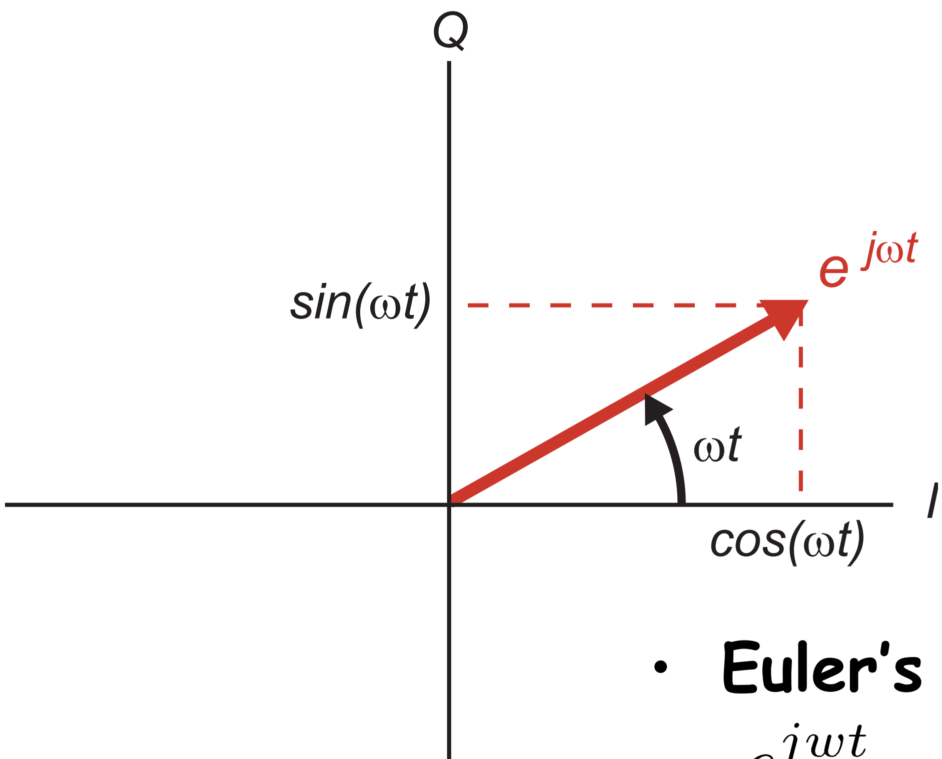
Fourier Series and Fourier Transform

- Complex exponentials
- Complex version of Fourier Series
- Time Shifting, Magnitude, Phase
- Fourier Transform

The Complex Exponential as a Vector



Note:

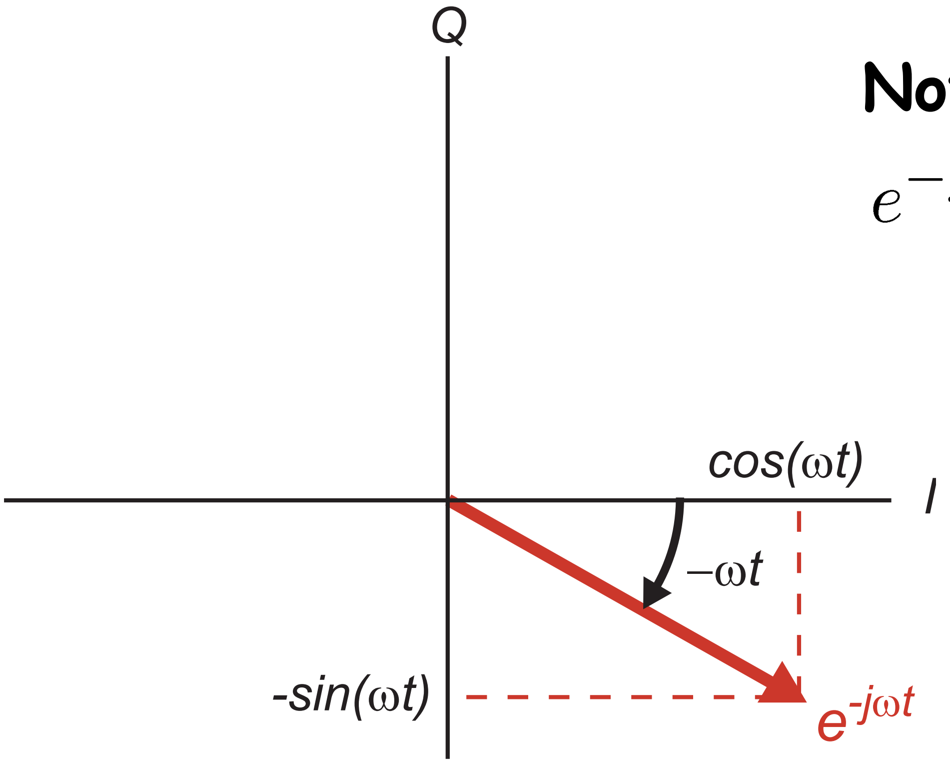
$$j = \sqrt{-1}$$

- **Euler's Identity:**

$$e^{jwt} = \cos(wt) + j\sin(wt)$$

- Consider *I* and *Q* as the *real* and *imaginary* parts
  - As explained later, in communication systems, *I* stands for *in-phase* and *Q* for *quadrature*
- As t increases, vector rotates *counterclockwise*
  - We consider e^{jwt} to have *positive* frequency

