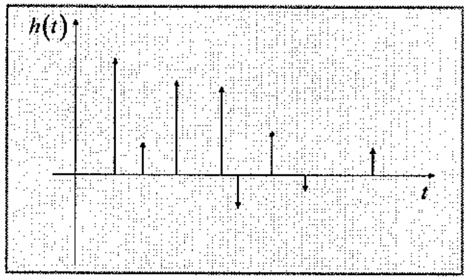
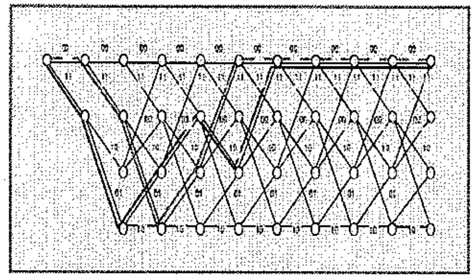
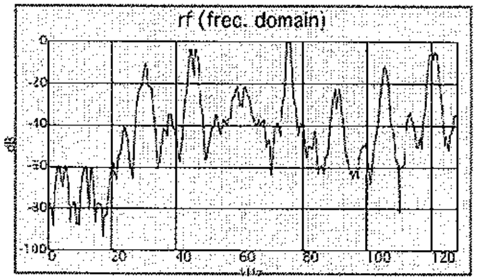
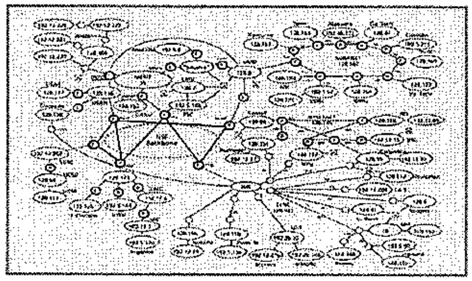
1









INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2009
Lecture #13

- Filters using zeros
- Filters using poles
- Notch Filter Demo

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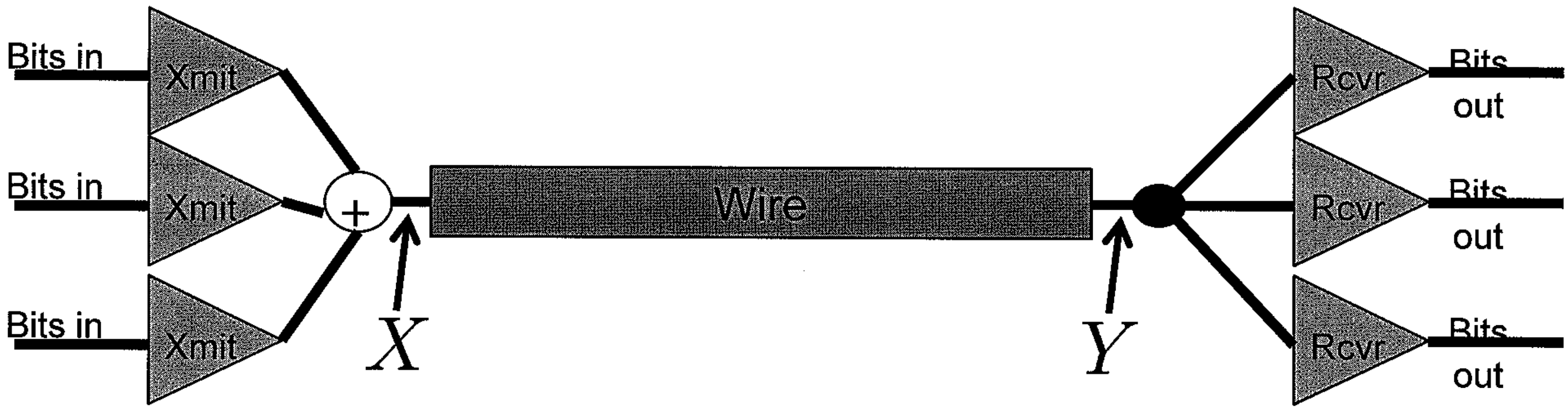
New Problem - Resource Sharing



- Frequency Division Multiplexing Strategy
  - Represent each channel with a different frequency
    - For LTI systems, frequencies do not mix

$$x[n] = A_1 e^{j\Omega_1 n} + ... + A_K e^{j\Omega_K n}$$

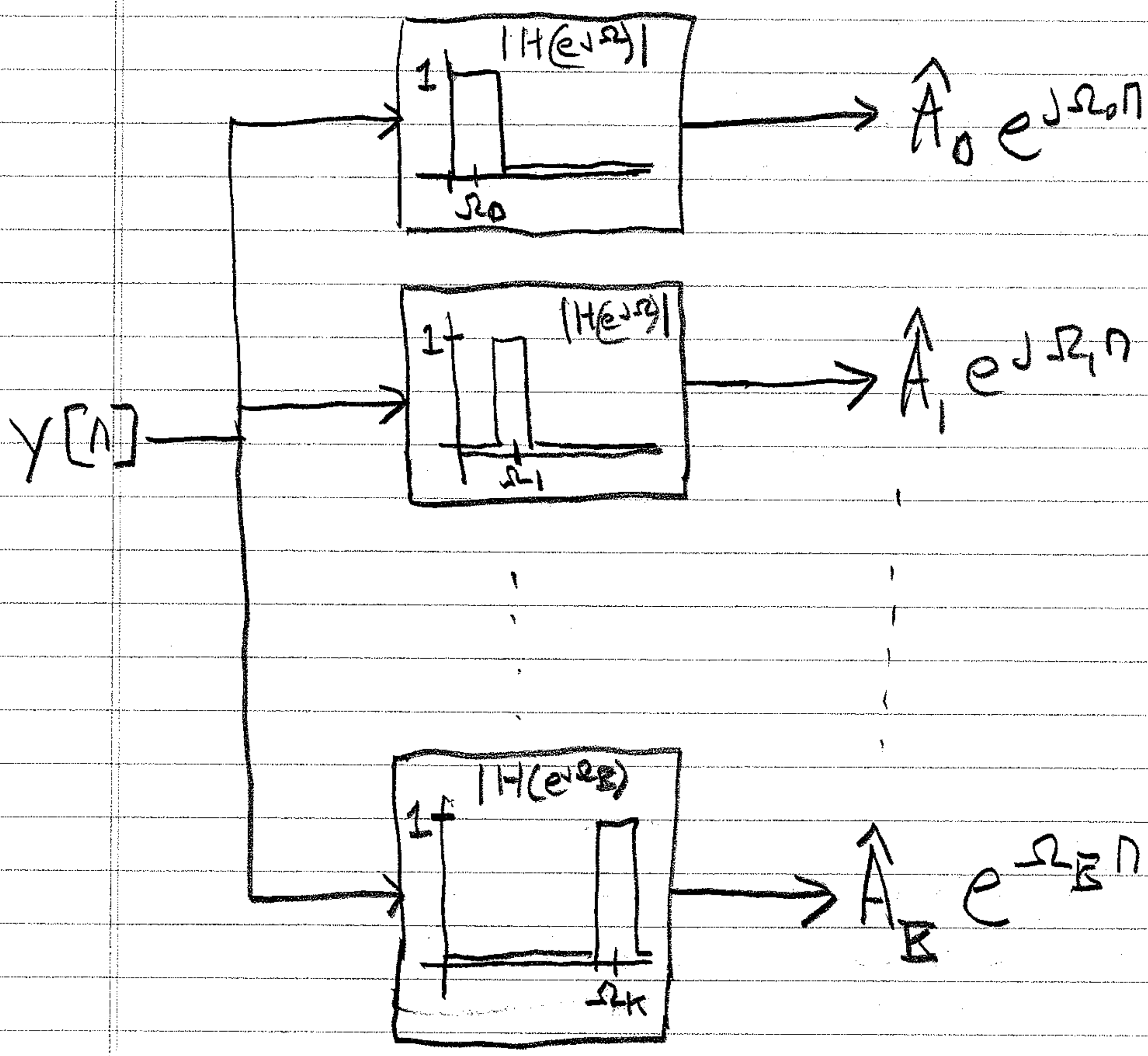
$$\Downarrow$$

$$y[n] = H(e^{j\Omega_1}) A_1 e^{j\Omega_1 n} + ... + H(e^{j\Omega_K}) A_K e^{j\Omega_K n}$$

  - Now need to separate the different frequencies
    - Use Filters to separate Y in to different channels
    - LTI systems with specific frequency responses

