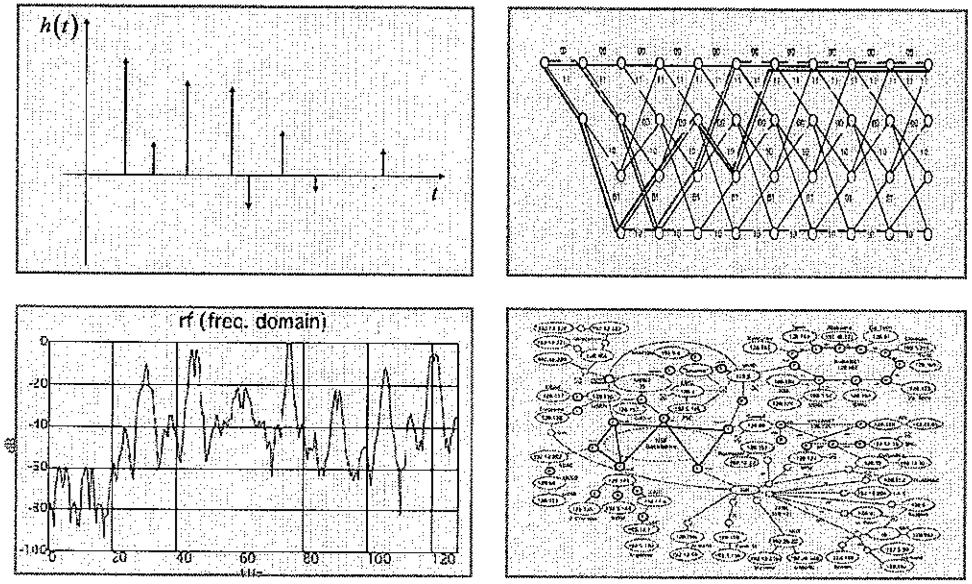

INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2009 Lecture #12

- Frequency Division Multiplexing
- Why Complex Exponentials
- Frequency Response and Filters
- Zeros and Poles

New Problem - Resource Sharing

- Frequency Division Multiplexing Strategy
  - Represent each channel with a different frequency
    - For LTI systems, frequencies do not mix

$$x[n] = A_1 e^{j\Omega_1 n} + \ldots + A_K e^{j\Omega_K n}$$

$$\Downarrow$$

$$y[n] = H(e^{j\Omega_1}) A_1 e^{j\Omega_1 n} + \ldots + H(e^{j\Omega_K}) A_K e^{j\Omega_K n}$$

  - Now need to separate the different frequencies
    - Use Filters to separate Y in to different channels
    - LTI systems with specific frequency responses

Eternal Complex Exponentials

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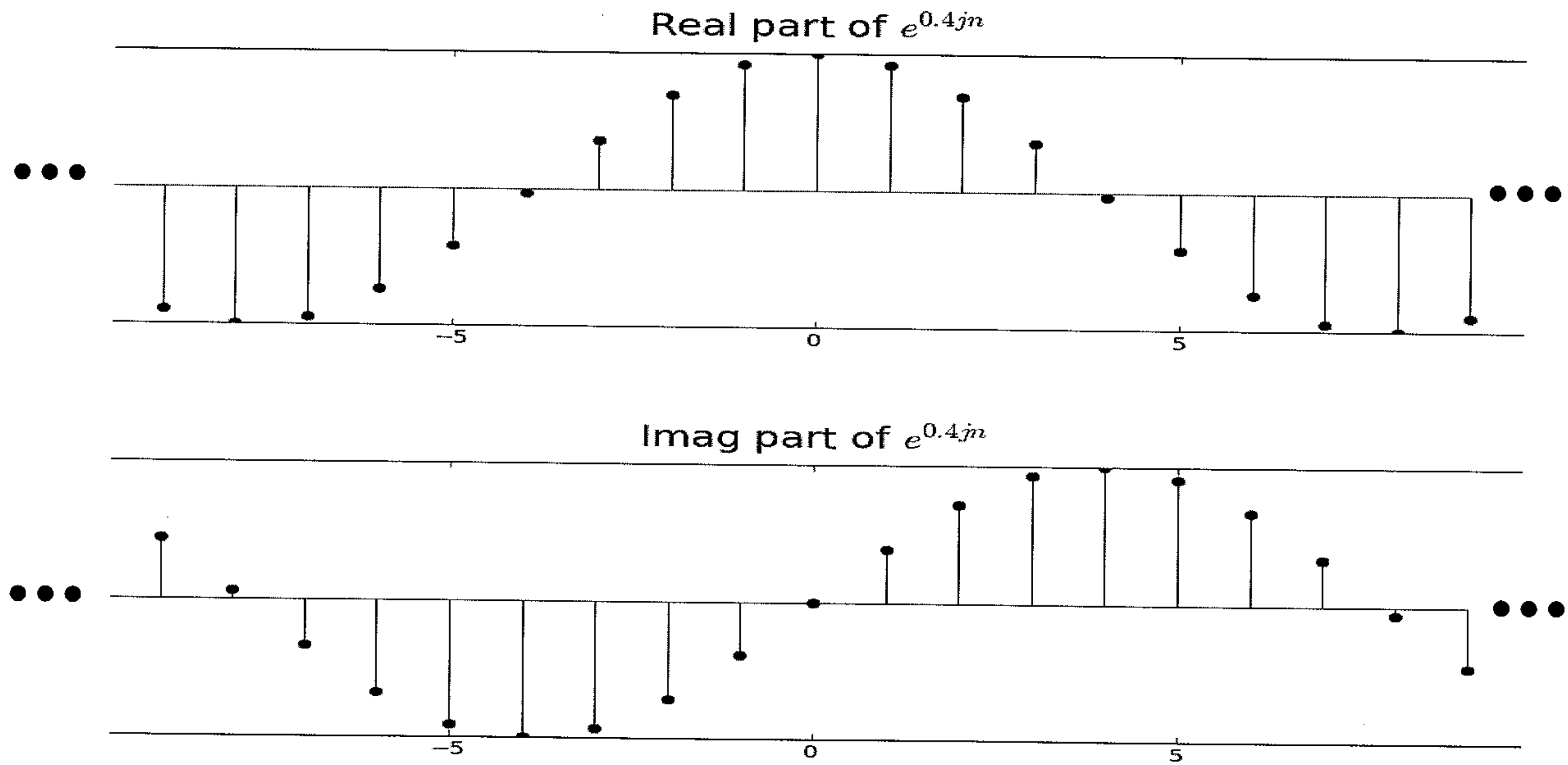
$$x[n] = e^{0.4jn} = \cos 0.4n + j \sin 0.4n$$