The Concept of Negative Frequency

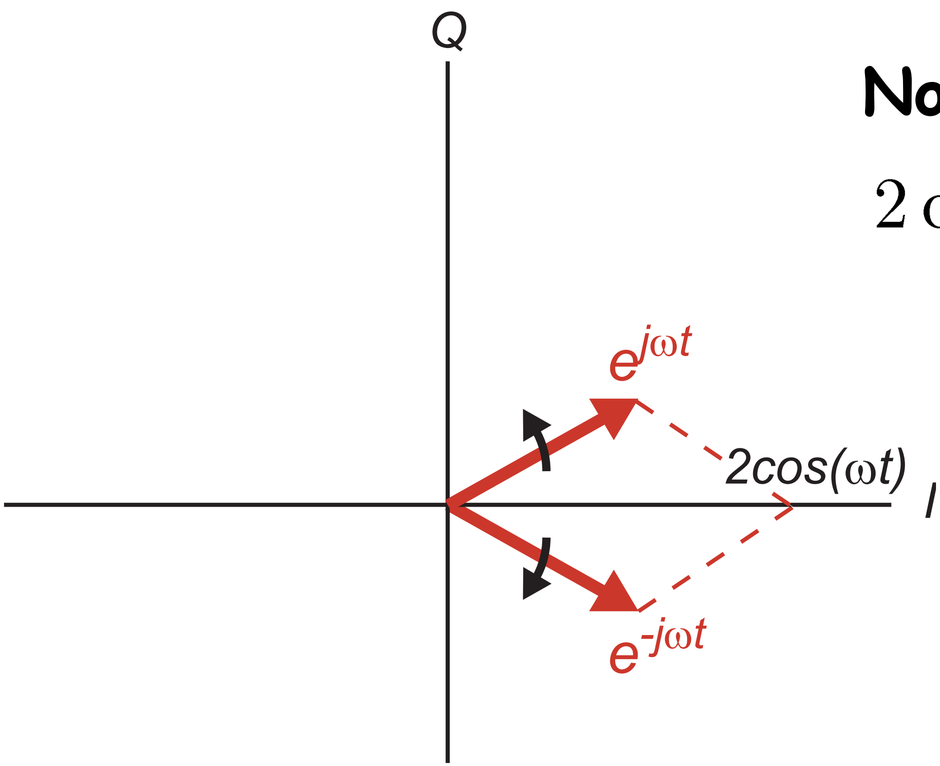


Note:

$$e^{-jwt} = \cos(wt) - j\sin(wt)$$

- As t increases, vector rotates *clockwise*
  - We consider e^{-jwt} to have *negative* frequency
- Note: *A-jB* is the *complex conjugate* of *A+jB*
  - So, e^{-jwt} is the complex conjugate of e^{jwt}

Add Positive and Negative Frequencies

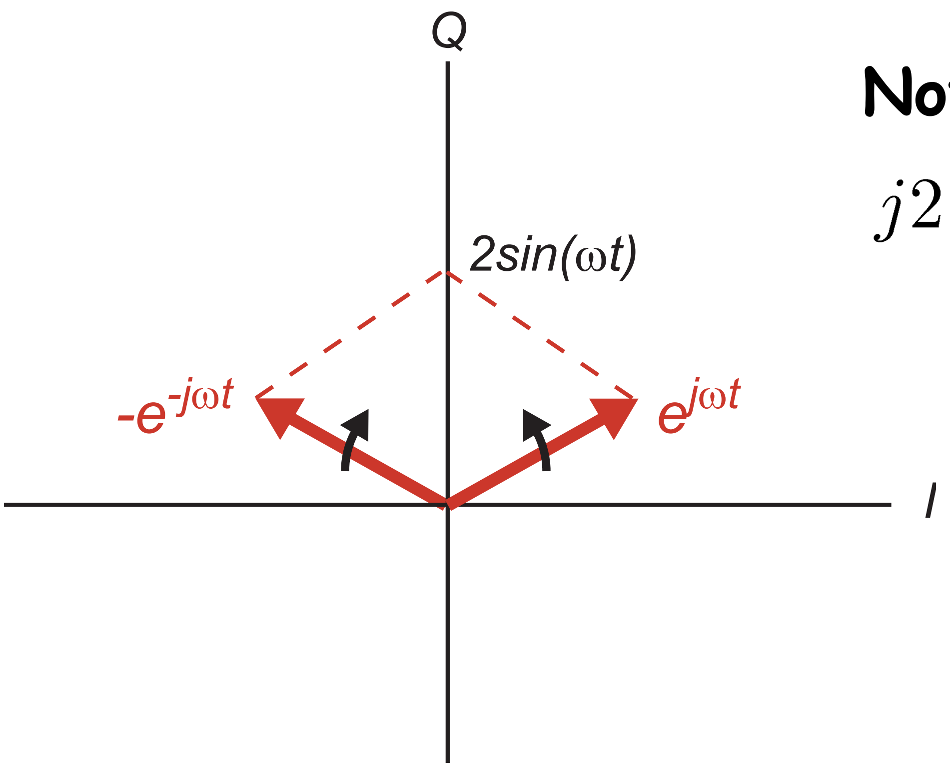


Note:

$$2\cos(wt) = e^{jwt} + e^{-jwt}$$

- As t increases, the *addition* of *positive* and *negative* frequency complex exponentials leads to a *cosine* wave
  - Note that the resulting cosine wave is purely *real* and considered to have a *positive* frequency

