Name: These R. Solutions

DEPARTMENT OF EECS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.02 Spring 2012

Quiz I

March 13, 2012

"×" your section	Section	Time	Recitation Instructor	TA
☐	1	10-11	Vincent Chan	Jared Monnin
☐	2	11-12	Vincent Chan	Anirudh Sivaraman
☐	3	12-1	Sidhant Misra	Sungwon Chung
☐	4	1-2	Sidhant Misra	Omid Aryan / Nathan Lachenmyer
☐	5	2-3	Katrina LaCurts	Sunghyun Park
☐	6	3-4	Katrina LaCurts	Muyiwa Ogunnika

Please read and follow these instructions:

0. Please write your name in the space above and × your section.
1. There are **15 questions** (some with multiple parts) and **6 pages** in this quiz booklet.
2. Answer each question according to the instructions given, within **120 minutes**.
3. **Please answer legibly. Explain your answers, especially when we ask you to.** If you find a question ambiguous, write down your assumptions. Show your work for partial credit.
4. Use the empty sides of this booklet if you need scratch space. *If you use the blank sides for answers, make sure to say so!*

One two-sided "crib sheet" and a calculator allowed. No other aids.

PLEASE NOTE: SOME STUDENTS WILL TAKE THE MAKE-UP QUIZ TOMORROW AT 9.30 AM. PLEASE DON'T DISCUSS THIS QUIZ WITH ANYONE IN THE CLASS, UNLESS YOU'RE SURE THEY HAVE TAKEN IT WITH YOU TODAY.

Do not write in the boxes below

1-3(x/21)	4 (x/15)	5-7 (x/18)	8-10 (x/11)	11-12 (x/11)	13-15 (x/24)	Total (x/100)

I Green Eggs and Hamming

Please see pset answers.

II *Errare humanum est* (To err is human)

1. [6 points]: A binary symmetric channel has bit-flip probability ε. Suppose we take a stream of S bits in which zeroes and ones occur with equal probability, divide it into blocks of k bits each, and apply an $(n, k, 3)$ code to each block. To correct bit errors, we decode each block of n bits at the receiver using maximum-likelihood decoding. Calculate the probability that the stream of S bits is decoded correctly, assuming that any n-bit block with two or more errors will not be decoded correctly. Show your work.

Solution: The probability of zero errors occuring in each block is $(1-\varepsilon)^n$. The error of 1 error occuring is $n\varepsilon(1-\varepsilon)^{n-1}$. There are S/k blocks, assuming (reasonably) that S is an integral multiple of k, so the total probability of the bits being decoded correctly is:

$$((1-\varepsilon)^n + n\varepsilon(1-\varepsilon)^{n-1})^{S/k}$$

2. [2+3 = 5 points]: For an (n, k, d) linear block code, fill in the blanks below in terms of one of more of n, k, d.

A. The maximum number of bit errors that can always be **detected** is $d-1$.

B. $2^{n-k} \geq 1 + n + \binom{n}{2} + \ldots + \binom{n}{t}$, where $t =$ $\lfloor \frac{d-1}{2} \rfloor$ (the number of correctable bits.) .

3. [1+2+4 = 7 points]: Consider the following variant of the rectangular parity code with r rows and c columns. Each row has a row parity bit. Each column has a column parity bit. In addition, **each codeword has an overall parity bit**, to ensure that the number of ones in each codeword is even.

A. Code rate = __________. Hamming distance of code = __________.

Solution: The code rate is equal to $\frac{k}{n} = \frac{rc}{(r+1)(c+1)} = \frac{rc}{rc+r+c+1}$.
The Hamming distance is 4 (A single bit-flip produces a change in the 3 parity bits plus the original bit that the error occured in.

B. Ben Bitdiddle takes each codeword of this code and *removes* the first row parity bit from each codeword. Note that the overall parity bit is calculated by **including** the row parity bit he eliminated. Ben then transmits these codewords over a noisy channel. What is the largest number of bit errors in a codeword that this new code can always **correct**? Explain.

Solution: The overall parity bit includes information about the row bit that was removed. We now have a Hamming distance of $d = 3$ between codewords, so we can correct up to $\frac{d-1}{2} = 1$ error.

III The Matrix Reloaded

Please see pset answers.

