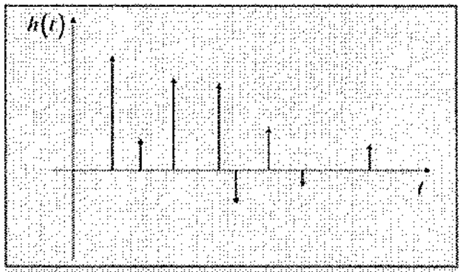
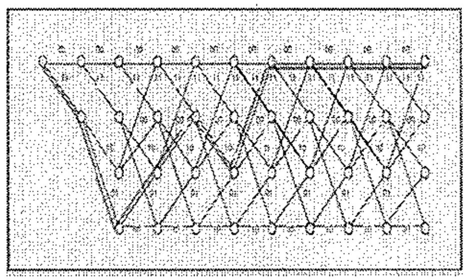
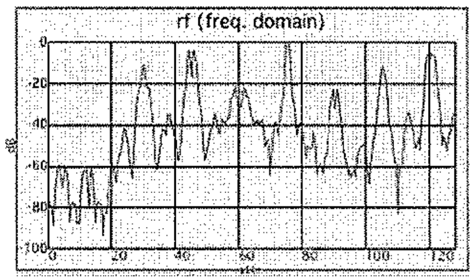
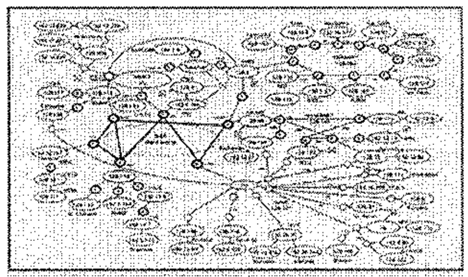








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2009
Lecture #11

- Summary of progress
- The problem of multiplexing
- Eternal Signals and LTI Systems
- Discrete-Time Sines and Cosines

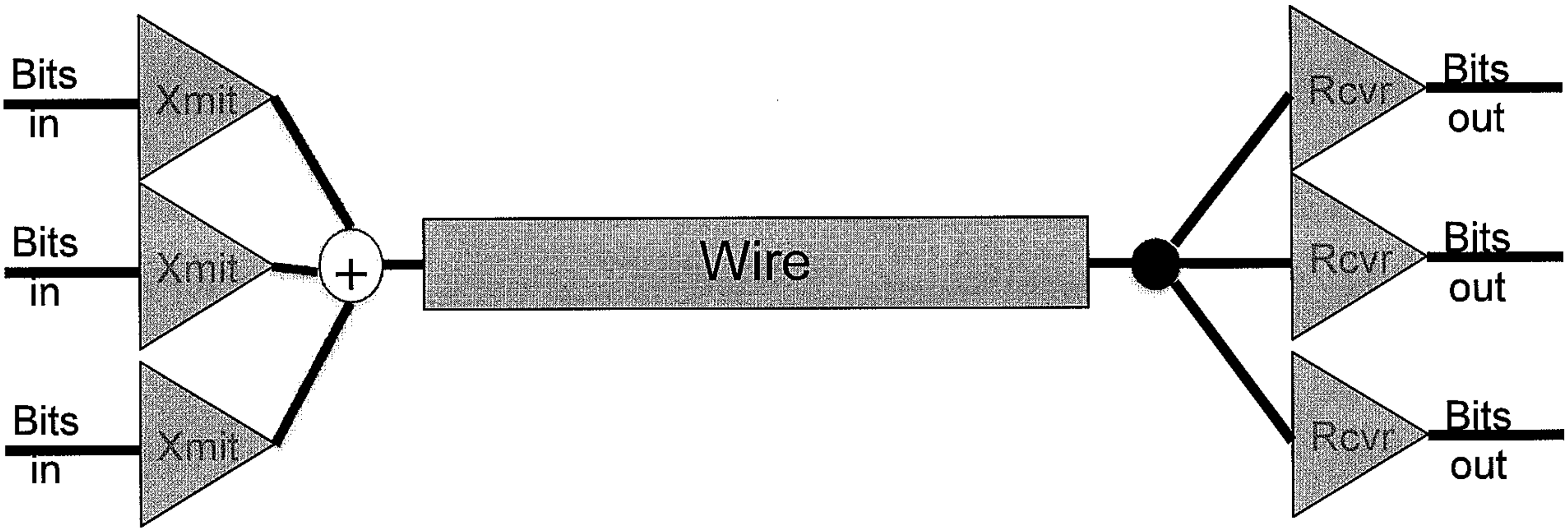
2

So Far - Low BER Transmission on Wires

- Problems and Analysis Techniques:
  - Intersymbol Interference (ISI)
    - LTI Systems and Unit Sample Responses
    - Eye Diagrams
  - Noise
    - Probability Density and Cumulative Distribution Functions
    - Normal (Gaussian) random variables
- Solution Approaches
  - Techniques based on LTI models
    - Deconvolution and Decision Feedback Equalization
  - Error Detection and Correction Codes
    - Parity bits, Reed-Solomon Codes, Viterbi Algorithm