For Demultiplexing

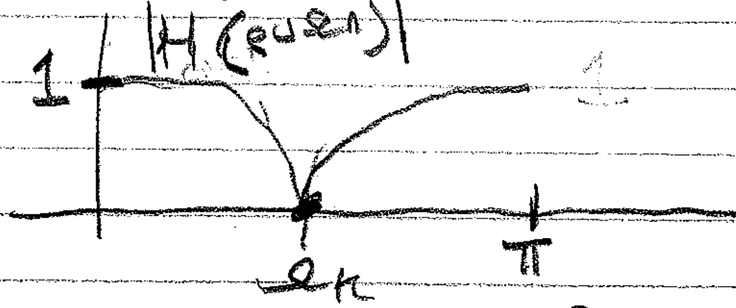
$$y[n] = \underbrace{\sum_{K=0}^{K-1} \hat{A}_K e^{j\Omega_K n}}_{\text{Wire output with many channels}}$$



Zeros based Approach

Suppose want to eliminate

$A_k e^{j\Omega_k n}$



$$w[n] = \sum_{m=0}^{2} h[m]\, y[n-m]$$

$$= h[0]\, y[n] + h[1]\, y[n-1] + h[2]\, y[n-2]$$

For a notch

$$H(e^{j\Omega}) = \sum_{m=0}^{2} h[m]\, e^{-j\Omega m}$$

$$= h[0] + h[1] z + h[2] z^2$$

$$z = e^{-j\Omega m}$$

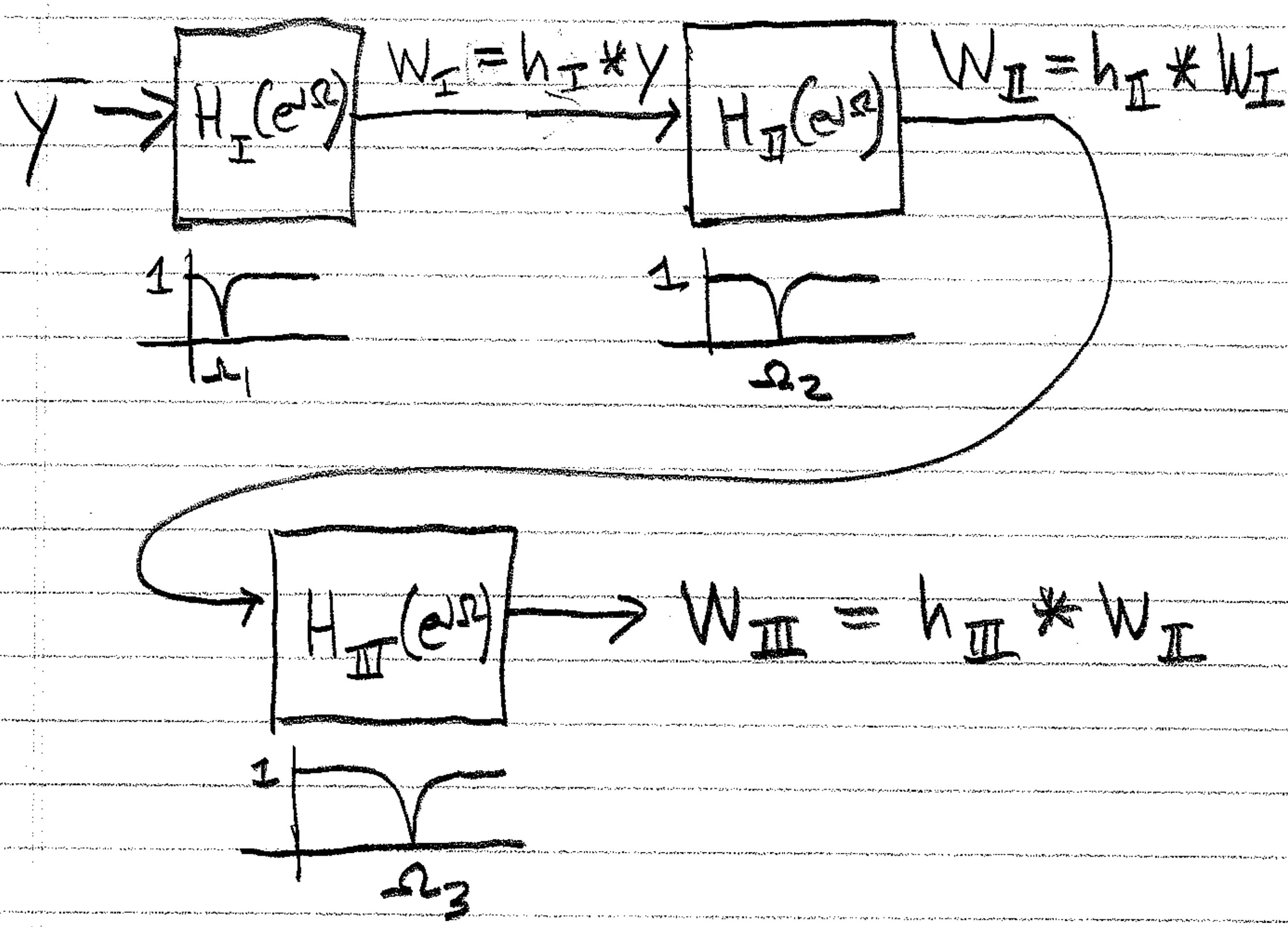
$$= \left(z - e^{+j\Omega_k}\right)\left(z - e^{-j\Omega_k}\right)$$

$$= \underset{h[0]}{1} - \underbrace{2\cos\Omega_k}_{h[1]} z + \underset{h[2]}{1} \cdot z^2$$

Note can also scale $h[n]$

$$\max_{\Omega} \left| H^{\text{scaled}}(e^{j\Omega}) \right| = 1 \Rightarrow h^{\text{scaled}}[n] = \frac{h[n]}{\max_{\Omega} |H(e^{j\Omega})|}$$

To Eliminate Several Frequencies



$$w_{III} = h_{III} * (h_{II} * (h_I * y))$$

$$= (h_{III} * h_{II} * h_I) * y$$

Multiplying Frequency Responses

$\Longleftrightarrow$ Convolving unit sample Responses

$$H(e^{j\Omega}) = H_1(e^{j\Omega}) \cdot H_2(e^{j\Omega}) H_3(e^{j\Omega}) \ldots$$

$$h = h_1 * h_2 * h_3 \ldots$$

Channel ~~5~~ 4, 8 Zeros Filter

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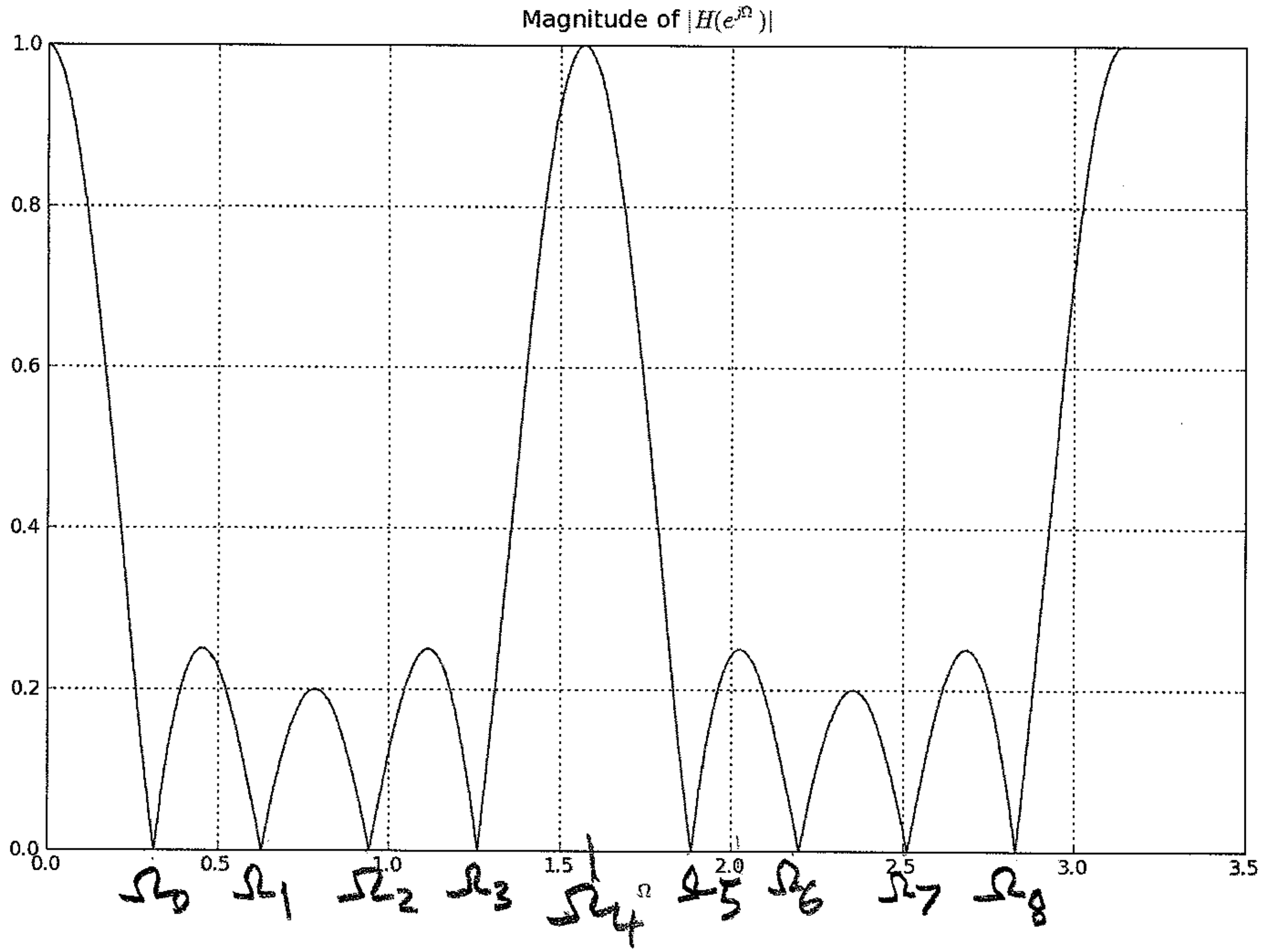