Frequency Response

- From convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] e^{j\Omega(n-m)}$$

Reorganizing

$$y[n] = \left(\sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m}\right) e^{j\Omega n}$$

A *complex* number if the sum converges

$$y[n] = H\left(e^{j\Omega}\right) e^{j\Omega n}$$

$H(e^{j\Omega})$, $-\pi < \Omega \leq \pi$, is the frequency response

Why Complex Exponentials?

Suppose $x[n] = \cos \Omega n$

$$y_c[n] = \sum_{m=-\infty}^{\infty} h[m] \underbrace{\cos \Omega(n-m)}$$

How to Simplify?

Suppose $x[n] = e^{j\Omega n}$

$$\mathrm{Re}(x[n]) = \mathrm{Re}(e^{j\Omega n}) = \cos \Omega n$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] e^{j\Omega(n-m)}$$

$$= \sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m} e^{j\Omega n}$$

$$R(y[n]) = \mathrm{Re}\left(\sum h[m] e^{j\Omega(n-m)}\right)$$

$$= \sum h[m] \mathrm{Re}\left(e^{j\Omega(n-m)}\right)$$

$$= \sum h[m] \cos \Omega(n-m)$$

Same Result!

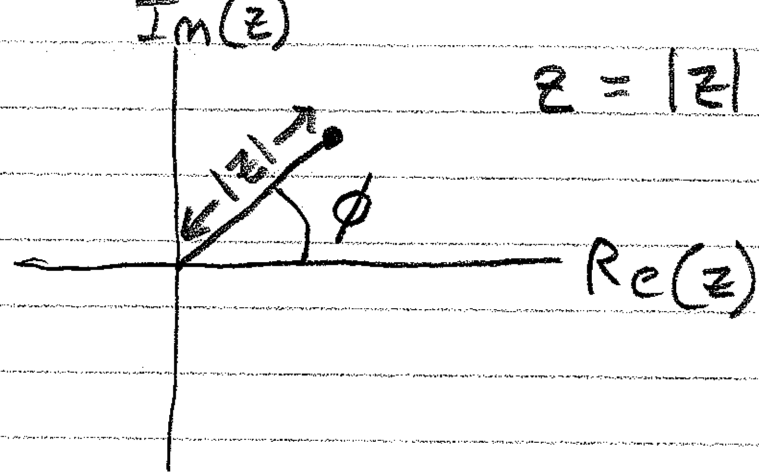
$$\Rightarrow \quad y[n] = H(e^{j\Omega}) e^{j\Omega n}$$

$$y_c[n] = \mathrm{Re}(y[n]) = \mathrm{Re}(H(e^{j\Omega}) e^{j\Omega n})$$

Magnitude and Phase

$$H(e^{j\Omega}) = \underbrace{|H(e^{j\Omega})|}_{\text{Magnitude}} \; \underbrace{e^{j\phi}}_{\text{phase or angle}}$$

Picture



$$z = |z| e^{j\phi}$$

Apply to $x[n] = e^{j\Omega n}$ case

$$y[n] = H(e^{j\Omega}) e^{j\Omega n}$$

$$= |H(e^{j\Omega})| e^{j\phi(\Omega)} e^{j\Omega n}$$

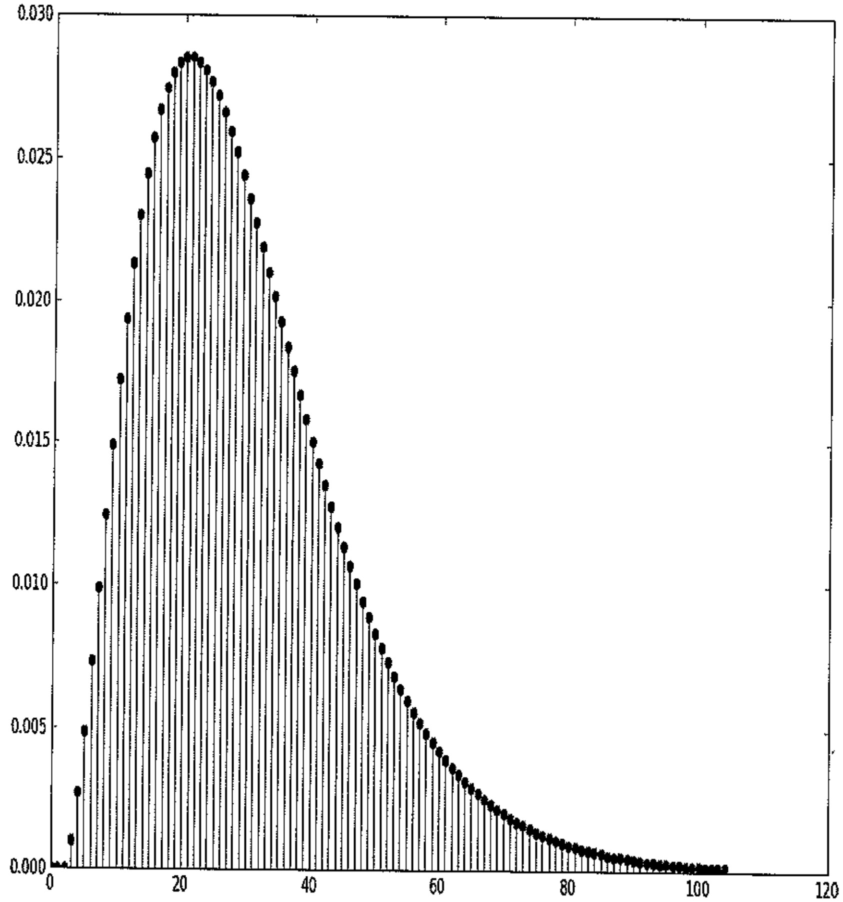
$$= |H(e^{j\Omega})| e^{j(\Omega n + \phi(\Omega))}$$