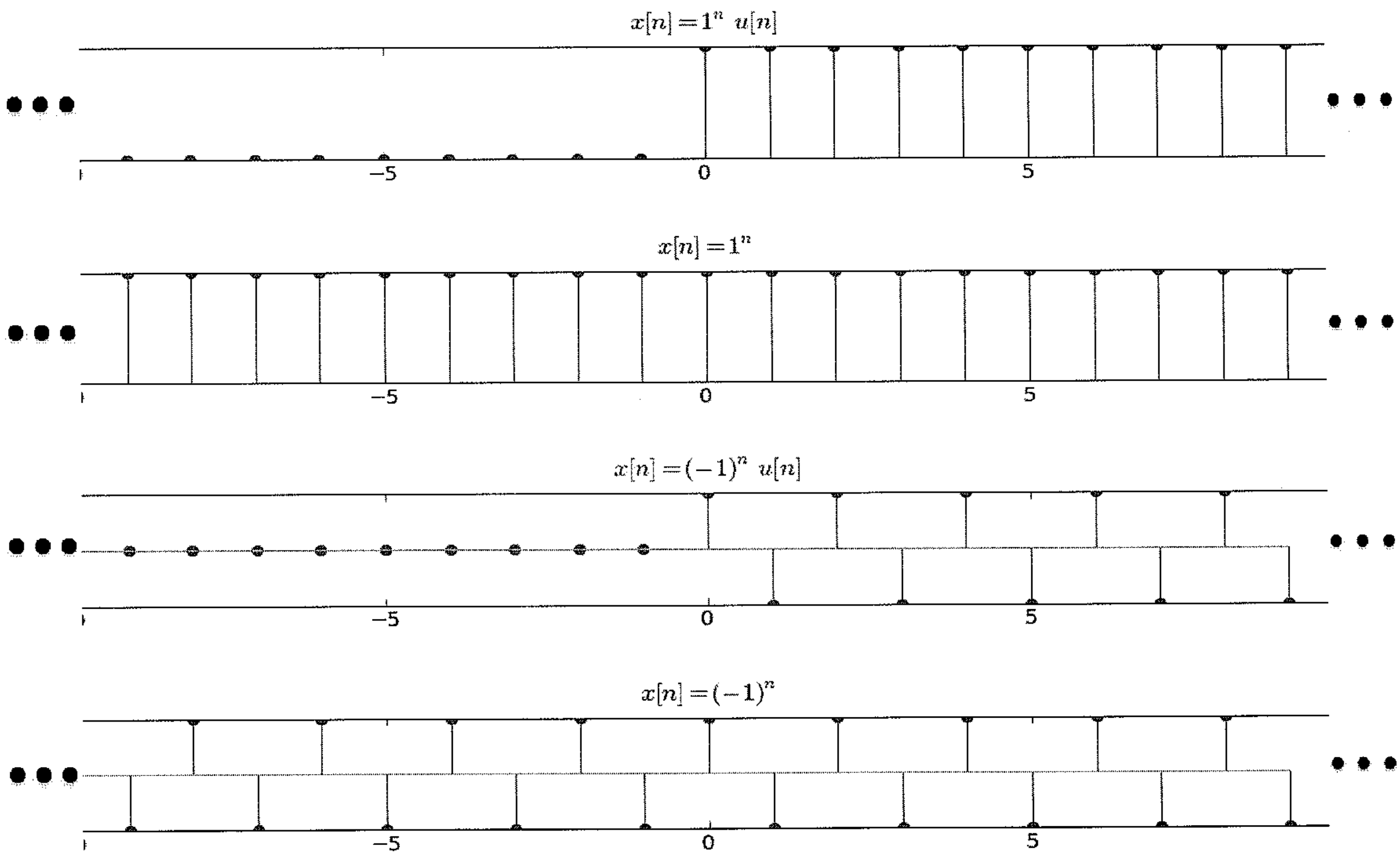
3

New Problem - Resource Sharing



- Two Approaches
  - Time Division Multiplexing (TDM)
    - Each Xmit-Rcvr pair gets a time slot (how to decide?)
    - Used by wired internet
  - Frequency Division Multiplexing (FDM)
    - Each Xmit-Rcvr gets part of the spectrum (explained later)
    - Like Broadcast TV and Radio, many wireless devices
- Next Two Weeks on FDM