Scaled

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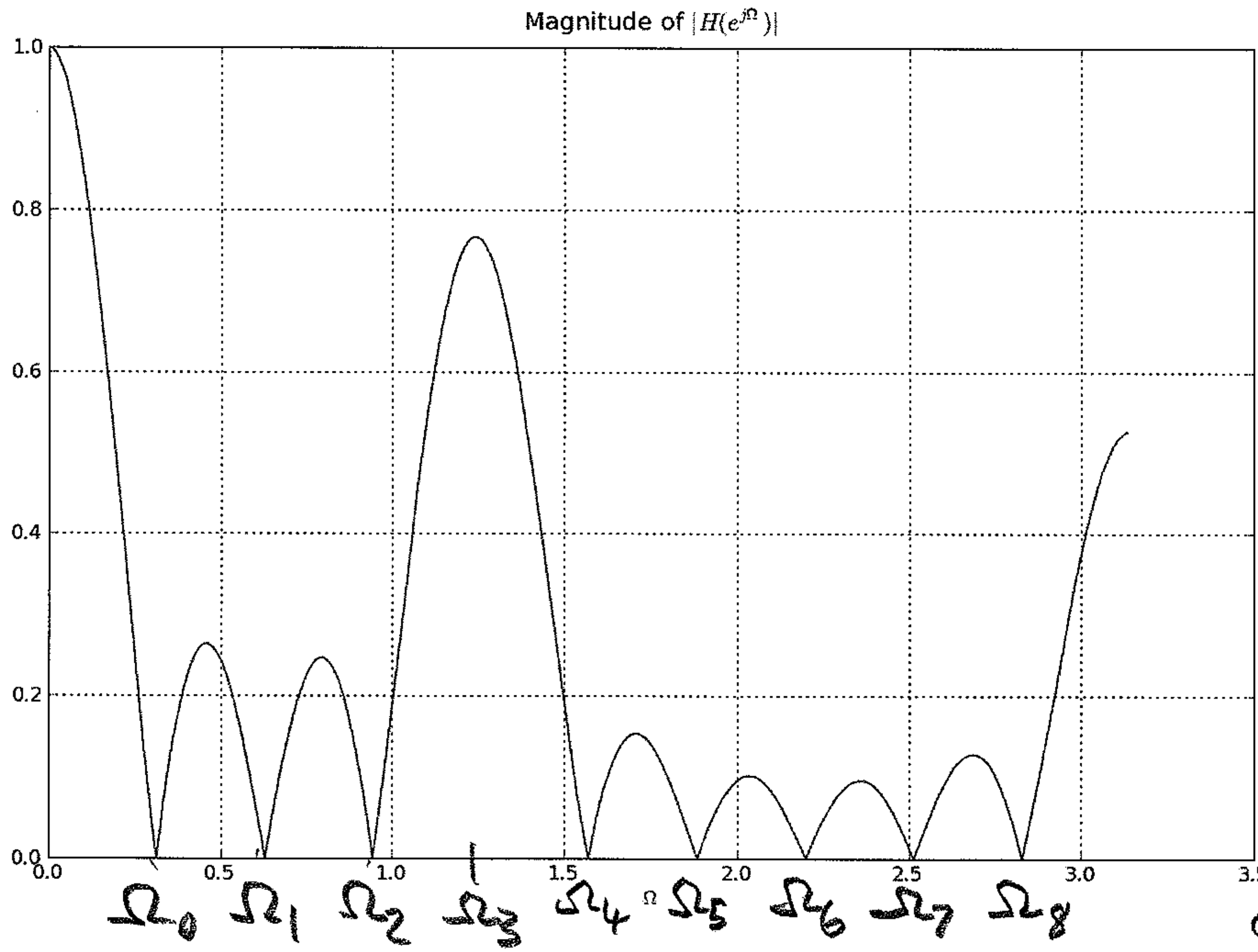
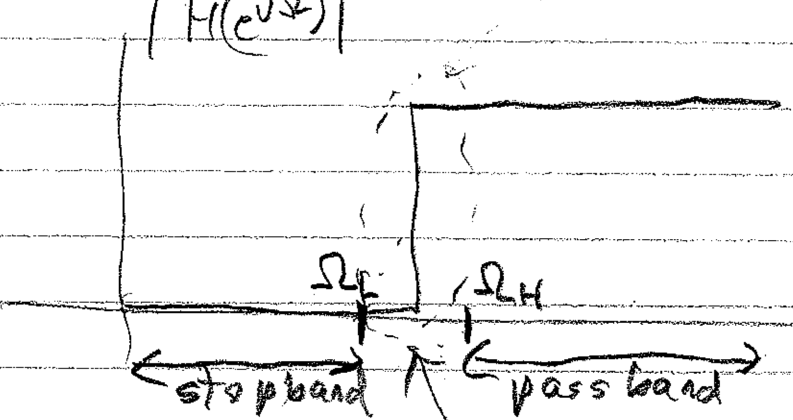
Channel ~~4~~ 3, 8-pair Zeros Filter