$$Re(y[n]) = |H(e^{j\Omega})| Re\left(e^{j(\Omega n + \phi(\Omega))}\right)$$

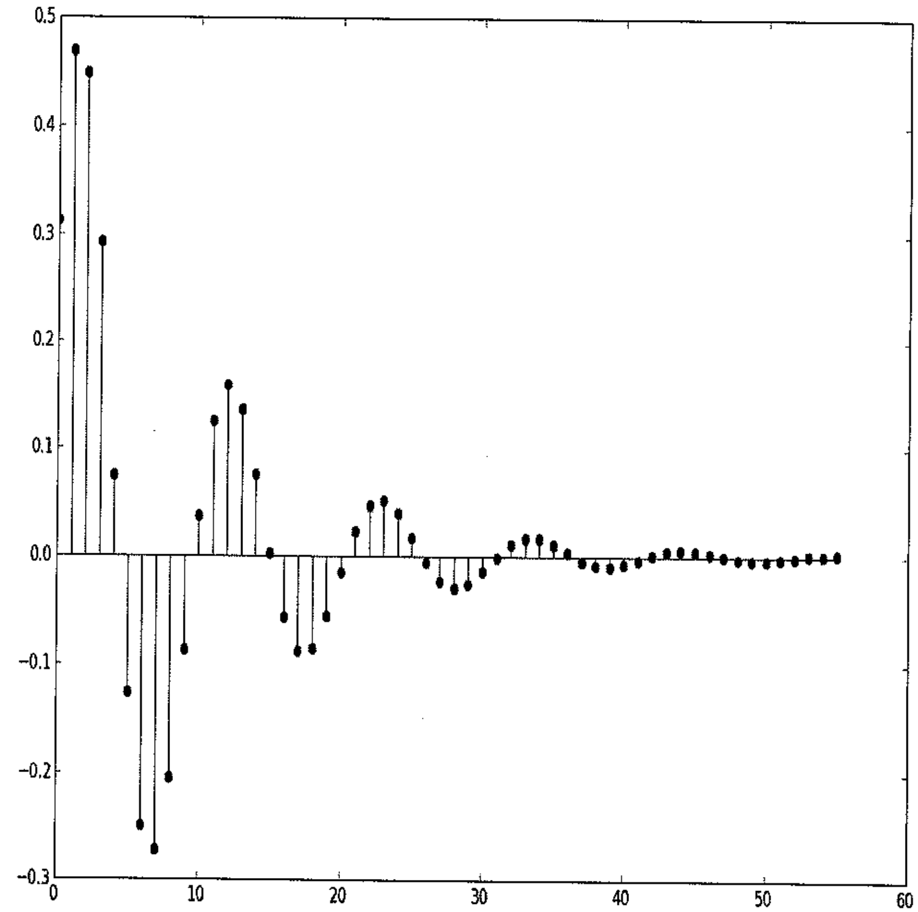
$$= \cos(\Omega n + \phi(\Omega))$$

Recall Channel Unit Sample Responses

Slow Channel