Subtract Positive and Negative Frequencies

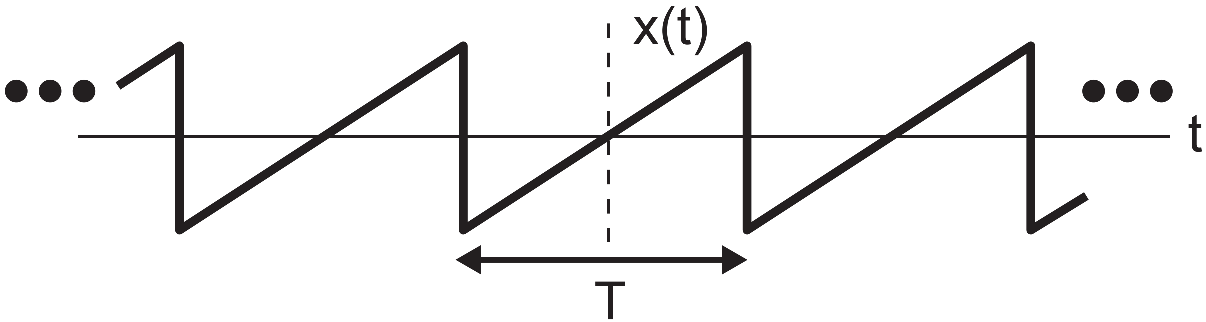


Note:

$$j2\sin(wt) = e^{jwt} - e^{-jwt}$$

- As t increases, the *subtraction* of *positive* and *negative* frequency complex exponentials leads to a *sine* wave
  - Note that the resulting sine wave is purely *imaginary* and considered to have a *positive* frequency

Fourier Series



- **The Fourier Series is compactly defined using complex exponentials**

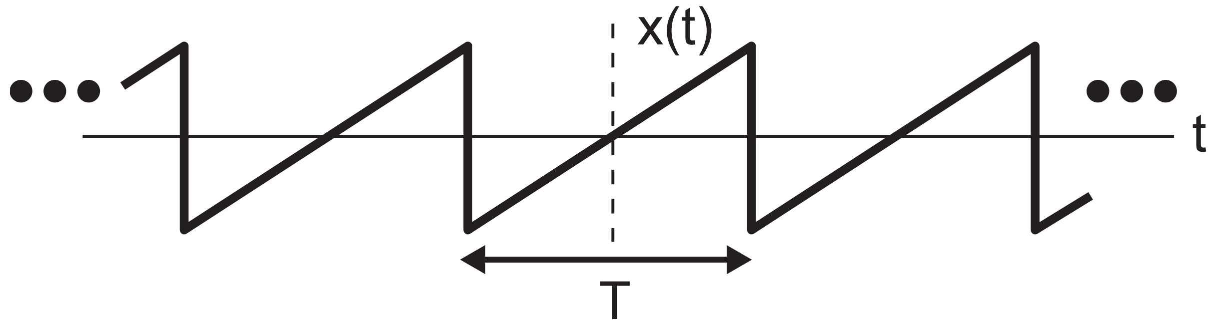
$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jnw_o t}$$

$$\hat{X}_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{-jnw_o t}dt$$

- **Where:**

$$w_o = \frac{2\pi}{T} \qquad \hat{X}_n = A_n + jB_n$$

From The Previous Lecture



- **The Fourier Series can also be written in terms of cosines and sines:**

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_o t) + b_n sin(nw_o t)$$

where for $n > 0$:

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\cos(nw_o t)dt,\ b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin(nw_o t)dt$$

and where : $\quad w_o = \frac{2\pi}{T}, \quad a_0 = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt$

Compare Fourier Definitions

- **Let us assume the following:** $\hat{X}_n = A_n + jB_n$

$$A_n = A_{-n} \qquad B_n = -B_{-n} \qquad B_0 = 0$$

- **Then:**

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jnw_o t} = \sum_{n=-\infty}^{\infty} A_n e^{jnw_o t} + \sum_{n=-\infty}^{\infty} jB_n e^{jnw_o t}$$

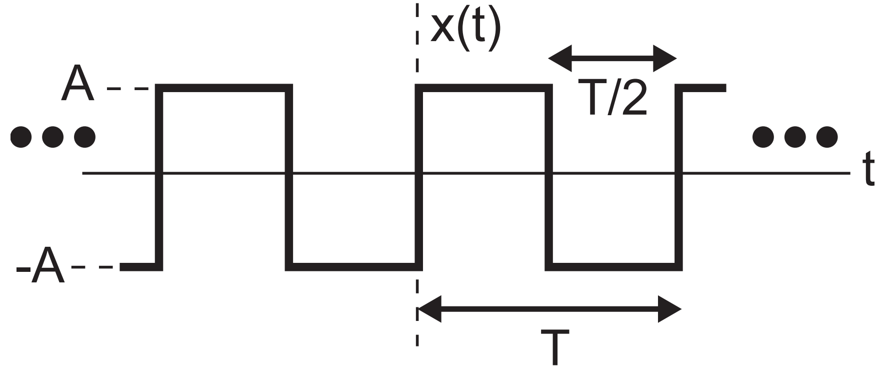
$$= A_0 + \sum_{n=1}^{\infty} A_n (e^{jnw_o t} + e^{-jnw_o t}) + \sum_{n=1}^{\infty} jB_n (e^{jnw_o t} - e^{-jnw_o t})$$

$$= A_0 + \sum_{n=1}^{\infty} 2A_n \cos(nw_o t) + \sum_{n=1}^{\infty} -2B_n \sin(nw_o t)$$

- **So:**

$$\boxed{A_0 = a_0} \qquad \boxed{2A_n = a_n} \qquad \boxed{-2B_n = b_n}$$

Square Wave Example



$$\hat{X}_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{-jnw_o t}dt$$

$$= \frac{1}{T}\int_{-T/2}^{0} -Ae^{-jnw_o t}dt + \frac{1}{T}\int_{0}^{T/2} Ae^{-jnw_o t}dt$$