Note Scaled

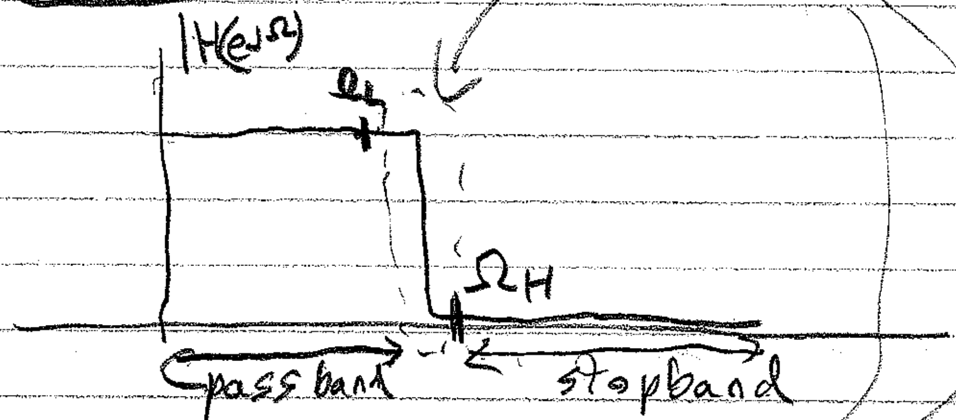
Filter Design Generally

Highpass

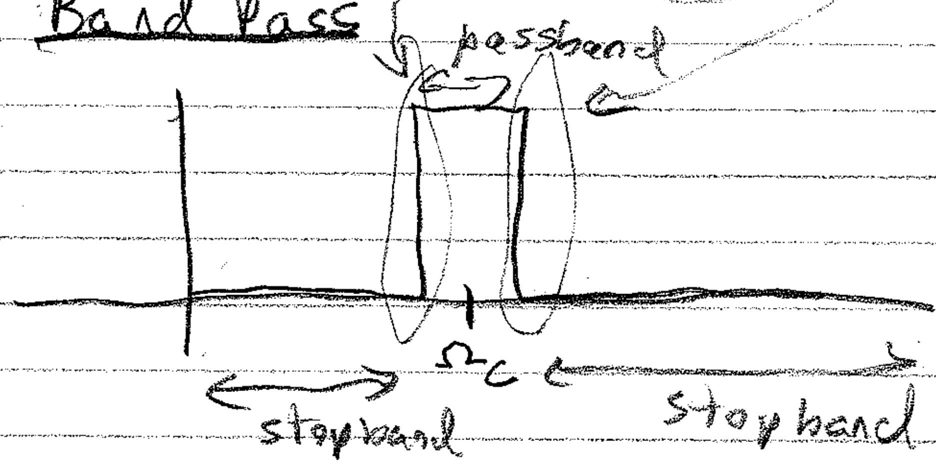


Transition Regions

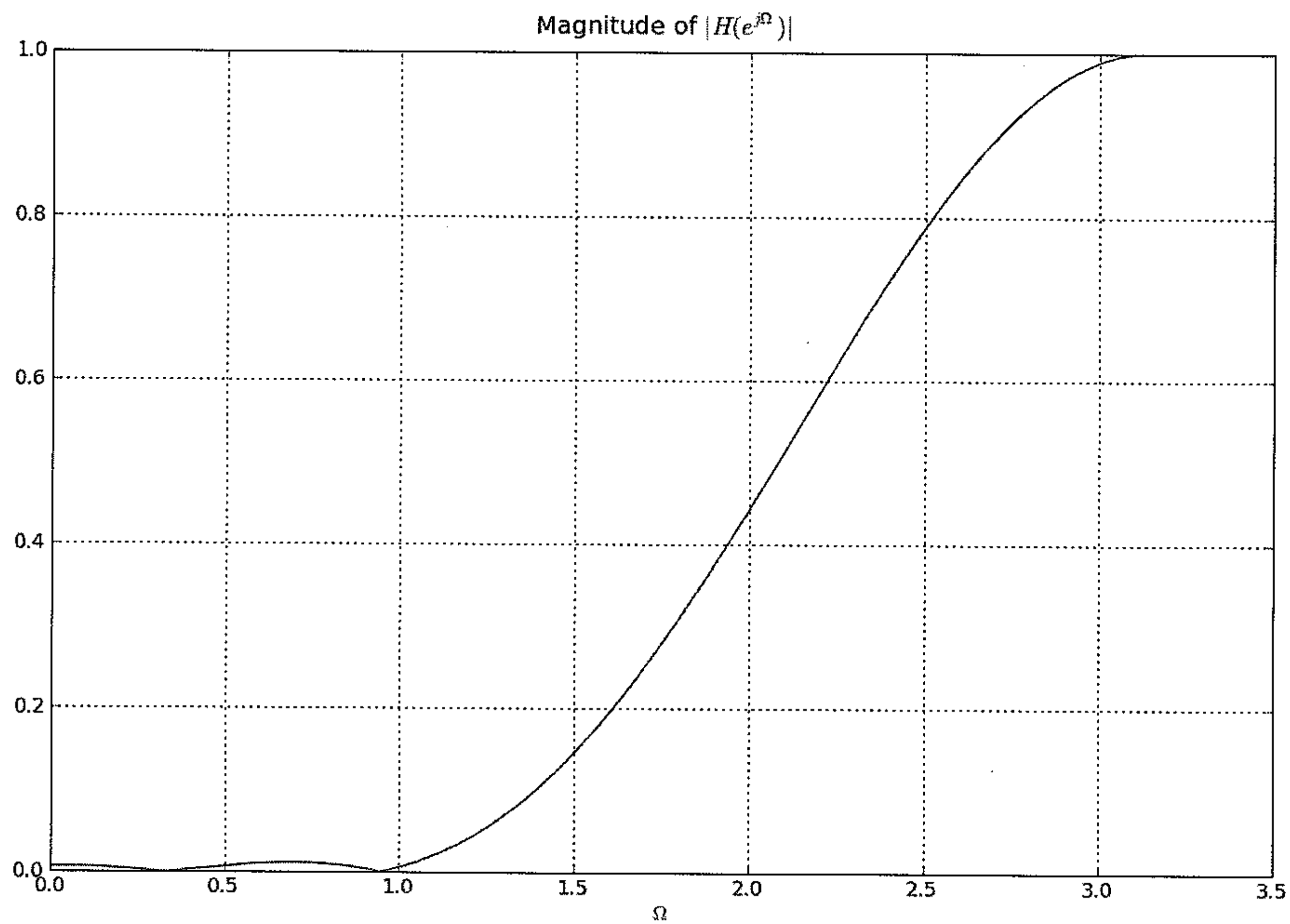
Low Pass



Band Pass



High Pass Filter, Two-pair Zeros



Low Pass Filter, Two-pair Zeros

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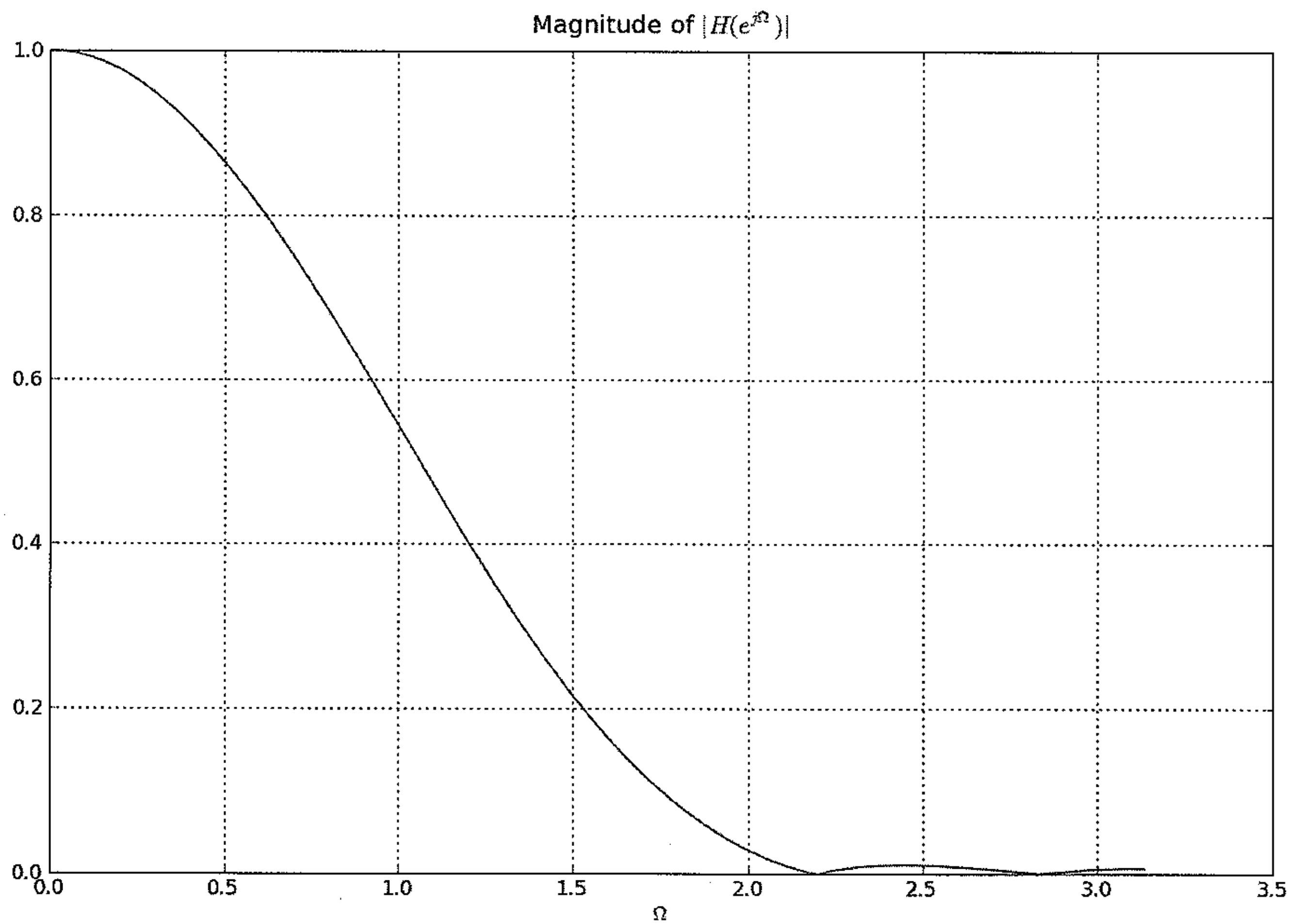

One More Idea

Using Difference Eqns

2nd order case

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

If $x[n] = e^{j\Omega n}$

then $y[n] = H(e^{j\Omega}) e^{j\Omega n}$

as dift eqn is L.T.I.