Fast Channel



Magnitude of Frequency Response

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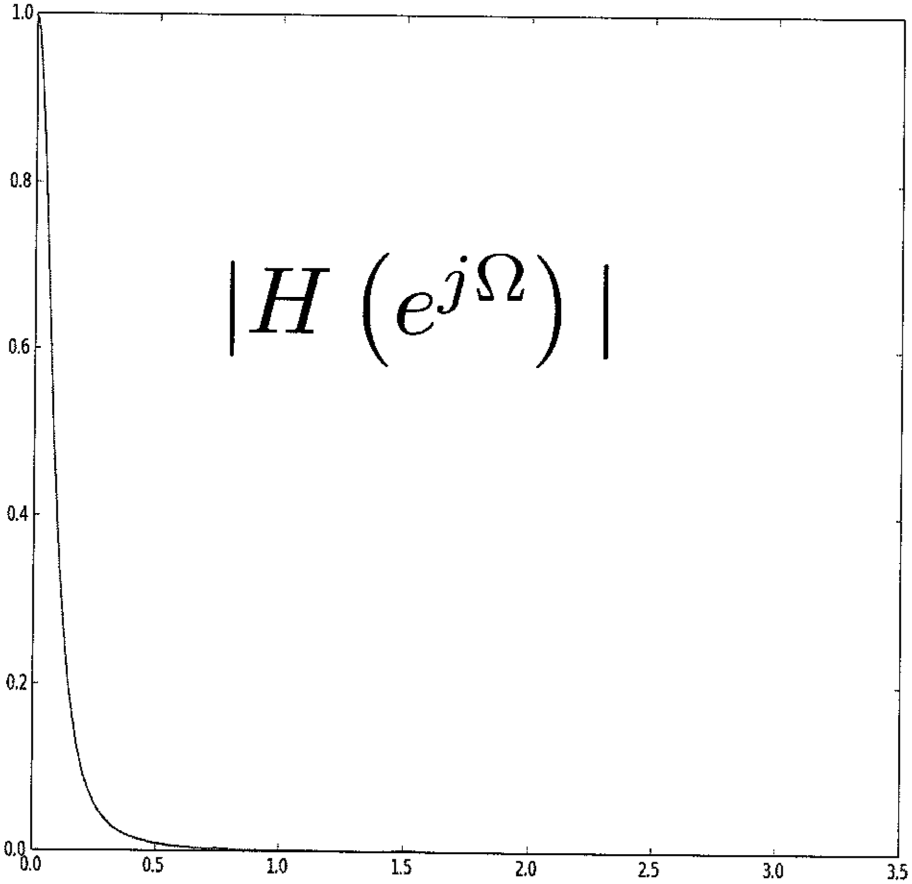
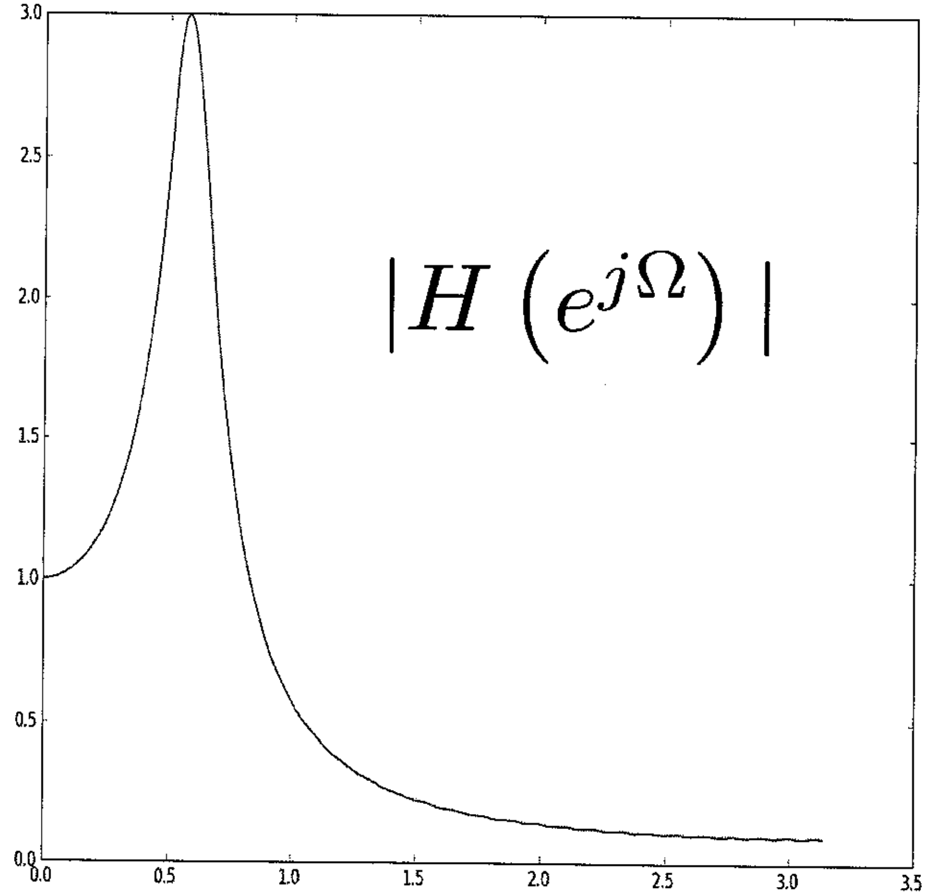
Slow Channel