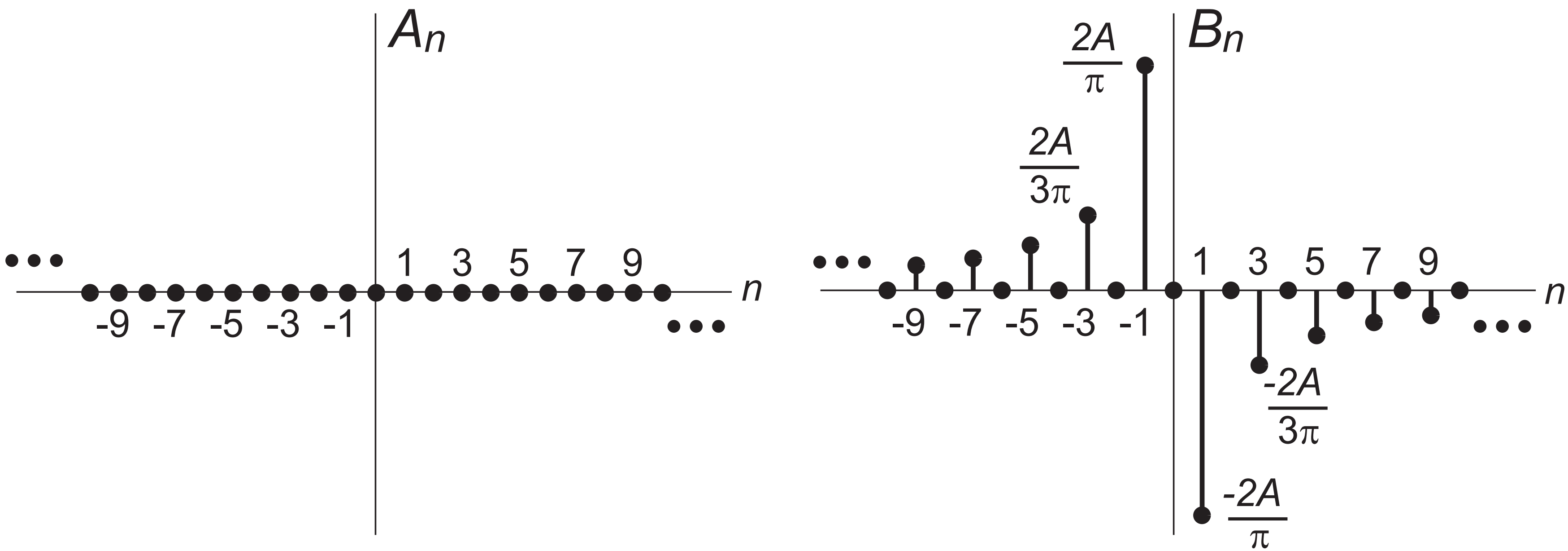
$$= \frac{1}{T}\frac{-A}{-jnw_o}(1 - e^{jnw_o T/2}) + \frac{1}{T}\frac{A}{-jnw_o}(e^{-jnw_o T/2} - 1)$$

$$= \frac{1}{T}\frac{2A}{jnw_o}(1 - \cos(nw_o T/2)) = \boxed{-j\frac{A}{n\pi}(1 - \cos(n\pi))}$$

Graphical View of Fourier Series

- **As in previous lecture, we can plot Fourier Series coefficients**
  - **Note that we now have *positive* and *negative* values of *n***
- **Square wave example:**

$$\hat{X}_n = A_n + jB_n \;=\; -j\frac{A}{n\pi}(1-\cos(n\pi)) = \begin{cases} 0 & (\text{even } n) \\ j\frac{-2A}{n\pi} & (\text{odd } n) \end{cases}$$