$$a_0 H(e^{j\Omega}) e^{j\Omega n} + a_1 H(e^{j\Omega}) e^{j\Omega(n-1)}$$
$$+ a_2 H(e^{j\Omega}) e^{j\Omega(n-2)}$$
$$=$$
$$b_0 e^{j\Omega n} + b_1 e^{j\Omega(n-1)} + b_2 e^{j\Omega(n-2)}$$

Dividing by $e^{j\Omega n}$

$$(a_0 + a_1 e^{-j\Omega} + a_2 e^{-2j\Omega}) H(e^{j\Omega})$$
$$= b_0 + b_1 e^{-j\Omega} + b_2 e^{-2j\Omega}$$

$$\Rightarrow H(e^{j\Omega}) = \left. \frac{b_0 + b_1 z + b_2 z^2}{a_0 + a_1 z + a_2 z^2} \right|_{z = e^{-j\Omega}}$$

Magnify Response for desired frequency

$$H(e^{j\Omega}) = \left. \frac{b_0 + b_1 z^2 + b_2 z^2}{a_0 + a_1 z + a_2 z^2} \right|_{z = e^{-j\Omega}}$$

desired frequency

$$H(e^{j\Omega_k}) \to \infty \text{ as } (a_0 + a_1 e^{-j\Omega_k} + a_2 e^{-2j\Omega_k}) \to 0$$

Factor into poles

$$a_0 + a_1 z + a_2 z^2 =$$

$$(z - \frac{1}{p_0})(z - \frac{1}{p_1})$$

~~Make $p_0 = e^{j\Omega_k} \Rightarrow p_1 = e^{-j\Omega_k}$?~~

real poly $\Leftrightarrow$ complex conjugate roots

No Difference equation is stable if $|p_i| < 1$!

$$p_0 = r e^{j\Omega_k} \qquad p_1 = r e^{-j\Omega_k}$$

$$0 < r < 1$$ Less than 1

$$|p_0| = |p_1| = r < 1$$

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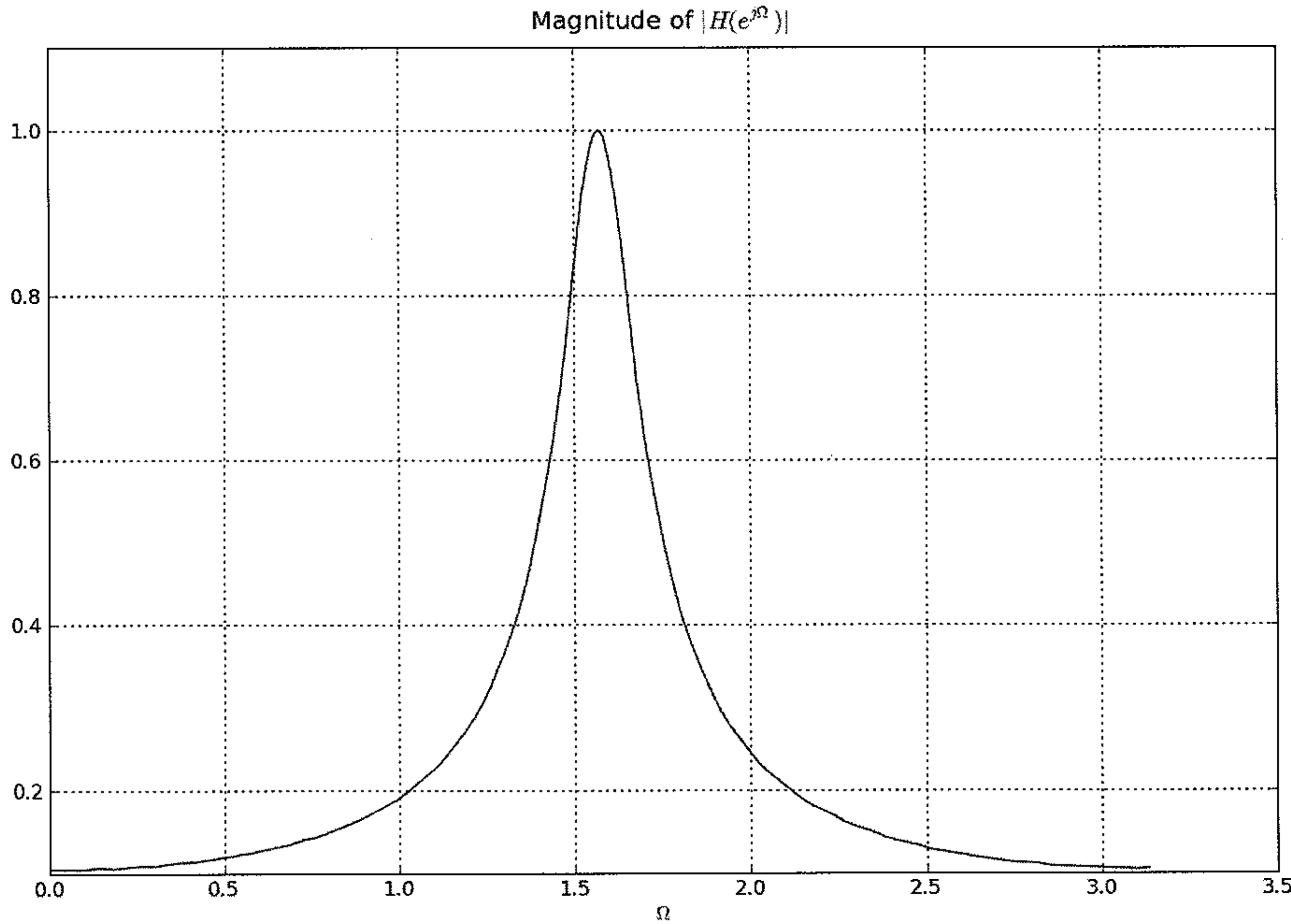
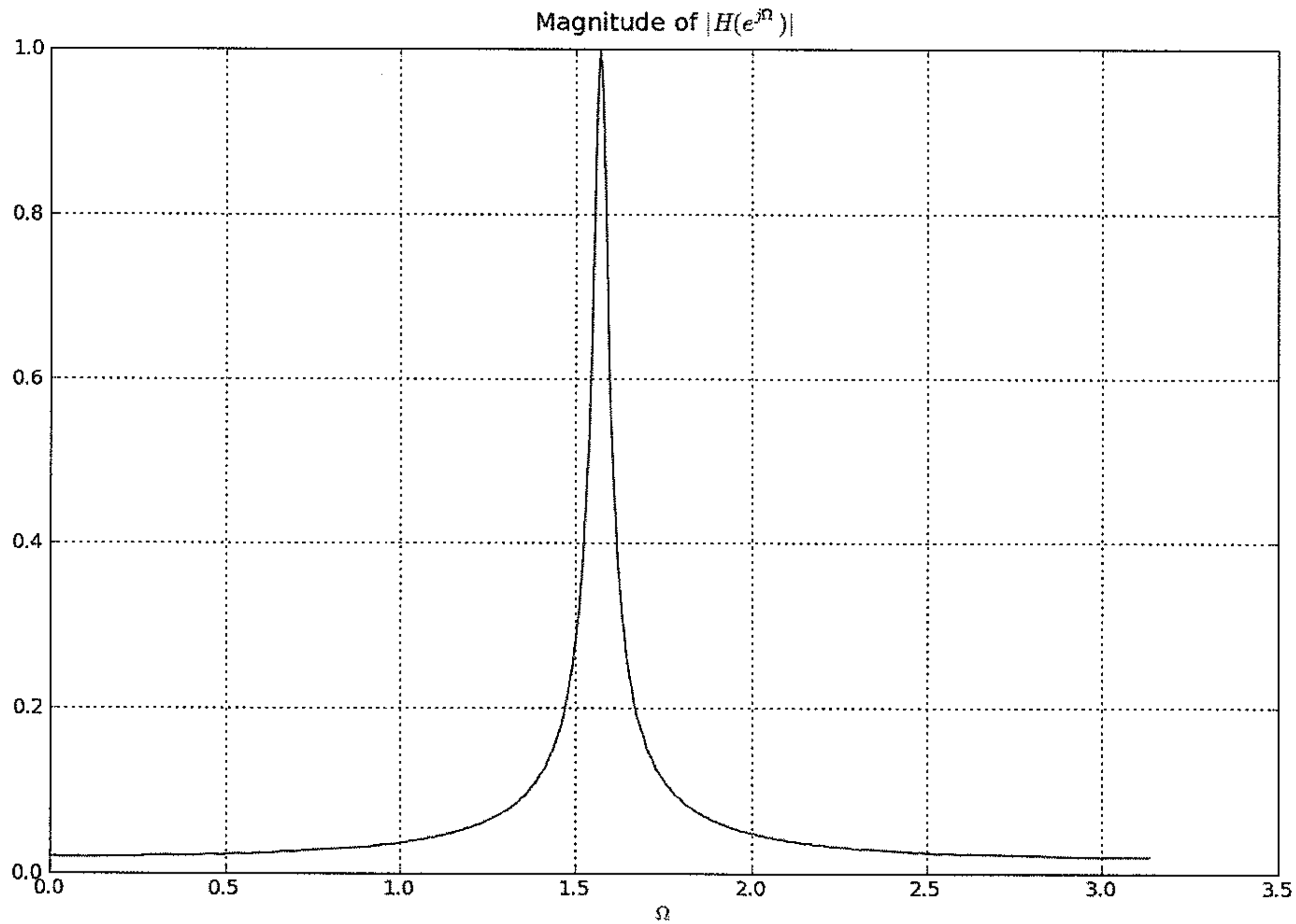
One-pair Poles Bandpass Filter r=0.9