$$\left|H\left(e^{j\Omega}\right)\right|$$

Fast Channel

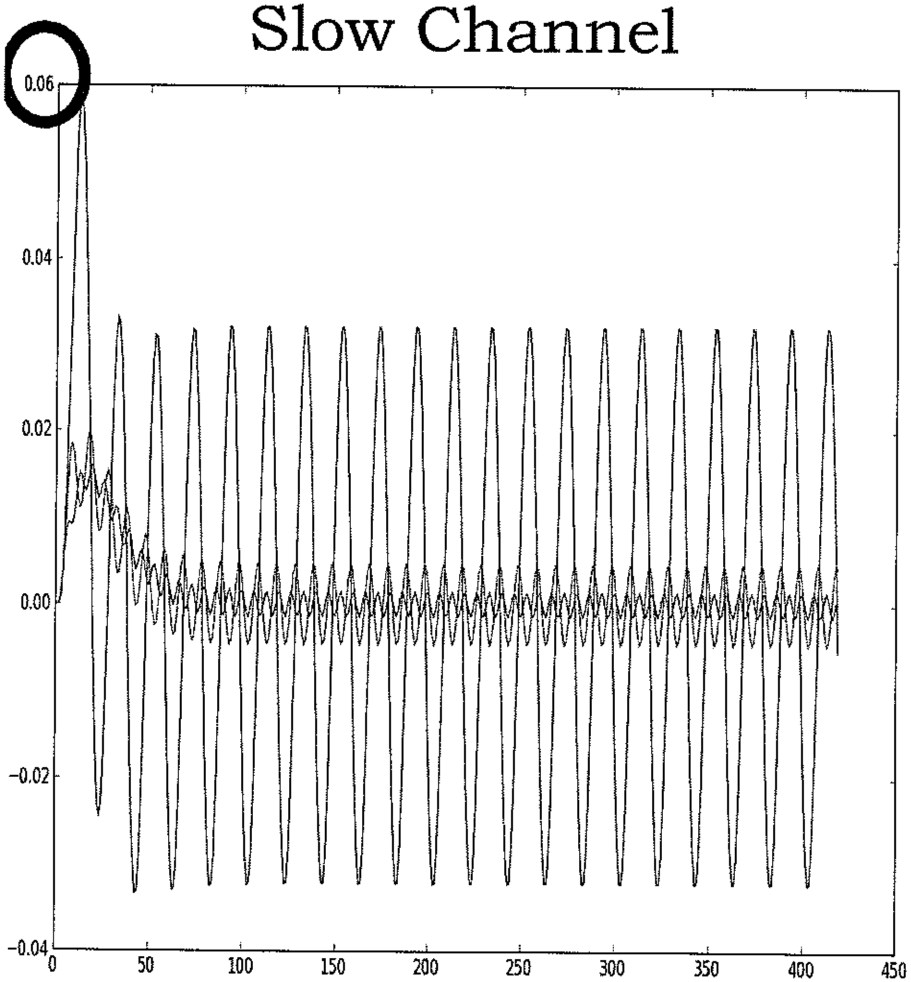


$$\left|H\left(e^{j\Omega}\right)\right|$$

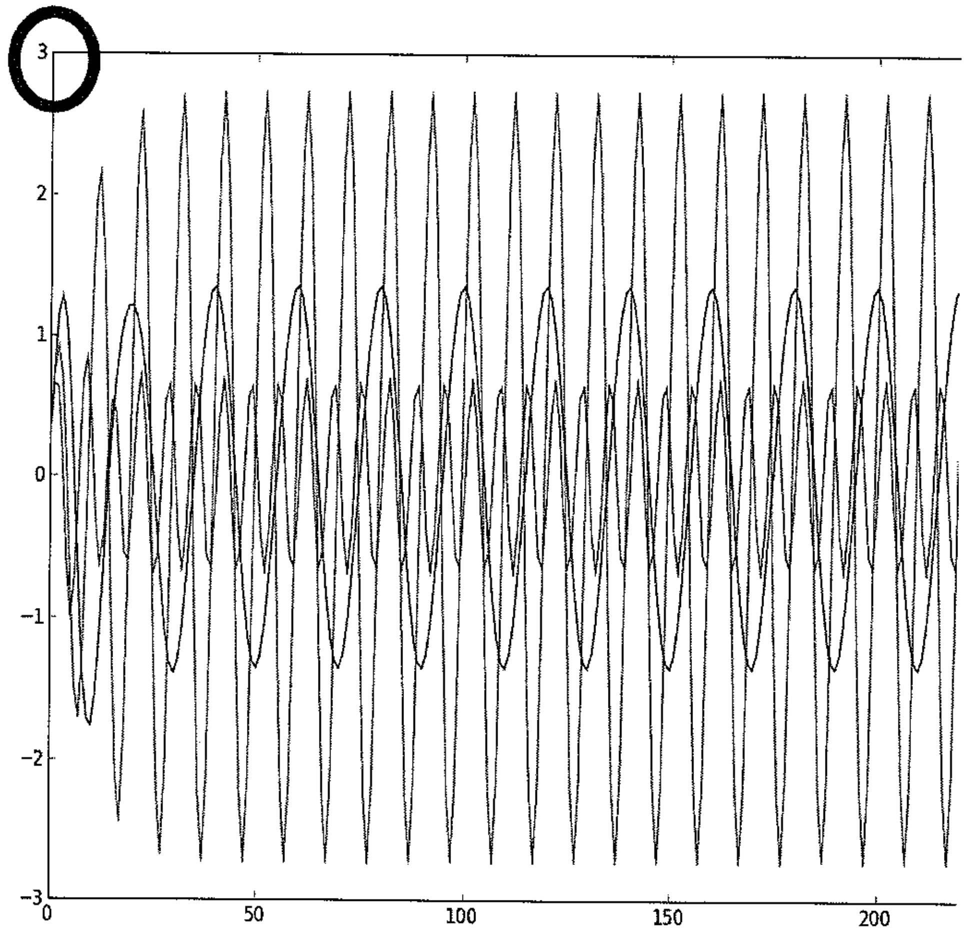
Response to Cosine starting at zero

$$x[n] = \cos(\Omega n)u[n] \;\; \Omega = \left\{\frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}\right\}$$