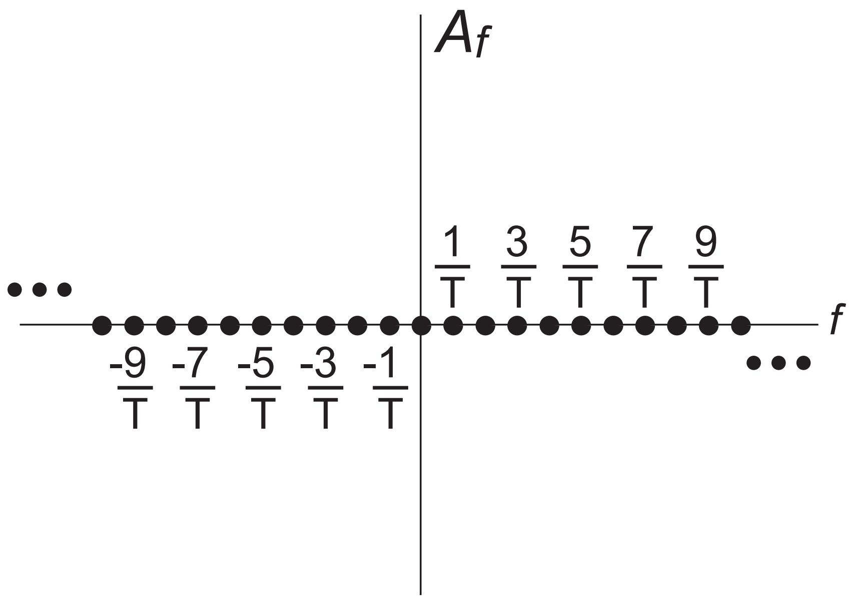
Indexing in Frequency

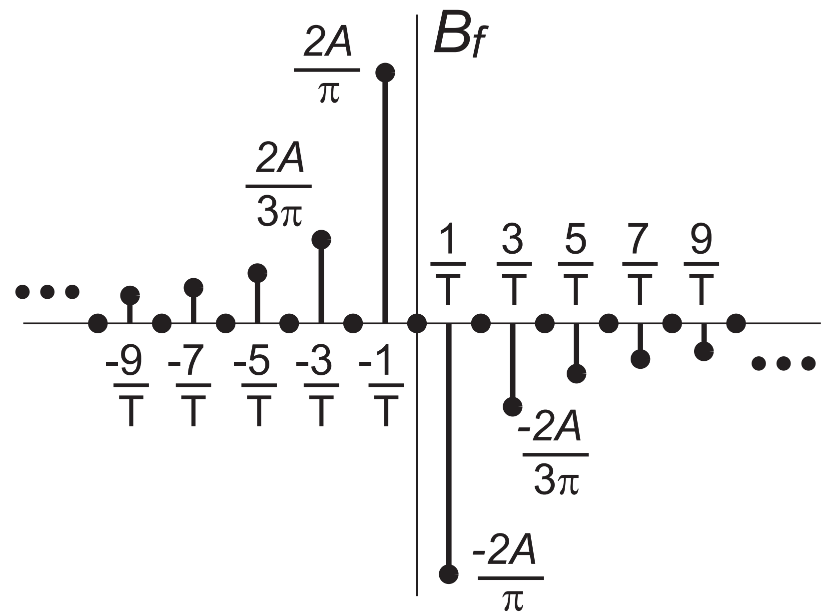
- **A given Fourier coefficient, $\hat{X}_n$, represents the weight corresponding to frequency *nw***

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jnw_o t}$$

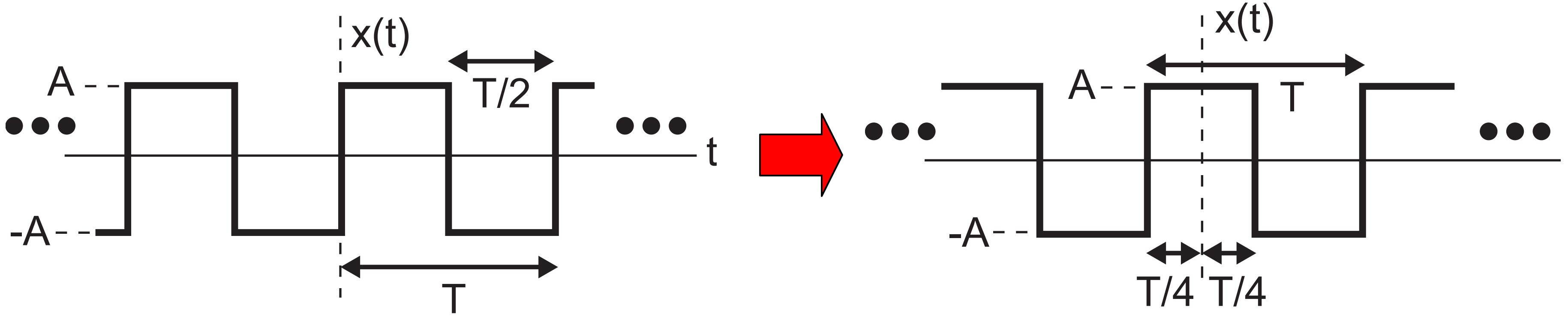
- **It is often convenient to index in *frequency (Hz)***

$$nw_o = 2\pi(nf_o) \; = \; 2\pi\left(n\frac{1}{T}\right)$$





The Impact of a Time (Phase) Shift



- **Consider shifting a signal *x(t)* in time by T_d**

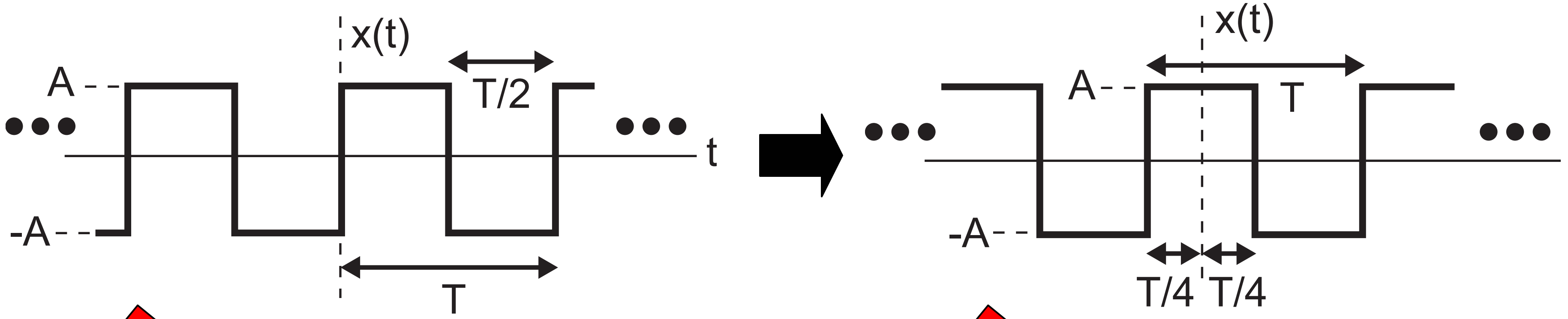
$$\hat{Y}_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t-T_d)e^{-jnw_o t}dt$$

- **Define:** $\tau = t - T_d \;\Rightarrow\; d\tau = dt$

- **Which leads to:** $\hat{Y}_n = \frac{1}{T}\int_{t_o+T_d}^{t_o+T+T_d} x(\tau)e^{-jnw_o(\tau+T_d)}d\tau$

$$= e^{-jnw_oT_d}\left(\frac{1}{T}\int_{t_o}^{t_o+T} x(\tau)e^{-jnw_o\tau}d\tau\right) = \boxed{e^{-jnw_oT_d}\hat{X}_n}$$