$p_0 = \underbrace{0.9}_{r} e^{j\frac{\pi}{2}}$ $p_1 = \underbrace{0.9}_{r} e^{-j\pi/2}$

One-pair poles Bandpass Filter r=0.98

(14)



$p_0 = \underbrace{0.98}_{r<1 \text{ (barely)}} e^{j\frac{\pi}{2}}$ $\qquad p_1 = \underbrace{0.98}_{r} e^{-j\pi/8}$

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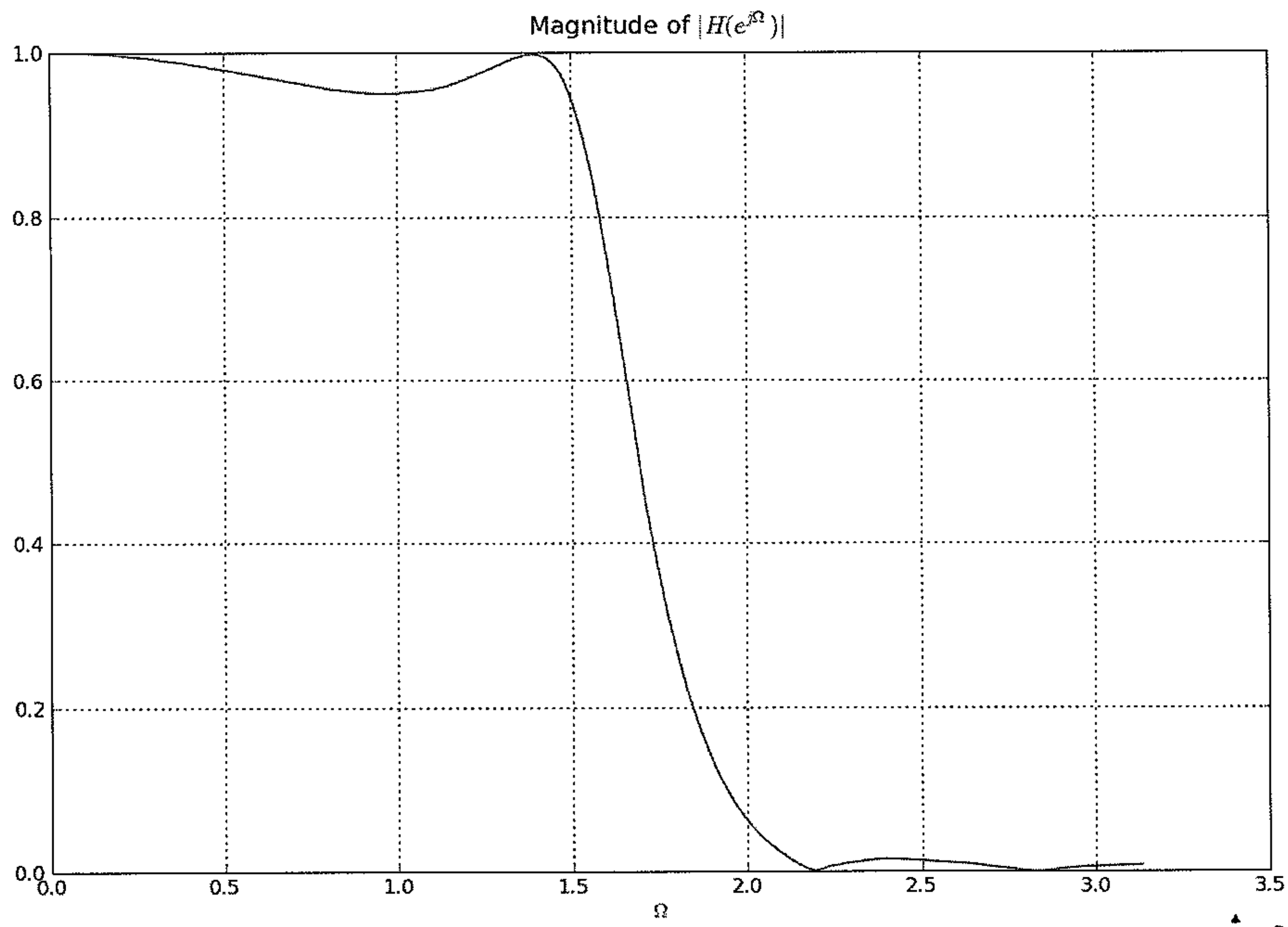
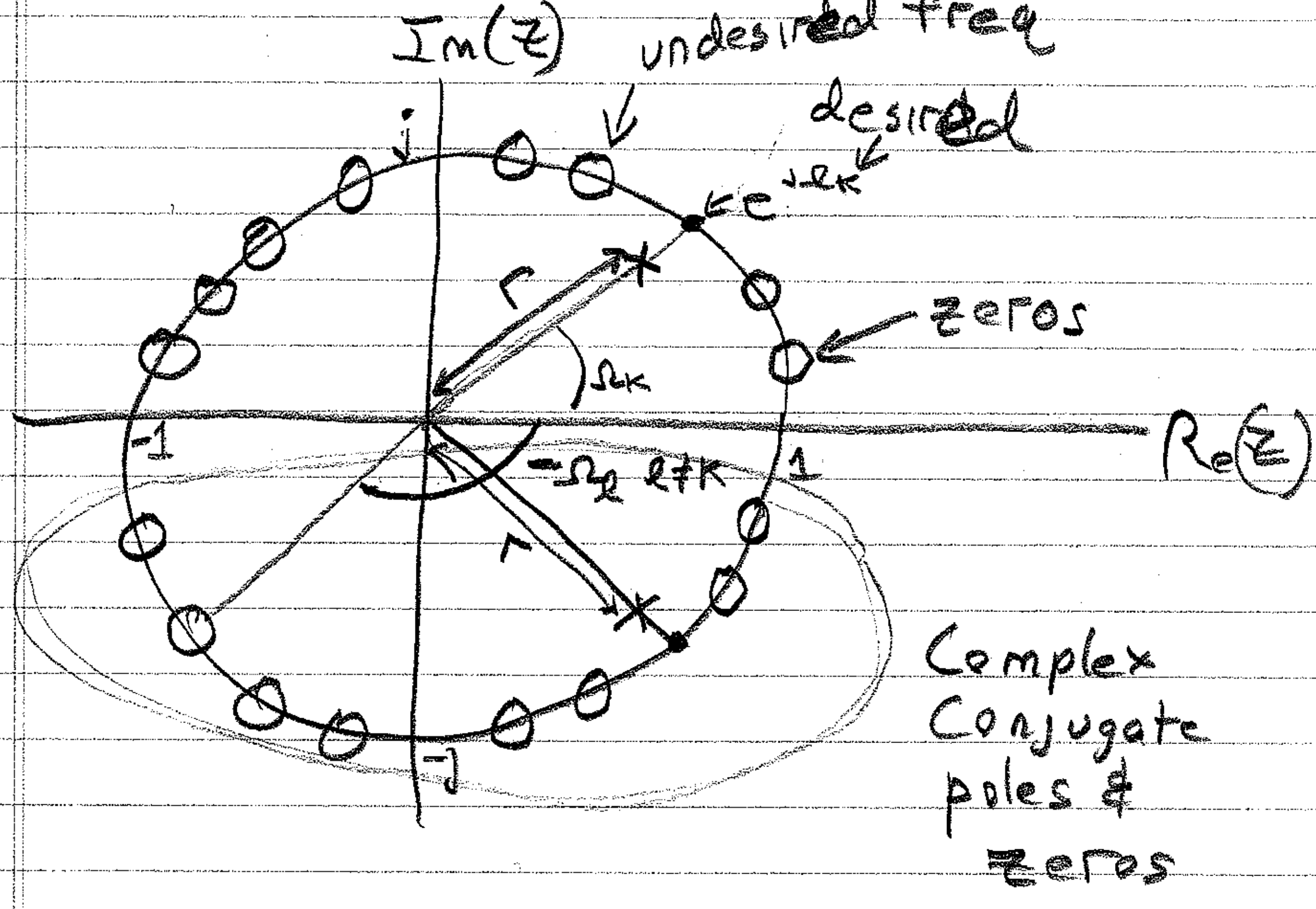
One pole-pair, two zero pairs LPF