Slow Channel



Fast Channel



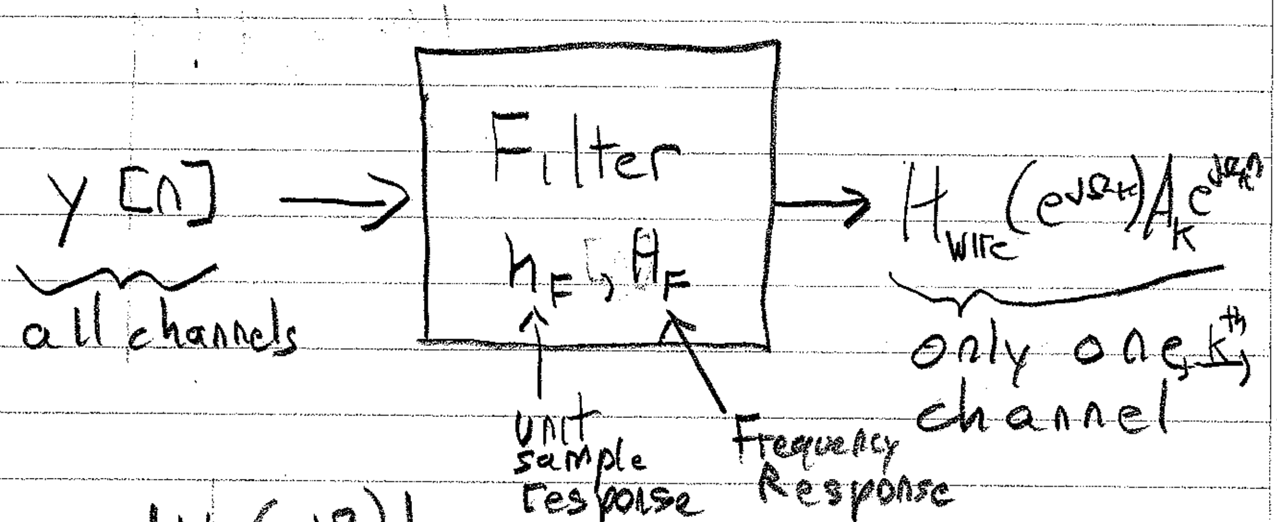
Filter Problem

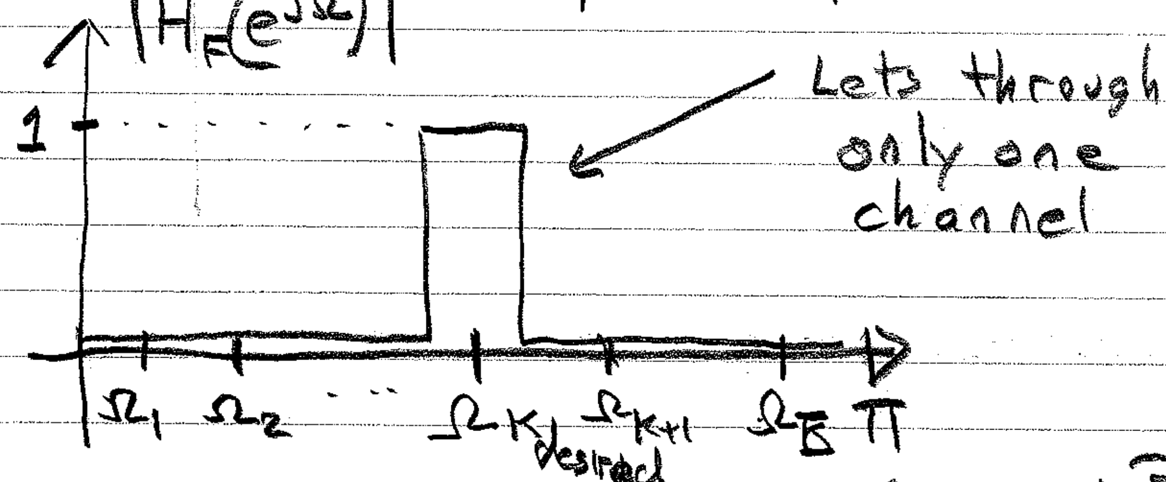
$$y[n] = \sum_{k=0}^{k=K} H_{wire}(e^{j\Omega_k}) A_k e^{j\Omega_k n}$$

(A_k ← k^{th} Message)

Using K Ω's to carry K different channels

Need a System to pick out <u>One</u> Message





What is $h_F[n]$ (U.S.R.)?

Using Zeros

$$H(e^{j\Omega n}) = \sum_{m=0}^{M} h[m]\, e^{-j\Omega m}$$

$$= h[0]\left(\cancel{e^{-j\Omega}}^{\,z}\right)^0 + h[1]\left(\cancel{e^{-j\Omega}}^{\,z}\right)$$

$$h[2]\left(\cancel{e^{-j\Omega}}\right)^{2} \; (z^2) + \ldots + h[M]\left(\cancel{e^{-j\Omega}}^{\,z}\right)^M$$

$$z = e^{-j\Omega}$$

<u>polynomial in z, has roots</u>

$$H(z) = (z - q_0)(z - q_1)\ldots(z - q_M)$$

If $z = q$ then $H(z) = 0$

Pick q's to be <u>undesired</u> freqs

$$q_m = e^{j\Omega_k} \quad k \neq k_{desired}$$

<u>**Wait**</u> $q_m = e^{j\Omega_k} = \cos\Omega_k + j\sin\Omega_k$

Polynomial with <u>real coeffs</u> ⇒ roots are complex-conj pairs

$$q_{m'} = e^{-j\Omega_k} = \cos\Omega_k - j\sin\Omega_k$$

<u>Need</u> $M = 2\cdot(K-1) =$ # undesired frequencies!

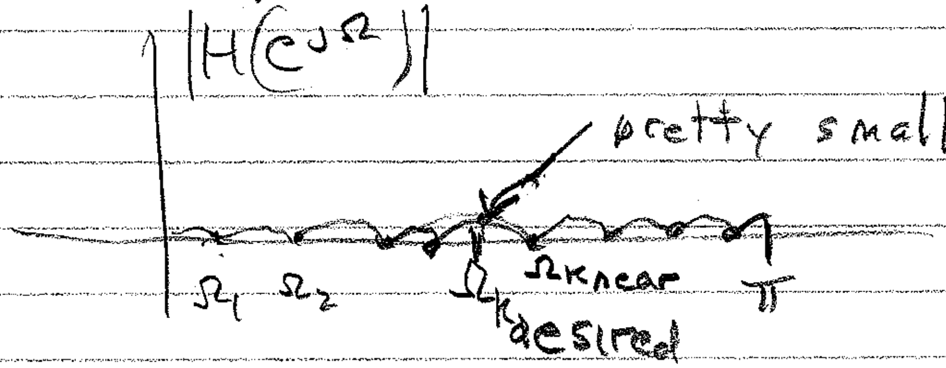
Another Issue

$$H(e^{j\Omega_{kd}}) = \prod_{k \neq k_d} (e^{j\Omega_{kd}} - e^{j\Omega_k}) \cdot \prod_{k = k_d} (e^{j\Omega_{kd}} - e^{-j\Omega_k})$$

(product; k_d = k desired)

$$|H(e^{j\Omega_{kd}})| = \prod |e^{j\Omega_{kd}} - e^{j\Omega_k}| \prod |e^{j\Omega_{kd}} - e^{-j\Omega_k}|$$

If an undesired Ω_k is near Ω_{kd}, $|H(e^{j\Omega_{kd}})|$ will be small!