Square Wave Example of Time Shift



$$\hat{X}_n = -j\frac{A}{n\pi}(1-\cos(n\pi))$$

$$= \begin{cases} 0 & (\text{even } n) \\ -j\frac{2A}{n\pi} & (\text{odd } n) \end{cases}$$

$$\hat{Y}_n = e^{-jnw_oT_d}\hat{X}_n$$

$$= e^{-jnw_o(-T/4)}\hat{X}_n$$

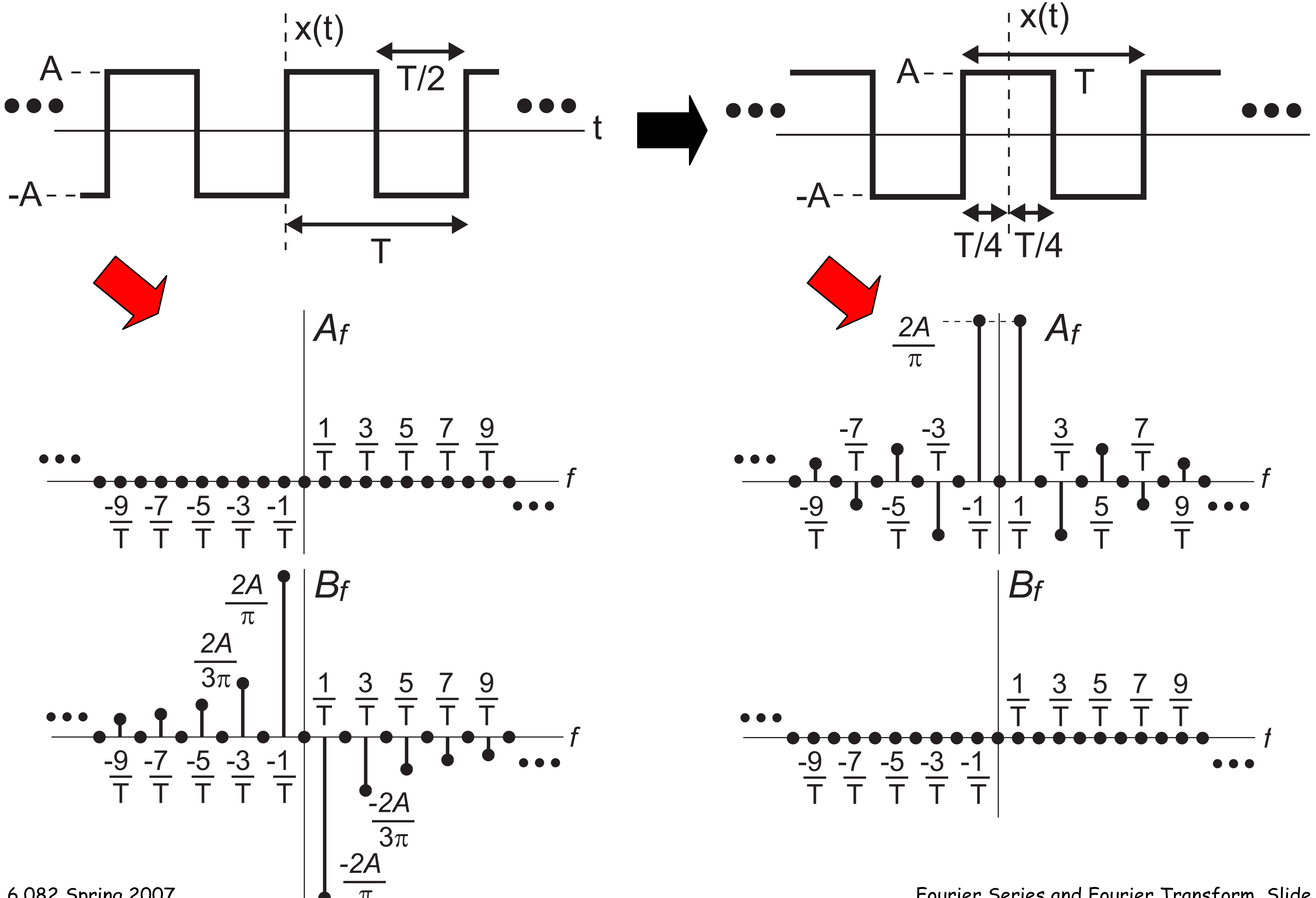
$$= e^{-jn(2\pi/T)(-T/4)}\hat{X}_n$$

$$= e^{jn\pi/2}\hat{X}_n$$

- **To simplify, note that** $\hat{X}_n = 0$ ***except* for odd *n***

$$\Rightarrow \hat{Y}_n = \begin{cases} & 0 \quad (\text{even } n) \\ j\sin(n\pi/2)\left(-j\frac{2A}{n\pi}\right) = & \sin(n\pi/2)\frac{2A}{n\pi} \quad (\text{odd } n) \end{cases}$$