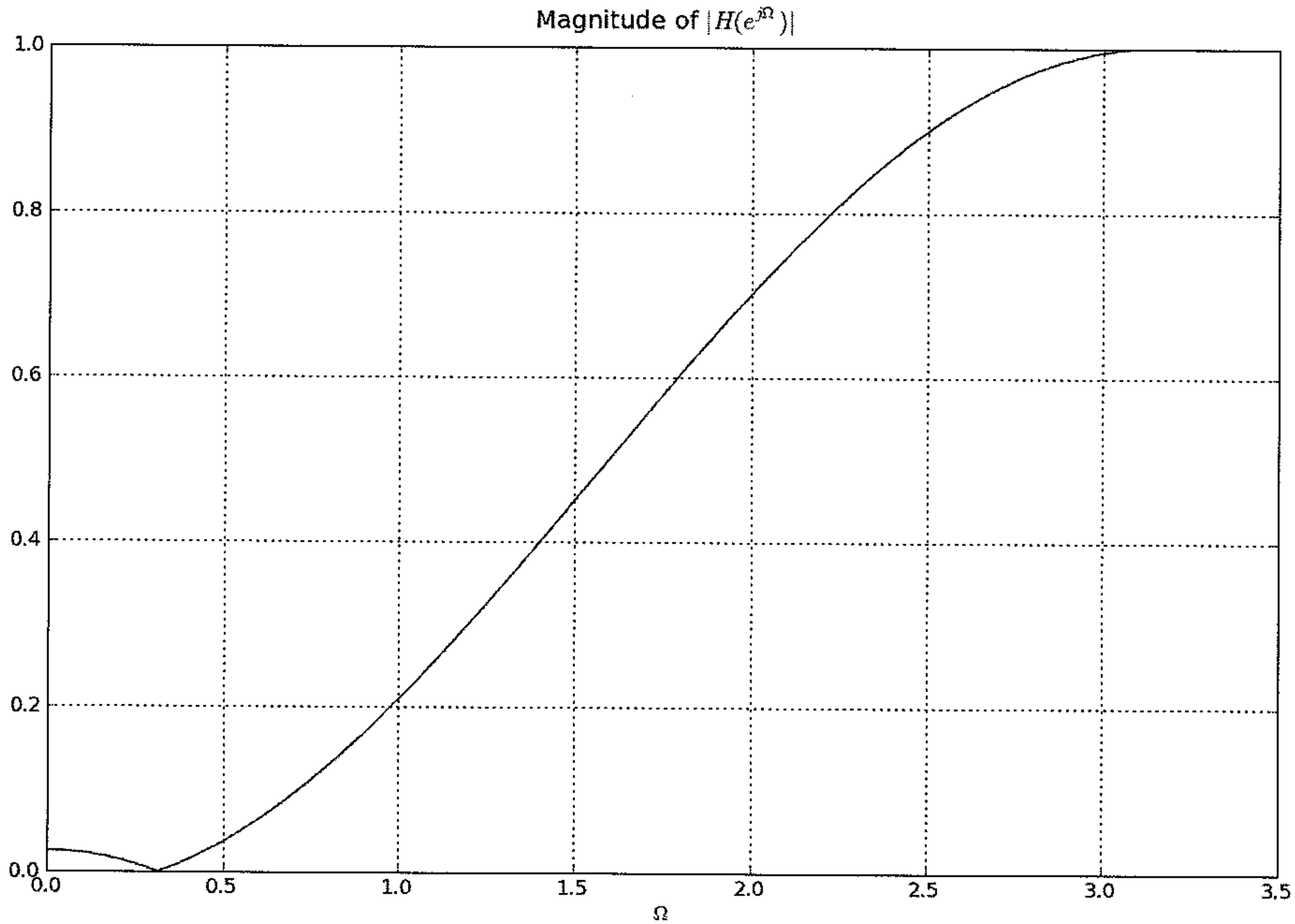


$p_0 = 0.8e^{j\frac{\pi}{2}}$ q_0, q_1 at $e^{j0.7\pi}, e^{j0.9\pi}$

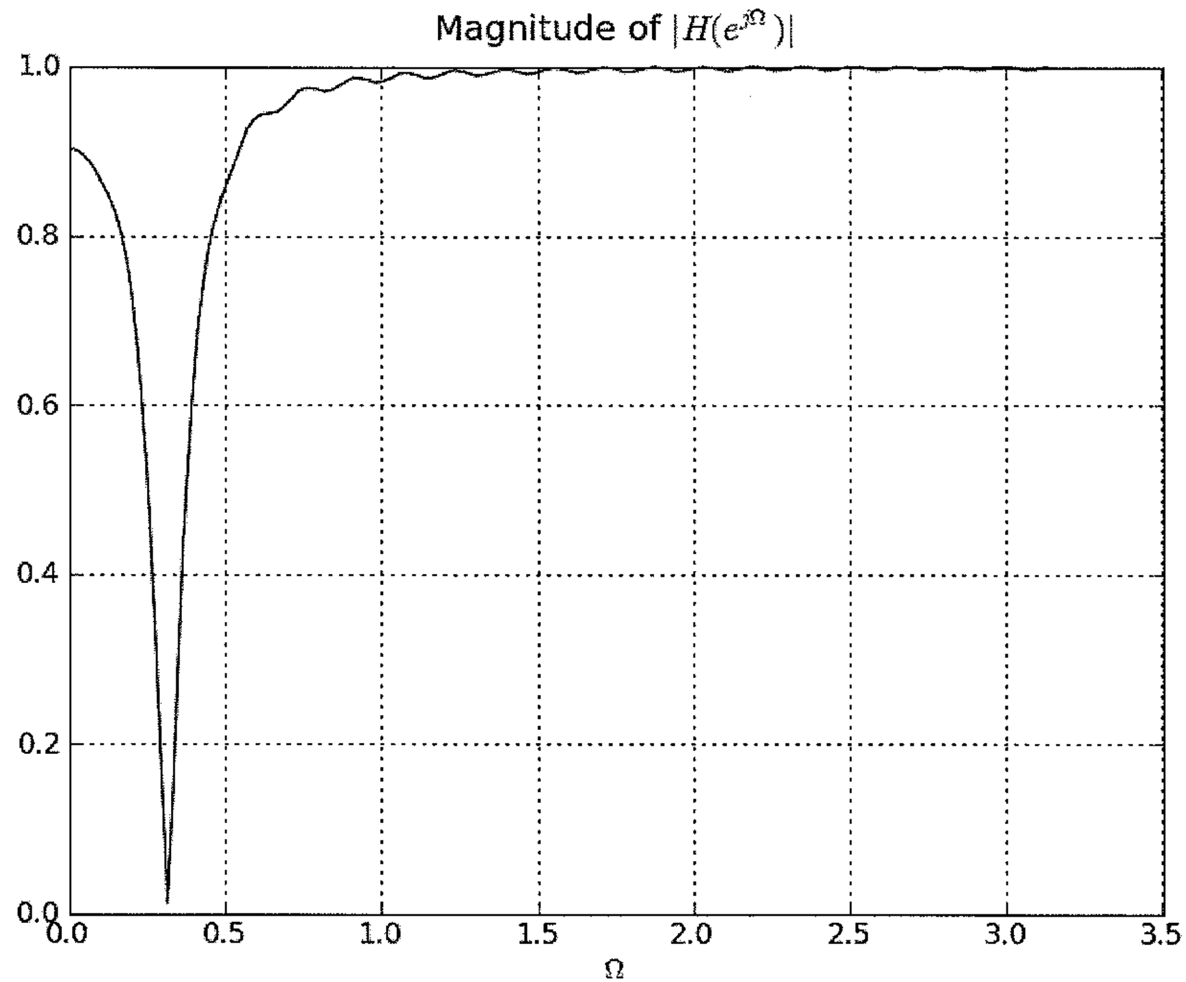
Unit Circle Picture



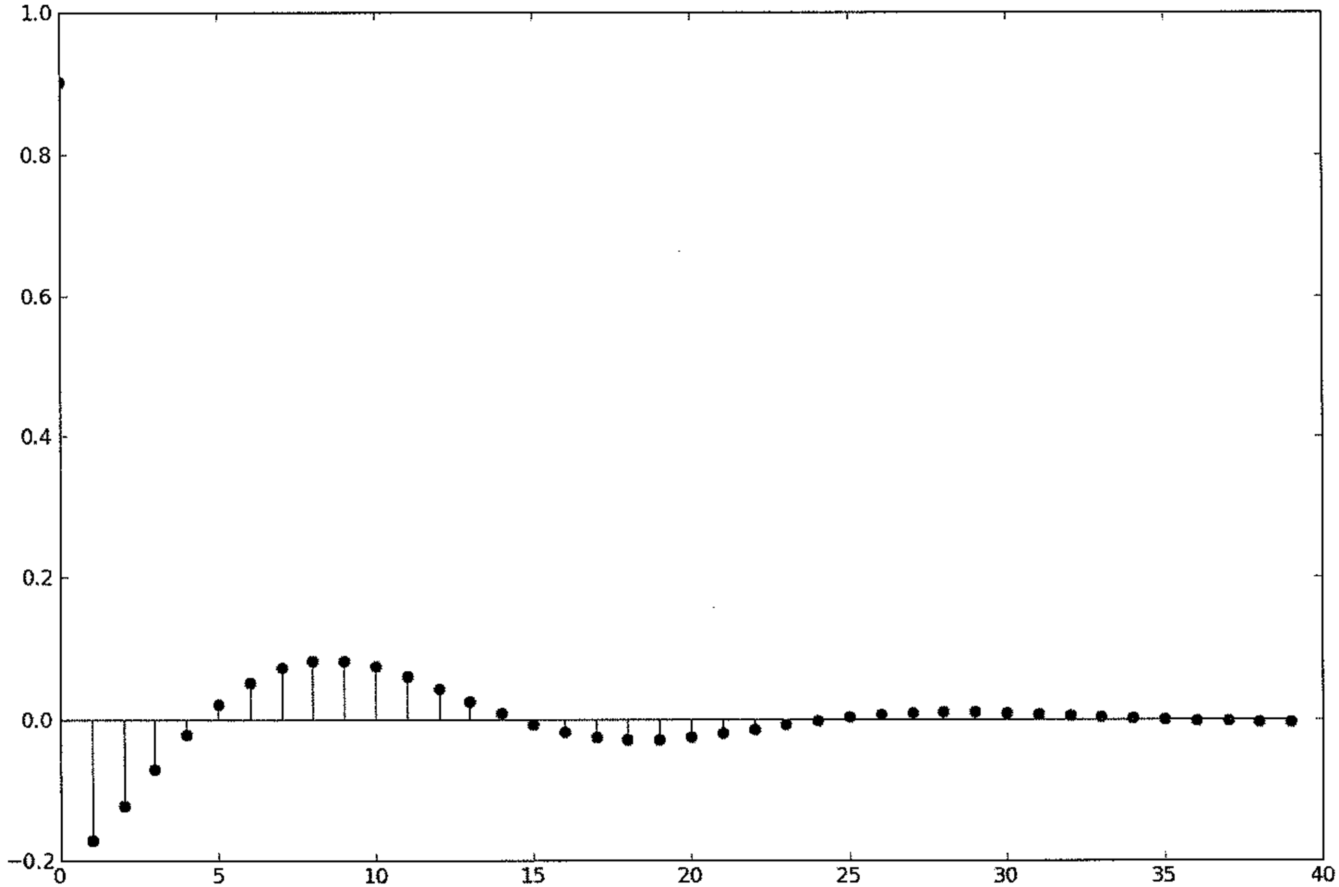
One zero-pair notch filter



One Zero-pair, One Pole-pair Notch



Notch Unit Sample Response



Examples

First Scaling for Zeros

Consider

$$y[n] = h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + h[3]\,x[n-3] + h[4]\,x[n-4]$$

Design h so that $H(e^{j\Omega}) = 0$ at $\Omega = \frac{\pi}{6}, \frac{5\pi}{6}$

then scale so that $\max_{\Omega} |H(e^{j\Omega})| = 1$

Compose two systems

$$H_1(e^{j\Omega}) = \left(z - e^{j\frac{\pi}{6}}\right)\left(z - e^{-j\frac{\pi}{6}}\right)\Big|_{z=e^{j\Omega}}$$