Consider Samples for $-\infty < n < \infty$



Eternal Signals

LTI Systems for Eternal Signals

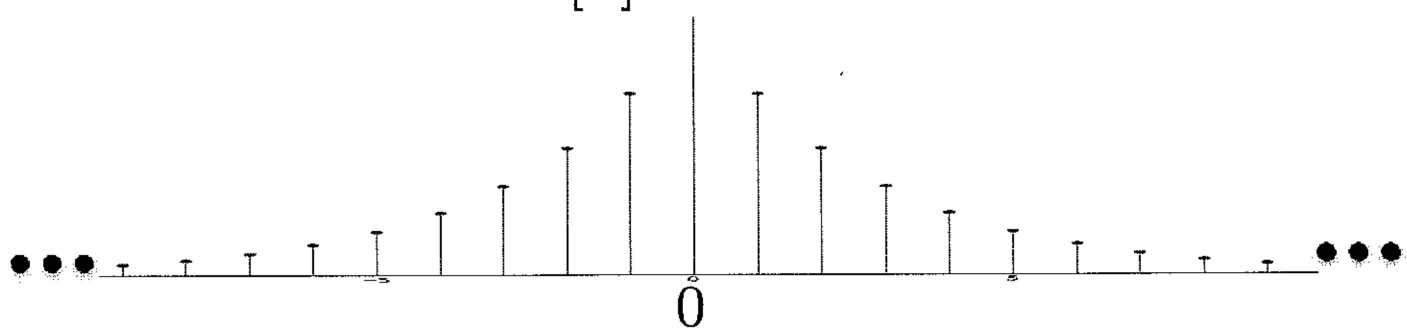
- Convolution Changes a little

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

x[n] may be eternal

h[m] may not be causal

$h[n] = 0.7^{|n|}$



6

Suppose $x[n] = z^n, \quad -\infty < n < \infty$

- Find y using convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$$

Reorganizing

$$y[n] = \left(\sum_{m=-\infty}^{\infty} h[m] z^{-m}\right) z^n$$

Just a number if sum converges = *H(z)*

$$y[n] = H(z) z^n$$

y[n] has the same form as x[n]!

(7)

So What

- Suppose $x[n] = A_1 z_1^n + A_2 z_2^n$

$$y[n] = \left(\sum_{m=-\infty}^{\infty} h[m] z_1^{-m}\right) A_1 z_1^n + \left(\sum_{m=-\infty}^{\infty} h[m] z_2^{-m}\right) A_2 z_2^n$$

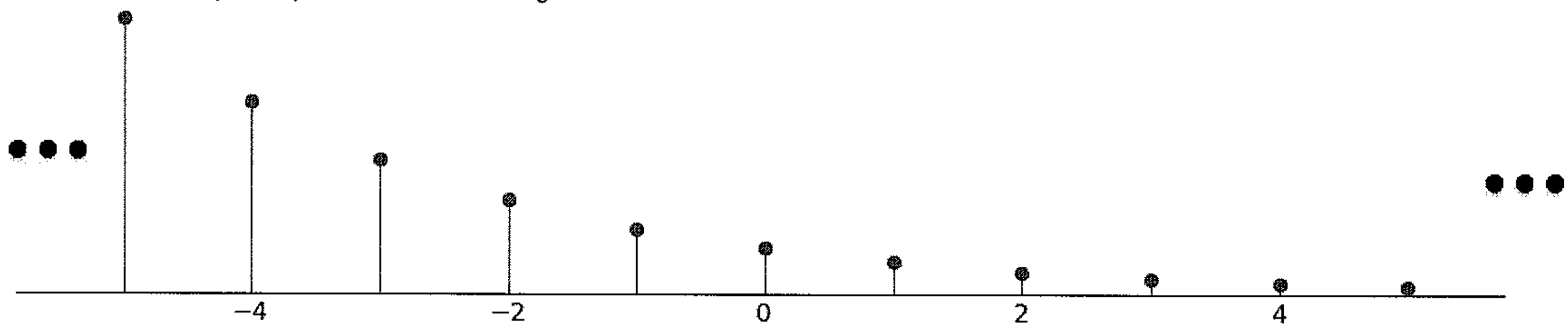
$$\Downarrow$$

$$y[n] = H(z_1) A_1 z_1^n + H(z_2) A_2 z_2^n$$

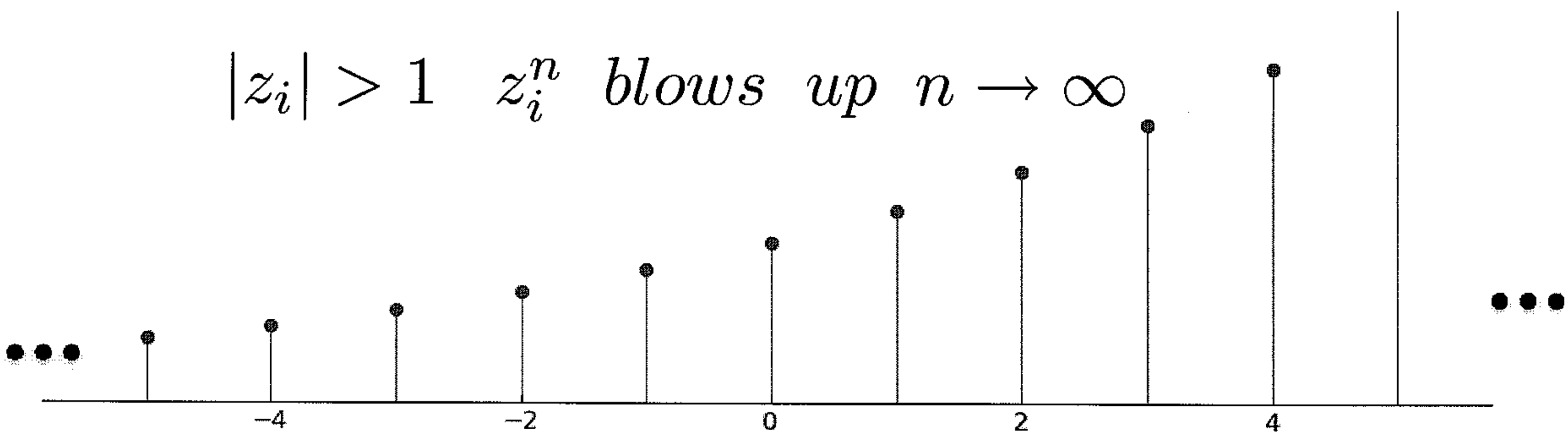
- If we can separate z_1^n part from z_2^n part in y[n]
  - Can recover two messages
- Could then use lots of z_i's
  - Multiple channels!