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Summary and Larger Picture

- Frequency Division Multiplexing
  - K channel's each using a different frequency

$$z_k = e^{j\Omega_k}, \quad 0 \le \Omega_k \le \pi$$

- Filtering
  - Design $|H(e^{j\Omega})|$
  - Use Zeros to eliminate undesired frequencies
    - Must be in complex-conjugate pairs $e^{j\Omega_k}, e^{-j\Omega_k}$
  - Use Poles to magnify desired frequency
    - Magnitude < 1, and conjugate pairs $re^{j\Omega_k}, re^{-j\Omega_k} \quad 0 < r < 1$
- After Spring Break
  - Encoding Information using different frequencies
    - What happens when we use

$$x[n] = A_1[n]e^{j\Omega_1 n} + A_2[n]e^{j\Omega_2 n}$$

    - Do the modulated complex exponentials still stay separated?

Recitation Post Lec 12 Examples

$$x[n] = A_1 \cos\frac{\pi}{6}n + A_2 \cos\frac{\pi}{2}n$$

Design a filter to eliminate one freq

$$\sum_{m=0}^{2} h[m]\, e^{-j\Omega m}$$ (Need 2 zeros because of complex conjugate)

$$= H(e^{j\Omega})$$

$$= (z - z_{z_1})(z - z_{z_2}) \qquad z = e^{-j\Omega}$$

$$= (z - e^{+j\Omega})(z - e^{-j\Omega})$$

$$= z^2 - (e^{+j\Omega} + e^{-j\Omega})z + 1$$

$$= 1 - 2\cos\Omega\, z + 1$$

$$= h[0] + h[1]z + h[2]z^2$$

Matching terms

$$h[0] = 1 \qquad h[1] = -2\cos\Omega$$

$$h[2] = 1$$

$$\Omega = \frac{\pi}{6} \qquad h_{\pi/6}[1] = -2\frac{\sqrt{3}}{2} = -\sqrt{3} \qquad h_{\pi/6}[2] = 1$$

$$\Omega = \frac{\pi}{2} \qquad h_{\pi/2}[1] = -2 \cdot 0 = 0$$

Frequency Responses

$$|H_{\pi/6}(e^{j\Omega})| = |1 - \sqrt{3}\, e^{-j\Omega} + e^{-2j\Omega}|$$

(zeros at $\frac{\pi}{6}, -\frac{\pi}{6}$ case)

See plots

$$|H_{\pi/2}(e^{j\Omega})| = |1 + e^{-2j\Omega}|$$

zeros at $\frac{\pi}{2}, -\frac{\pi}{2}$ case

Design a Filter to zero both $\frac{\pi}{6}$ and $\frac{\pi}{2}$

$$\sum_{i=0}^{4} h[m]\, e^{-j\Omega m} = H(e^{j\Omega})$$

(Need 4 zeros)

$$= (z - z_{z_1})(z - z_{z_2})(z - z_{z_3})(z - z_{z_4})$$

$$= (z - e^{j\Omega_1})(z - e^{-j\Omega_1})(z - e^{j\Omega_2})(z - e^{-j\Omega_2})$$

$$= (z^2 - 2\cos\Omega_1 z + 1)(z^2 - 2\cos\Omega_2 z + 1)$$

$\Omega_1 = \pi/6 \quad \Omega_2 = \pi/2$

$$= (z^2 - \sqrt{3}\, z + 1)(z^2 - 1)$$

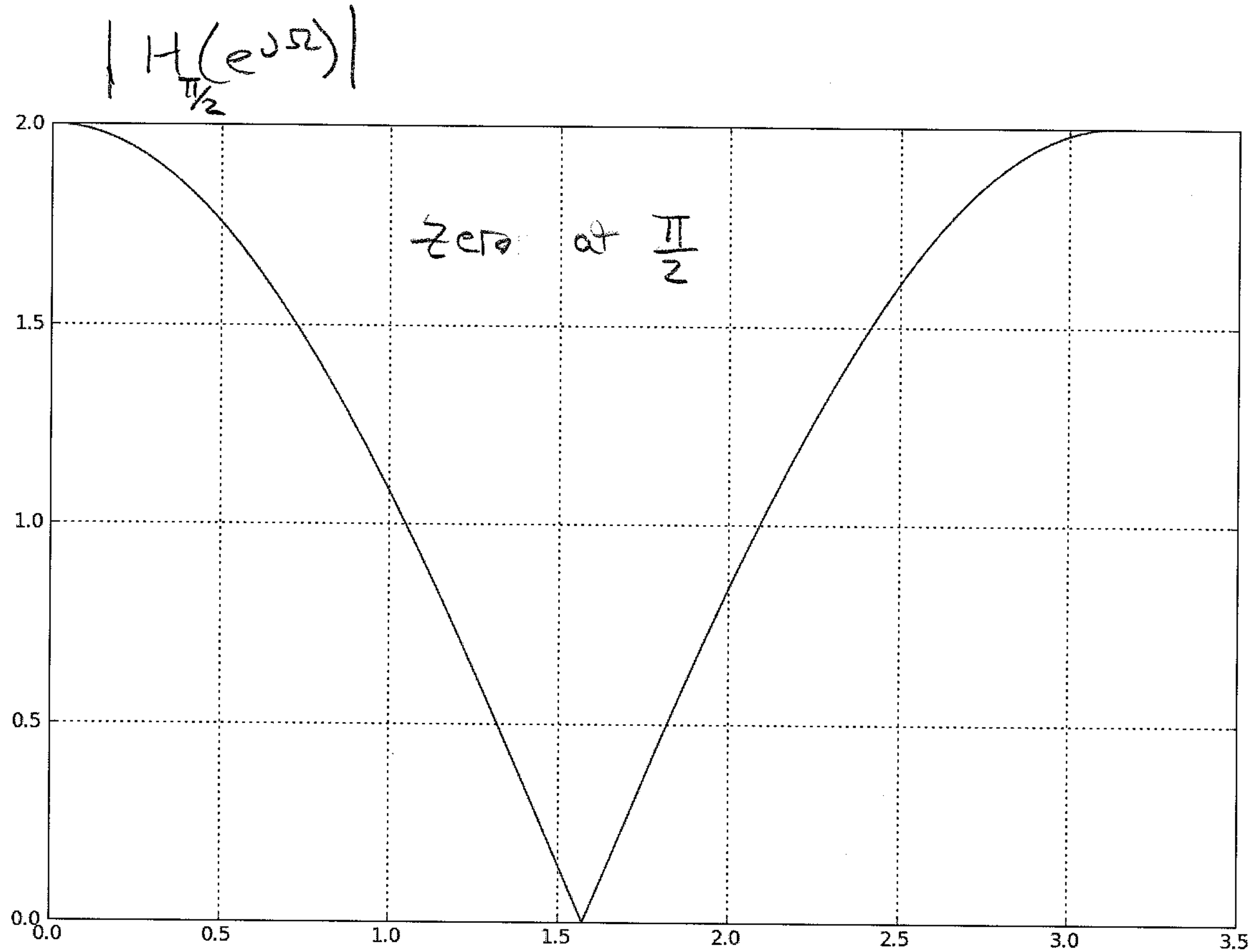
$$= 1 - \sqrt{3}\, z + 2 z^2 - \sqrt{3}\, z^3 + z^4$$