Graphical View of Fourier Series



Magnitude and Phase

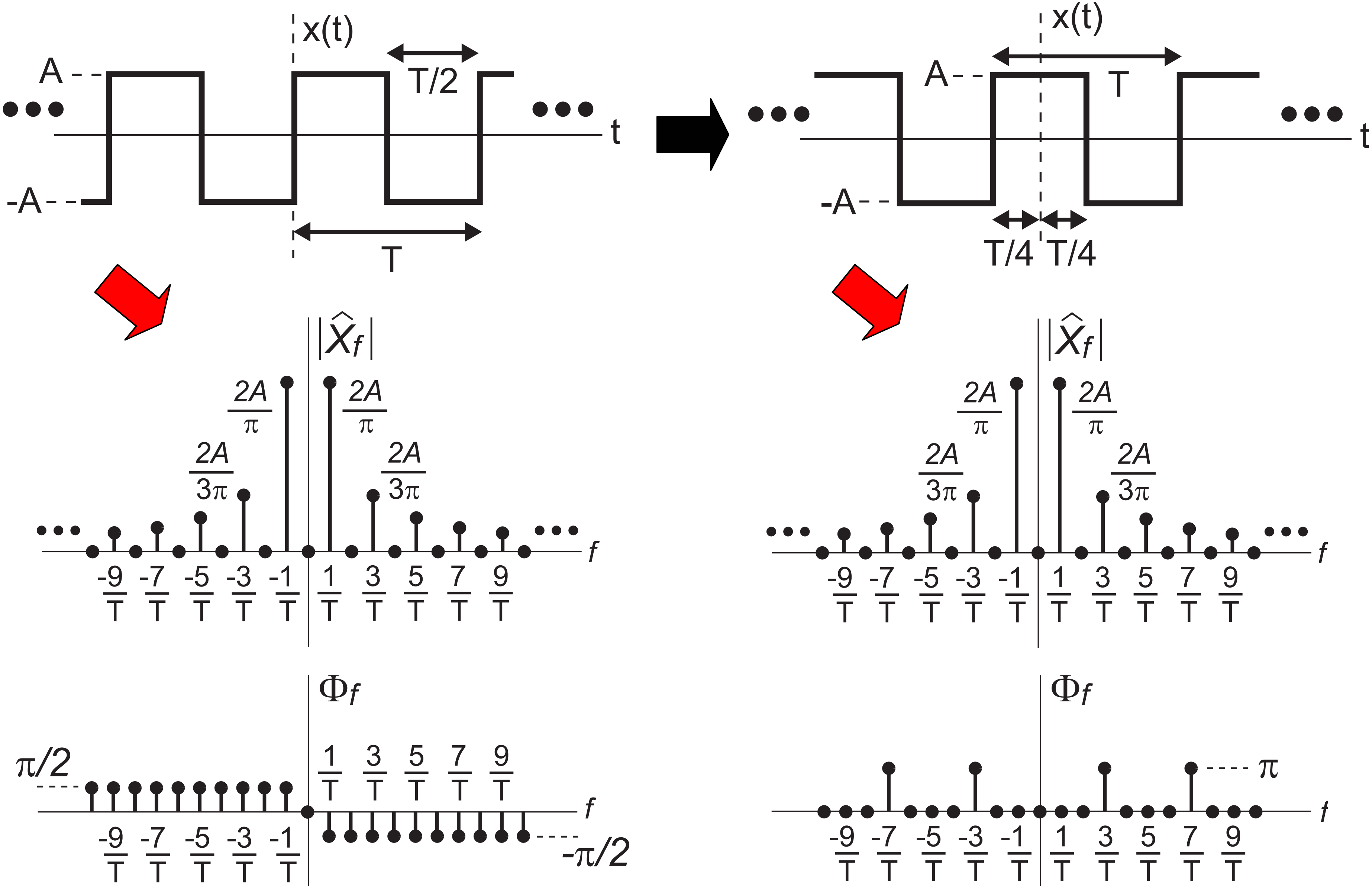
- **We often want to ignore the issue of time (phase) shifts when using Fourier analysis**
  - **Unfortunately, we have seen that the A_n and B_n coefficients are very sensitive to time (phase) shifts**
- **The Fourier coefficients can also be represented in term of magnitude and phase**

$$\hat{X}_n = A_n + jB_n \;=\; |\hat{X}_n|e^{j\Phi_n}$$

- **where:**

$$|\hat{X}_n| = \sqrt{A_n^2 + B_n^2} \qquad\qquad \Phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

Graphical View of Magnitude and Phase



Does Time Shifting Impact Magnitude?

- **Consider a waveform *x(t)* along with its Fourier Series**

$$x(t) \;\Leftrightarrow\; \hat{X}_n$$

- **We showed that the impact of time (phase) shifting *x(t)* on its Fourier Series is**

$$x(t-T_d) \;\Leftrightarrow\; e^{-jnw_oT_d}\hat{X}_n$$

- **We therefore see that time (phase) shifting does *not* impact the Fourier Series magnitude**

$$\left|e^{-jnw_oT_d}\hat{X}_n\right| = \left|e^{-jnw_oT_d}\right|\left|\hat{X}_n\right| \;=\; \left|\hat{X}_n\right|$$

Parseval's Theorem

- The squared magnitude of the Fourier Series coefficients indicates *power* at corresponding frequencies
  - *Power* is defined as: $$\frac{1}{T}\int_{t_o}^{t_o+T} x^2(t)dt$$

$$\frac{1}{T}\int_{t_o}^{t_o+T} x^2(t)dt = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)\sum_{n=-\infty}^{\infty}\hat{X}_n e^{jnw_o t}dt$$

$$= \sum_{n=-\infty}^{\infty}\hat{X}_n \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{jnw_o t}dt$$

$$= \sum_{n=-\infty}^{\infty}\hat{X}_n \hat{X}_n^* = \sum_{n=-\infty}^{\infty}\left|\hat{X}_n\right|^2$$

Note:
*** means complex conjugate**

The Fourier Transform

- **The Fourier Series deals with *periodic* signals**

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jnw_o t}$$

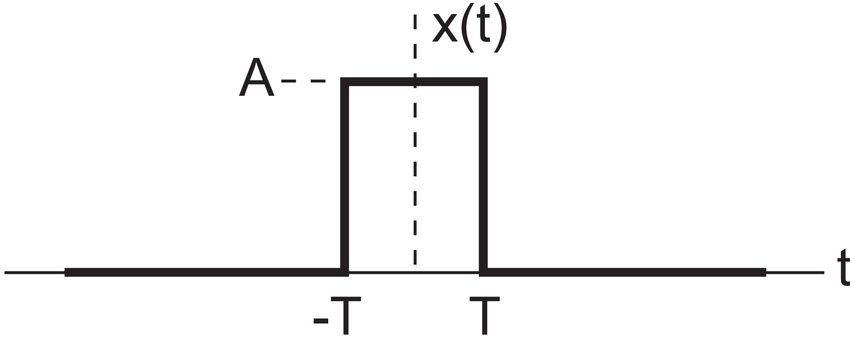
$$\hat{X}_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t) e^{-jnw_o t} dt$$