A problem with eternal signals

$$|z_i| < 1 \quad z_i^n \ \ blows \ \ up \ \ n \rightarrow -\infty$$



$$|z_i| > 1 \quad z_i^n \ \ blows \ \ up \ \ n \rightarrow \infty$$



Use $|z_i| = 1$

- How many z's are there with unit magnitude?
  - z=1, z=-1, any others?
- Can generate more z's using complex numbers

$$z = e^{j\Omega} = cos\Omega + j sin\Omega$$

- Magnitude is one for any frequency

$$|z| = |e^{j\Omega}| = |\cos\Omega + j\sin\Omega| = \sqrt{\cos^2\Omega + \sin^2\Omega} = 1$$

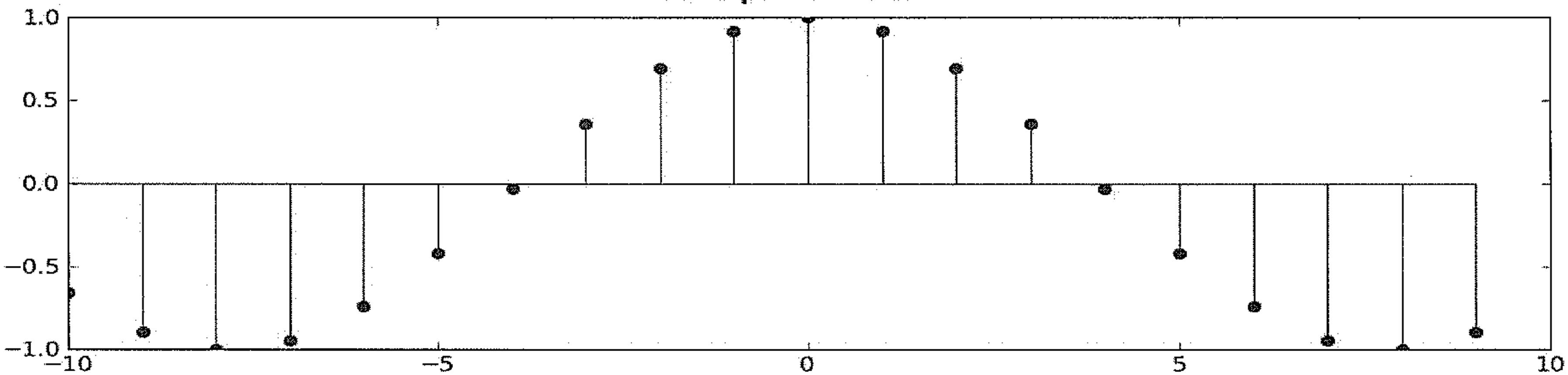
- Know how to evaluate z^n

$$z^n = e^{j\Omega n} = \cos\Omega n + j\sin\Omega n$$

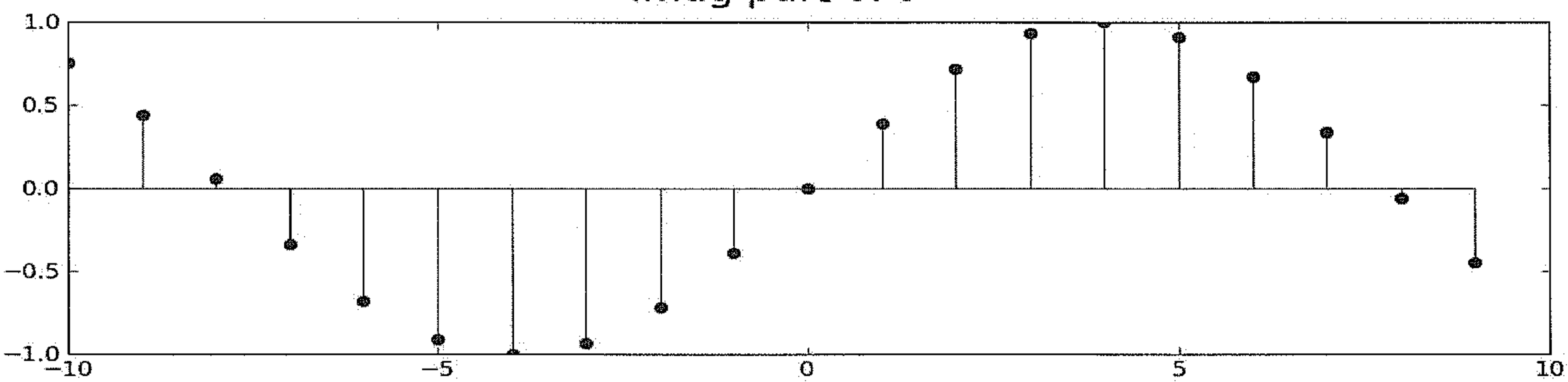
10

Example: $z^n = e^{0.4jn} = \cos 0.4n + j \sin 0.4n$

Real part of $e^{0.4jn}$



Imag part of $e^{0.4jn}$



$|e^{j\Omega n}| = 1$!

