketches of the magnitude $|H(\Omega)|$ and phase $\angle H(\Omega)$ of the frequency response, for Ω in the interval $0 \leq |\Omega| \leq \pi$. (Sketch the phase only in the frequency ranges where $|H\Omega)| > 0$.)

B. (4 points) Suppose $\Omega_m = \pi$, so $H(\Omega) = e^{-j3\Omega}$ for *all* Ω in $[-\pi, \pi]$, i.e., *all* frequency components make it through the channel. For this case, $y[n]$ can be expressed quite simply in terms of $x[.]$; find the relevant expression.
(If you get stuck on this part, move on, as the other parts don't depend on this.)

In terms of the input $x[.]$, the output $y[n] =$

C. (7 points) Suppose the input $x[n]$ to this channel is a periodic "rectangular-wave" signal with period 12. Specifically:

$$x[-1] = x[0] = x[1] = 1$$

and these values repeat every 12 steps, so

$$x[11] = x[12] = x[13] = 1$$

and more generally

$$x[12r - 1] = x[12r] = x[12r + 1]$$

for all integers r from $-\infty$ to ∞. At **all other times** n, we have $x[n] = 0$. (You might find it helpful to sketch this signal for yourself, e.g., for n ranging from -2 to 13.)

Find explicit values for the Fourier coefficients in the discrete-time Fourier series (DTFS) for this input $x[n]$, i.e., the numbers A_k in the representation

$$x[n] = \sum_{k=-6}^{5} A_k e^{j\Omega_k n} \ ,$$

where $\Omega_k = k(2\pi/12)$. Recall that

$$A_k = \frac{1}{12}\sum_{\langle n \rangle} x[n]e^{-j\Omega_k n} \ ,$$

where the summation is over **any** 12 **consecutive values** of n (as indicated by writing $\langle n \rangle$), so all you need to do is evaluate this expression for the particular $x[n]$ that we have.

Since $x[n]$ is an *even* function of n, all the A_k should be *purely real*, so be sure your expression for A_k makes clear that it is real. (Depending on how you proceed, you may or may not find it helpful to note that $\exp\{-j11(2\pi/12)\} = \exp\{j(2\pi/12)\}$.)

Check that your values for A_0 and $A_{-6} = A_6$ are correct, and be explicit about how you are checking.

$A_k =$

Check on A_0:

Check on A_{-6}:

D. (6 points) Suppose the cutoff frequency of the channel is $\Omega_m = \pi/4$ (which is the case you sketched in part **A**), and the input is the $x[n]$ specified in part **C**. Compute the values of all the nonzero Fourier coefficients of the channel output $y[n]$, i.e., find the values of the nonzero numbers B_k in the representation

$$y[n] = \sum_{k=-6}^{5} B_k e^{j\Omega_k n} ,$$

where $\Omega_k = k(2\pi/12)$. Don't forget that $H(\Omega) = e^{-j3\Omega}$ in the passband of the filter, $0 \leq |\Omega| < \Omega_m$. Also, depending on how you solve the problem, you may encounter the factor $\cos(\pi/6)$, which you can leave as is, or replace by $\sqrt{3}/2$ if that seems simpler.

Values of the nonzero B_k are:

E. (4 points) Express the $y[n]$ in part **D** as an explicit and real function of time n—any other representation of $y[n]$, even if correct, may only earn you partial credit. (If you were to sketch $y[n]$, which we are *not* asking you to do, you would discover that it is a low-frequency approximation to the $y[n]$ that would have been obtained if $\Omega_m = \pi$.)

$y[n] =$

V Modulation/Demodulation

5. [15 points]: The Boston news radio station WBZ AM broadcasts on a carrier frequency of 1030 kHz. Suppose its continuous-time (CT) carrier signal is $\sin(2\pi \times 1030 \times 10^3 t)$, where t is measured in seconds. Denote the CT audio signal that's modulated onto this carrier by $x(t)$, so that the CT signal transmitted by the radio station is

$$q(t) = x(t)\sin(2\pi \times 1030 \times 10^3 t)\ . \tag{1}$$

We use the symbols $q[n]$ and $x[n]$ to denote the discrete-time (DT) signals that would have been obtained by respectively sampling $q(t)$ and $x(t)$ in Eq.(1) at f_s samples/sec; more specifically, the signals are sampled at the discrete time instants $t = n(1/f_s)$. Thus

$$q[n] = x[n]\sin(\Omega_c n) \tag{2}$$

for an appropriately chosen value of the angular frequency Ω_c. Assume that $x[n]$ is periodic with some period N, and that $f_s = 4 \times 10^6$ samples/sec, corresponding to a sampling frequency of 4 MHz.

A. (3 points) Determine the value (in radians/sample) of Ω_c in Eq.(2), restricting your answer to a value in the range $[-\pi, \pi]$. (You can assume in what follows that the period N of $x[n]$ is such that $\Omega_c = k_c(2\pi/N)$ for some integer k_c; this is a detail, and needn't concern you unduly.)

$\Omega_c =$

B. (3 points) With the notation $\Omega_k = k(2\pi/N)$, determine the range of values of Ω_k for which the Fourier coefficients of $x[n]$ can be nonzero, assuming the FCC insists that the transmitted signal be confined to a frequency band of 5 kHz on either side of the 1030 kHz carrier, i.e., to the range from 1025 kHz to 1035 kHz. (Explain your reasoning; there is additional space on the next page.)

Range of Ω_k for which the Fourier coefficients of $x[n]$ can be nonzero:

Assume now that the receiver detects the CT signal $r(t) = 10^{-3}q(t - t_0)$, where $t_0 = 10^{-5}$ sec, and that it samples this signal at f_s samples/sec, thereby obtaining the DT signal

$$r[n] = 10^{-3}q[n - M] = 10^{-3}x[n - M]\sin\Big(\Omega_c(n - M)\Big) \tag{3}$$

for an appropriately chosen integer M.

C. (3 points) Determine the value of M in Eq. (3).

$M =$

D. (3 points) Noting your answer from part **B**, determine for precisely which intervals of the Ω frequency axis the Fourier series coefficients of the signal $q[n-M]$ in Eq.(3) are non-zero. (Explain your reasoning.)

Intervals on the Ω axis where the Fourier coefficients of $q[n-M]$ can be nonzero:

E. (3 points) The demodulation step to obtain the DT signal $x[n-M]$ from the received signal $r[n]$ in Eq.(3) now involves multiplying $r[n]$ by a DT carrier-frequency signal, followed by appropriate low-pass filtering. Which one of the following six carrier-frequency signals would you choose to multiply the received signal by? Circle it, and give a brief explanation.

- $\cos\Big(\Omega_c n\Big)$
- $\cos\Big(\Omega_c(n-M)\Big)$
- $\cos\Big(\Omega_c(n+M)\Big)$
- $\sin\Big(\Omega_c n\Big)$
- $\sin\Big(\Omega_c(n-M)\Big)$
- $\sin\Big(\Omega_c(n+M)\Big)$