$$= h[0] + h[1] z + h[2] z^2 + h[3] z^3 + h[4] z^4$$

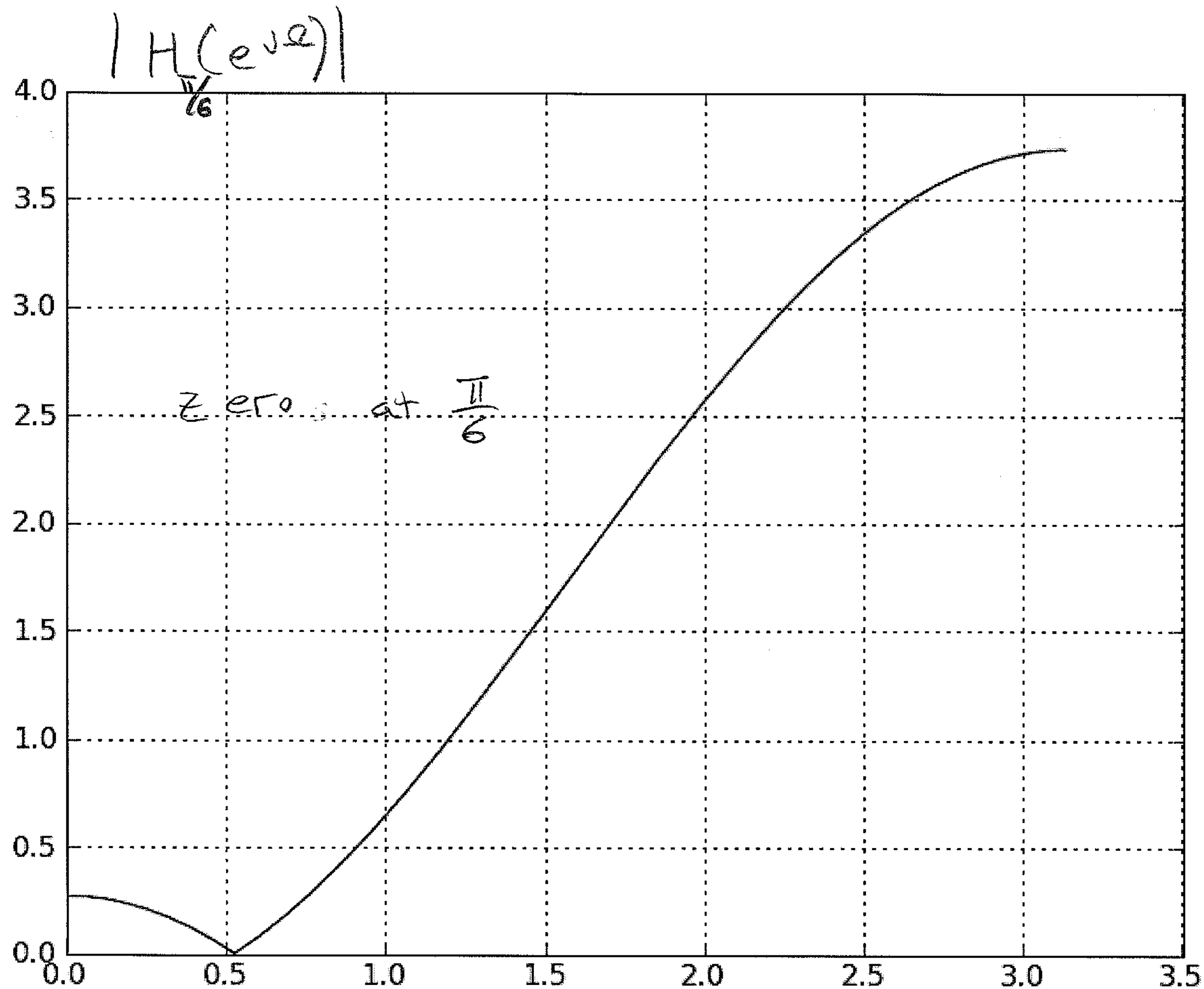
$h[0] = 1 \quad h[1] = -\sqrt{3} \quad h[2] = 2 \quad h[3] = -\sqrt{3} \quad h[4] = 1$

Note $h = h_{\pi/6} * h_{\pi/2}$ & $H(e^{j\Omega}) = H_{\pi/6}(e^{j\Omega}) H_{\pi/2}(e^{j\Omega})$

(16)


$|H_{\pi/2}(e^{j\Omega})|$
zero at $\frac{\pi}{2}$
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

(17)



$|H_{\pi/6}(e^{j\Omega})|$
zeros at $\frac{\pi}{6}$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

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$$|H_{Both}(e^{j\Omega})| = |H_{\pi/6}(e^{j\Omega})\,H_{\pi/2}(e^{j\Omega})|$$