11

Frequency Response

- From convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] e^{j\Omega(n-m)}$$

Reorganizing

$$y[n] = \left(\sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m}\right) e^{j\Omega n}$$

A *complex* number if the sum converges

$$y[n] = H\left(e^{j\Omega}\right) e^{j\Omega n}$$

$H(e^{j\Omega})$, $-\pi < \Omega < \pi$, is the frequency response

12

Summary and Next Time

- Frequency Division Multiplexing
  - Eternal z^n's do not mix when input to LTI systems
  - Need many eternal z^n's that do not blow up, $|z| = 1$
  - Use $z_i = e^{j\Omega_i}, \quad -\pi < \Omega_i \leq \pi$
- Next Week
  - How do we separate the different frequencies in y[n]
    - Filters and Fourier Analysis
- After Spring Break
  - Encoding Information using different frequencies
    - What happens when we use

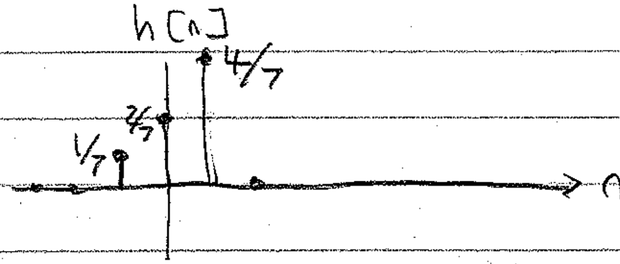
$$x[n] = A_1[n]e^{j\Omega_1 n} + A_2[n]e^{j\Omega_2 n}$$

    - Do the modulated complex exponentials still stay separated?

Example

$$x[n] = \left(\frac{1}{2}\right)^n$$

$$y[n] = \sum h[m] \, \frac{1}{2}^{(n-m)}$$

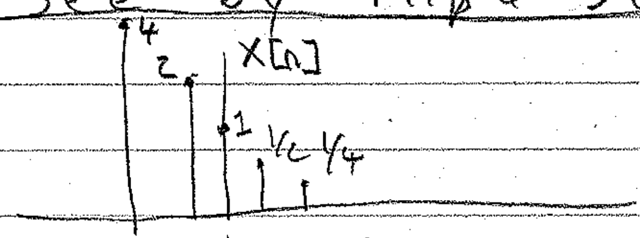


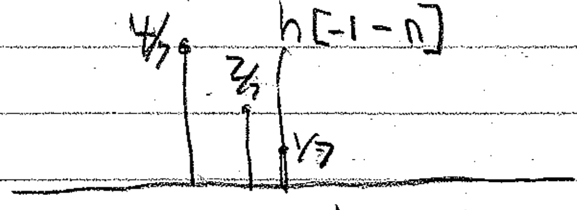
$$y[n] = \left(\frac{1}{7}\right)\frac{1}{2}^{(n+1)} + \left(\frac{2}{7}\right)\frac{1}{2}^{(n-0)} + \left(\frac{4}{7}\right)\left(\frac{1}{2}\right)^{n-1}$$

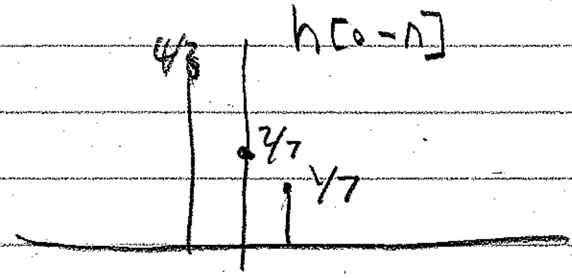
$$= \left(\frac{1}{2}\frac{1}{7} + \frac{2}{7}\cdot 1 + \frac{4}{7}\left(\frac{1}{2}\right)^{-1}\right)\left(\frac{1}{2}\right)^n$$

$$= \left(\frac{1}{14} + \frac{2}{7} + \frac{8}{7}\right)\left(\frac{1}{2}\right)^n = \frac{3}{2}\cdot\left(\frac{1}{2}\right)^n$$

Also see by flip & slide







$y[-1] =$

$\frac{1}{7}\cdot 1 + \frac{2}{7}2 + \frac{4}{7}\cdot 4$

$= 3 = \frac{3}{2}\left(\frac{1}{2}\right)^{-1}$

$y[0] =$

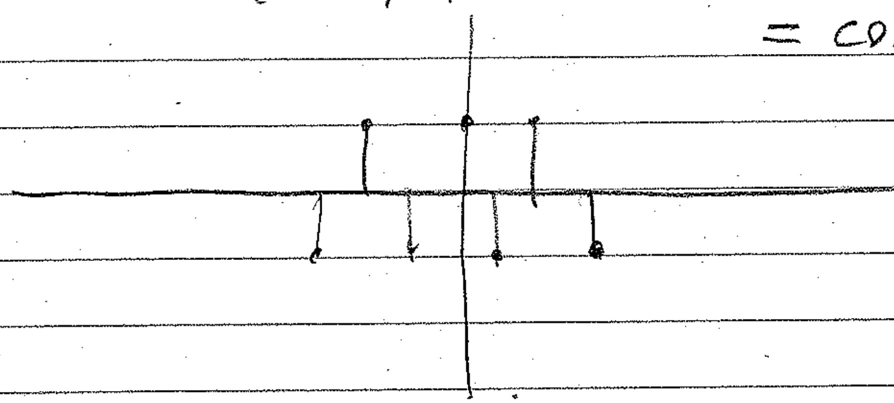
$\frac{1}{2}\frac{1}{7} + 1\cdot\frac{2}{7}$