IV Convolutionally yours

Dona Ferentes is debugging a Viterbi decoder for her client, The TD Company, which is building a wireless network to send gifts from mobile phones. She picks a rate-$1/2$ code with constraint length 4, no puncturing. Parity stream p_0 has the generator $g_0 = 1110$. Parity stream p_1 has the generator $g_1 = 1xyz$, but she needs your help determining x, y, z, as well as some other things about the code. In these questions, each state is labeled with the most-recent bit on the left and the least-recent bit on the right.

4. [4+7+4 = 15 points]: These questions are about the state transitions and generators.

A. From state 010, the possible **next states** are 001 and 101.

From state 010, the possible **predecessor states** are 100 and 101.

B. Given the following facts, find g_1, the generator for parity stream p_1. g_1 has the form $1xyz$, with the standard convention that the left-most bit of the generator multiplies the most-recent input bit.

Starting at state 011, receiving a 0 produces $p_1 = 0$.
Starting at state 110, receiving a 0 produces $p_1 = 1$.
Starting at state 111, receiving a 1 produces $p_1 = 1$.

Solution: 1011. We can use the above information to generate 3 linear equations:

$$\begin{aligned} y + z &= 0 \\ x + y &= 1 \\ 1 + x + y + z &= 1 \end{aligned}$$

Solving this 3×3 system gives x = 0, y = 1, and z = 1, giving the generator 1011. To solve these equations, it's important to remember that they are over $\mathbb{F}_{\not{2}}$. So from the first one, $z = y$, and substituting $x + y = 1$ into the third eqution, we get $1 + 1 + z = 1$, which means that $z = 1$. THerefore $y = 1$ and $x = 1 - y = 0$. Hence, the generator is 1011.

C. Dona has just completed the forward pass through the trellis and has figured out the path metrics for all the end states. Suppose the state with smallest path metric is 110. The traceback from this state looks as follows:

$$000 \leftarrow 100 \leftarrow 010 \leftarrow 001 \leftarrow 100 \leftarrow 110$$

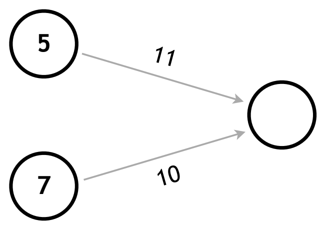
What is the most likely transmitted message? Explain your answer, and if there is not enough information to produce a unique answer, say why.

Solution: Looking at the leftmost bit that is added at each transition, we can trace the transmitted message as 10011.

5. [8 points]: During the decoding process, Dona observes the voltage pair $(0.9, 0.2)$ volts for the parity bits p_0p_1, where the sender transmits 1.0 volts for a "1" and 0.0 volts for a "0". The threshold voltage at the decoder is 0.5 volts. In the portion of the trellis shown below, each edge shows the expected parity bits p_0p_1. The number in each circle is the path metric of that state.

A. With **hard-decision decoding**, give the branch metric near each edge and the path metric inside the circle.

received voltages: .9, .2

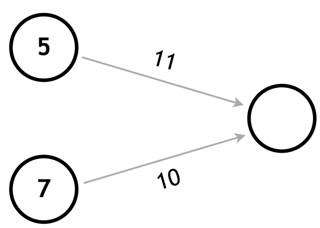


Solution: Branch metrics: 1 for the upper edge, 0 for the lower edge.
The path metric is $\min(5+1, 7+1) = 6$.

B. Timmy Dan (founder of TD Corp.) suggests that Dona use **soft-decision decoding** using the **squared Euclidean distance** metric. Give the branch metric near each edge and the path metric inside the circle.

Solution: Branch metric: $(1-0.9)^2 + (1-0.2)^2 = 0.65$ for the upper edge and $(1-0.9)^2 + (0-0.2)^2 = 0.05$ for the lower edge. The path metric is $\min(5.65, 7.05) = 5.65$.

received voltages: .9, .2



6. [1 free point!]: The real purpose behind Dona Ferentes decoding convolutionally is some awful wordplay with Virgil's classical Latin. What does *Timeo Danaos et dona ferentes* mean?

(Circle ALL that apply.)

A. Timmy Dan and Dona are friends.

B. It's time to dance with Dona Ferentes.

C. I fear the Greeks, even those bearing gifts.

D. I fear the Greeks, especially those bearing debt.

E. You *#@$*@!#s. This is the last straw; I'm reporting you to the Dean. If I'd wanted to learn this, I'd have gone to that school up the Charles!

Solution: **C.** Arguably also E.