- **The Fourier Transform deals with *non-periodic* signals**

$$x(t) = \int_{-\infty}^{\infty} X(j2\pi f) e^{j2\pi f t} df$$

$$X(j2\pi f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier Transform Example



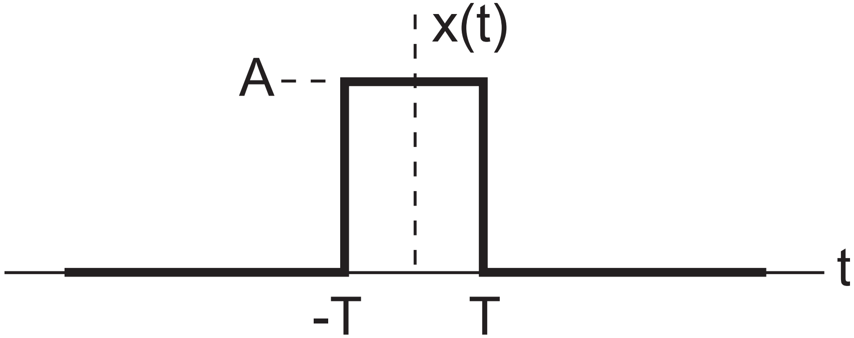
- **Note that *x(t)* is *not* periodic**
- **Calculation of Fourier Transform:**

$$X(j2\pi f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$= \int_{-T}^{T} A e^{-j2\pi f t} dt = \frac{A}{-j2\pi f} e^{-j2\pi f t} \Big|_{-T}^{T}$$

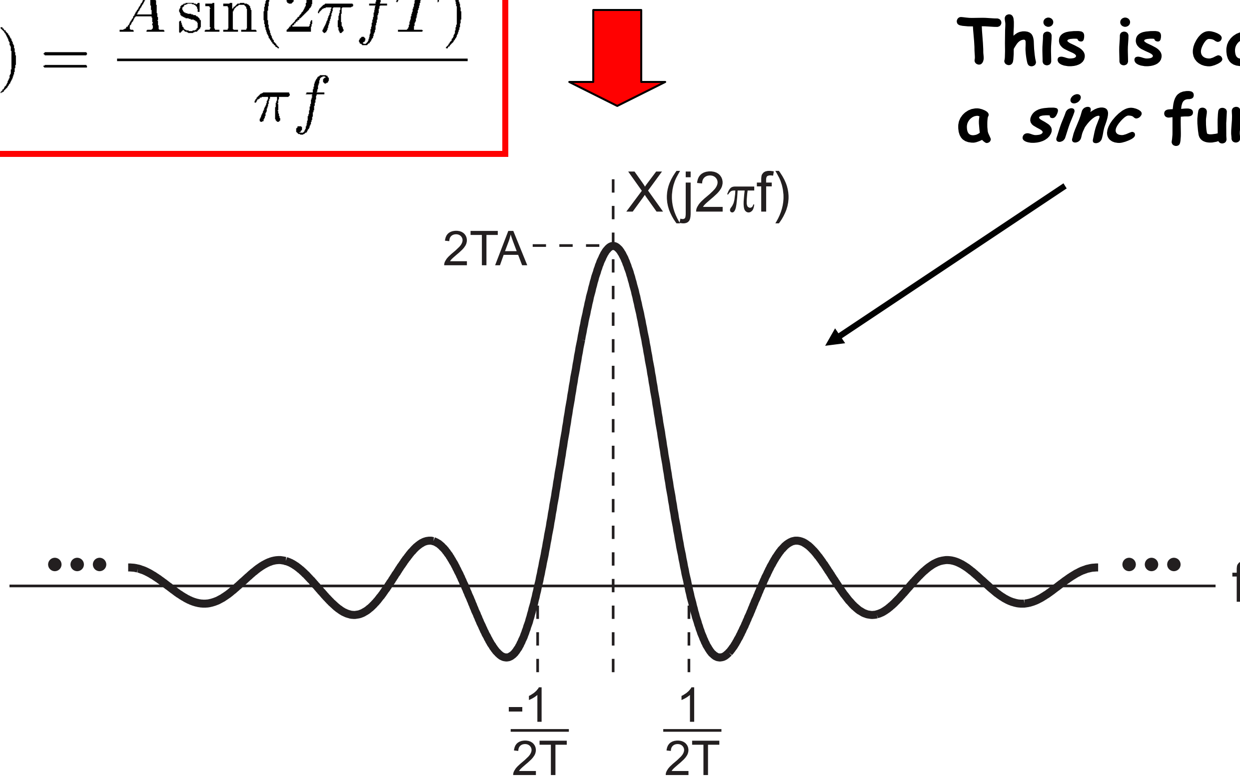
$$= \boxed{\frac{A \sin(2\pi f T)}{\pi f}}$$

Graphical View of Fourier Transform



$$X(j2\pi f) = \frac{A\sin(2\pi f T)}{\pi f}$$

This is called a *sinc* function



Summary

- The Fourier Series can be formulated in terms of complex exponentials
  - Allows convenient mathematical form
  - Introduces concept of positive and negative frequencies
- The Fourier Series coefficients can be expressed in terms of magnitude and phase
  - Magnitude is independent of time (phase) shifts of $x(t)$
  - The magnitude squared of a given Fourier Series coefficient corresponds to the power present at the corresponding frequency
- The Fourier Transform was briefly introduced
  - Will be used to explain modulation and filtering in the upcoming lectures
  - We will provide an intuitive comparison of Fourier Series and Fourier Transform in a few weeks …