$+ 2\cdot\frac{4}{7}$

$= 3/2$

Only consider $-\pi \leq \Omega \leq \pi$

Highest frequency $= e^{j\pi n} = \cos \pi n + j \sin \pi n$
$= \cos \pi n = (-1)^n$



$$e^{j(-\pi)n} = \cos(-\pi)n + j\sin(-\pi)n$$
$$= \cos \pi n = (-1)^n$$

Consider $\Omega = \pi + \Delta \quad \Delta > 0$ (outside $-\pi \to \pi$)

$$e^{j(\pi+\Delta)n} = \cos(\pi+\Delta)n + j\sin(\pi+\Delta)n$$

$$e^{j\pi n} e^{j\Delta n} = (-1)^n (\cos \Delta n + j \sin \Delta n)$$

$$e^{j(-\pi+\Delta)n} = e^{j(-\pi)n} e^{j\Delta n}$$
$$= (-1)^n (\cos \Delta n + j \sin \Delta n)$$

$$e^{j(\pi+\Delta)n} = e^{j(-\pi+\Delta)n}$$

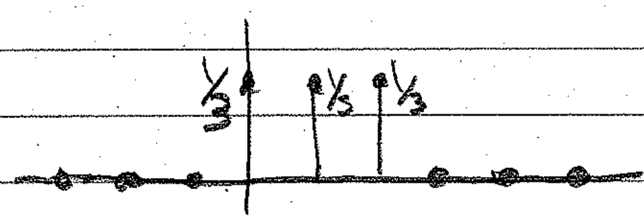
$\Omega = \pi + \Delta$ (outside $-\pi \to \pi$) same as $\Omega = \Delta - \pi$ (Inside $-\pi \to \pi$)

Example

$$y[n] = \sum_{m=0}^{\infty} h[m]\, x[n-m]$$

(∞ ← eternal x; $m=0$ ← causal h)

h[n]



Case 1 $\quad x[n] = \left(\frac{1}{2}\right)^n$

$$y[n] = \left(\sum_{m=0}^{\infty} h[m] \left(\frac{1}{2}\right)^{-m}\right)\left(\frac{1}{2}\right)^n$$

$$= \left(\frac{1}{3}\cdot\left(\frac{1}{2}\right)^0 + \frac{1}{3}\left(\frac{1}{2}\right)^{-1} + \frac{1}{3}\left(\frac{1}{2}\right)^{-2}\right)\left(\frac{1}{2}\right)^n$$

$$= \frac{7}{3}\left(\frac{1}{2}\right)^n \quad \text{\underline{for all} } n$$

Case 2 $\quad x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$y[0] = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^0 = \frac{1}{3} \neq \frac{7}{3}\left(\frac{1}{2}\right)^0$$

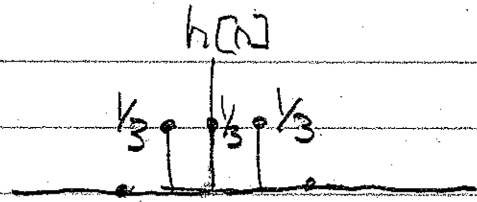
$$y[1] = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^0 + \frac{1}{3}\left(\frac{1}{2}\right)^1 = \frac{1}{2} \neq \frac{7}{3}\left(\frac{1}{2}\right)^1$$

$$y[2] = \frac{1}{3}\left(\frac{1}{2}\right)^0 + \frac{1}{3}\left(\frac{1}{2}\right)^1 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 = \frac{7}{12} = \frac{7}{3}\left(\frac{1}{2}\right)^2$$

$\vdots$

$y[n] = \frac{7}{3}\left(\frac{1}{2}\right)^n$ only for large n

Example



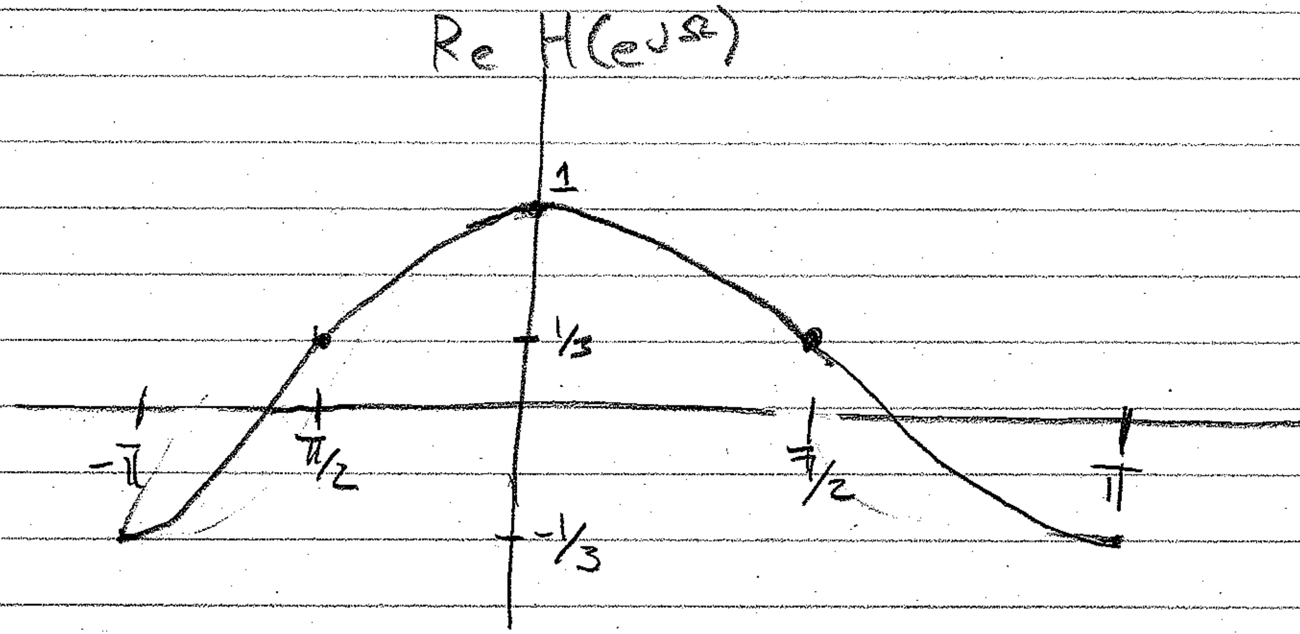
$$H(e^{j\Omega}) = \frac{1}{3}e^{j\Omega} + \frac{1}{3}e^{-j\Omega} + \frac{1}{3}$$

$$= \frac{1}{3}(\cos\Omega + j\sin\Omega)$$

$$+ \frac{1}{3}(\cos\Omega - j\sin\Omega)$$

$$+ \frac{1}{3}$$

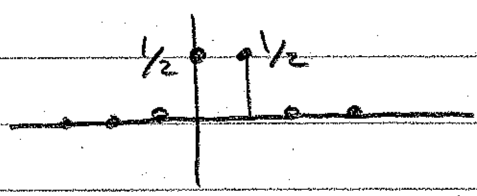
$$= \frac{2}{3}\cos\Omega + \frac{1}{3}$$



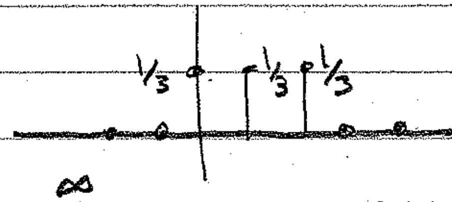
Example

Frequency Response in two cases

$h_1[n]$



$h_2[n]$

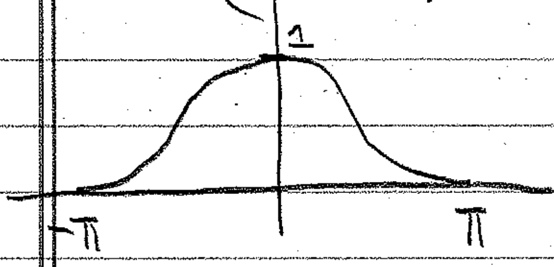


$$H(e^{j\Omega}) = \sum_{m=0}^{\infty} h[m]\, e^{-j\Omega m}$$

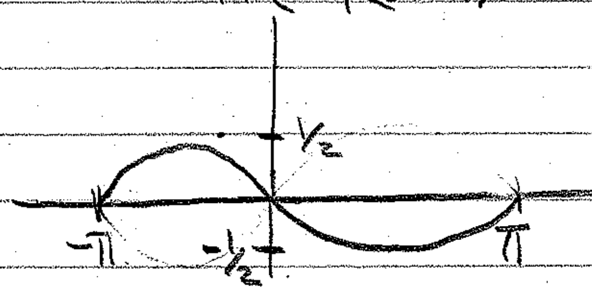
$$H_1(e^{j\Omega}) = h_1[0]\cdot e^{-j\Omega 0} + h_1[1]\, e^{-j\Omega\cdot 1}$$
$$= \tfrac{1}{2}(1 + \cos\Omega) - \tfrac{1}{2}j \sin\Omega$$

$$H_2(e^{j\Omega}) = \tfrac{1}{3} e^{-j\Omega 0} + \tfrac{1}{3} e^{-j\Omega 1} + \tfrac{1}{3} e^{-j\Omega\cdot 2}$$
$$= \tfrac{1}{3}(1 + \cos\Omega + \cos 2\Omega)$$
$$- \tfrac{1}{3}j(\sin\Omega + \sin 2\Omega)$$

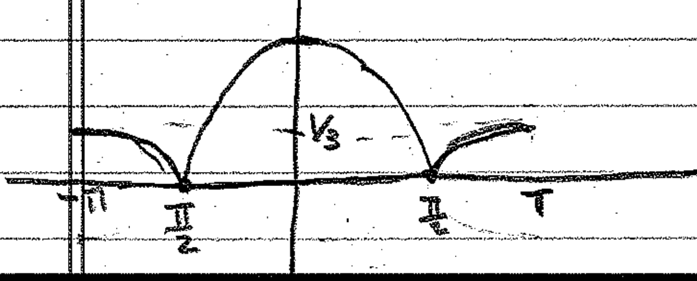
$Re(H_1(e^{j\Omega}))$



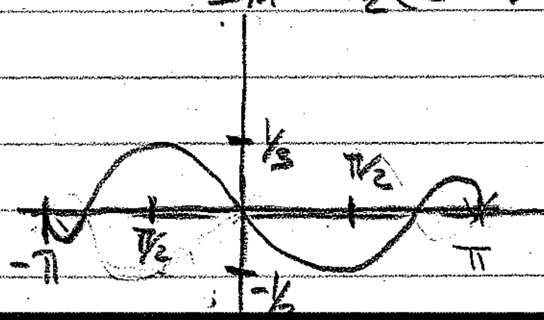
$Im(H_1(e^{j\Omega}))$



$Re\ H_2(e^{j\Omega})$



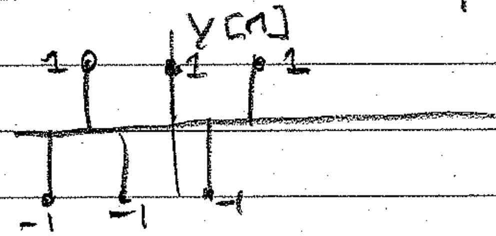
$Im\ H_2(e^{j\Omega})$

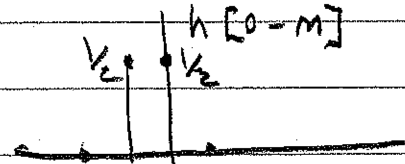


<u>Note</u> if $x[n] = (-1)^n = e^{j\pi n}$

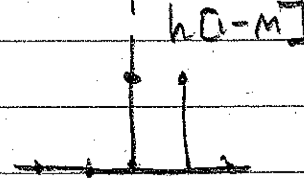
$$y_1[n] = H_1(e^{j\pi})\, e^{j\pi n} = 0 \cdot e^{j\pi n} = 0$$

<u>Makes sense</u> convolve h_1 with y





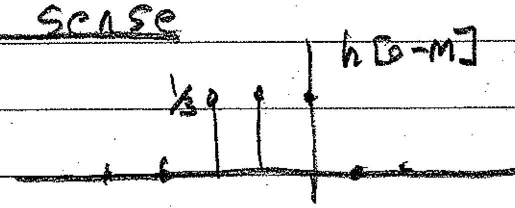
$\Rightarrow y[0] = 1 \cdot \frac{1}{2} + (-1)\frac{1}{2}$
$= 0$



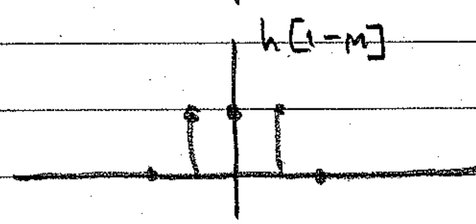
$y[n] = (-1)\frac{1}{2} + (1)\frac{1}{2}$
$= 0$

$$y_2[n] = H_1(e^{j\pi})\, e^{j\pi n} = \frac{1}{3} e^{j\pi n} = \frac{1}{3}(-1)^n$$

<u>Makes sense</u>



$y[0] = \frac{1}{3}$



$y[1] = -\frac{1}{3}$

$y[2] = \frac{1}{3}$

Suppose $x[n] = \cos \Omega n = \frac{1}{2}\left(e^{j\Omega n} + e^{-j\Omega n}\right)$

$$y[n] = \sum h[m] \cos \Omega(n-m)$$

$$y[n] = H(e^{j\Omega})\tfrac{1}{2}e^{j\Omega n} + H(e^{-j\Omega})\tfrac{1}{2}e^{-j\Omega n}$$

Consider $\Omega = \frac{\pi}{2}$ and system 2

$$H(e^{j\pi/2}) = 0 + -\tfrac{1}{3}j$$

$$H(e^{-j\pi/2}) = 0 + \tfrac{1}{3}j$$

$$y[n] = -\tfrac{1}{3}j\,\tfrac{1}{2}e^{j\frac{\pi}{2}n} - \tfrac{1}{3}j\,\tfrac{1}{2}e^{-j\frac{\pi}{2}n}$$

$$= -\tfrac{1}{6}j\left(\cos\tfrac{\pi}{2}n + j\sin\tfrac{\pi}{2}n - \left(\cos\tfrac{\pi}{2}n - j\sin\tfrac{\pi}{2}n\right)\right)$$

$$= -\tfrac{1}{6}j\left(2j\sin\tfrac{\pi}{2}n\right)$$

$$= \tfrac{1}{3}\sin\tfrac{\pi}{2}n$$

Same Amplitude Different phase

Suppose $h[n] =$

(Analyzed Before)

$$y[n] = \underbrace{H(e^{j\frac{\pi}{2}})}_{1/3}\tfrac{1}{2}e^{j\frac{\pi}{2}n} + \underbrace{H(e^{-j\frac{\pi}{2}})}_{1/3}\tfrac{1}{2}e^{-j\frac{\pi}{2}n}$$

$$= \tfrac{1}{6}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$$

$$= \tfrac{1}{3}\cos\tfrac{\pi}{2}n$$