$$= z^2 - 2\cos\tfrac{\pi}{6}\, z + 1$$

$$= z^2 - \sqrt{3}\, z + 1$$

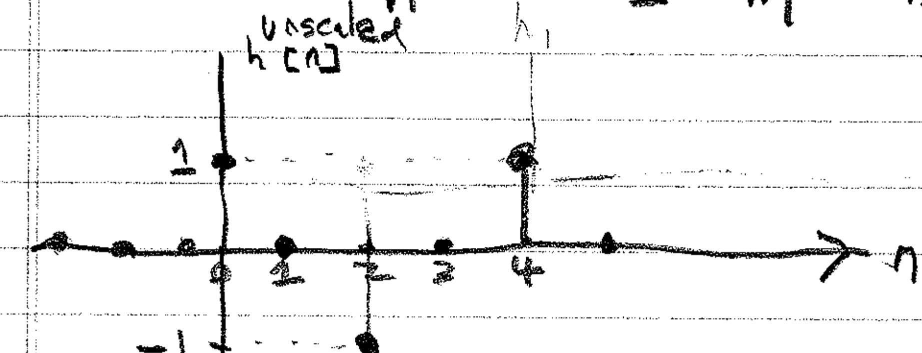
$$H_2(e^{j\Omega}) = \left(z - e^{j\frac{5\pi}{6}}\right)\left(z - e^{-j\frac{5\pi}{6}}\right)\Big|_{z=e^{j\Omega}}$$

$$= z^2 + \sqrt{3}\, z + 1$$

$$H^{\text{unscaled}}(e^{j\Omega}) = H_1(e^{j\Omega})\, H_2(e^{j\Omega})$$

$$h^{\text{unscaled}}[n]\, z^n = \left(z^2 - \sqrt{3}z + 1\right)\left(z^2 + \sqrt{3}z + 1\right)$$

$$h^{\text{unscaled}} = h_1 * h_2$$

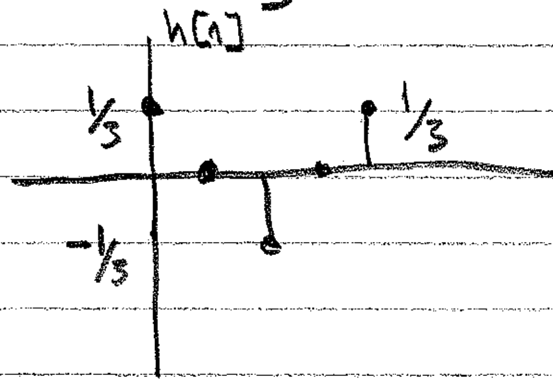


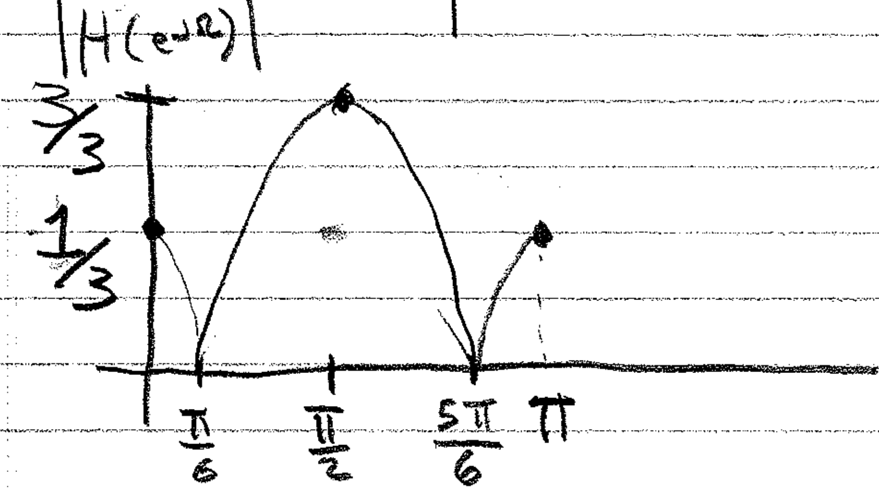
$$\max_{\Omega} \left| H^{\text{unscaled}}(e^{j\Omega}) \right| = \max_{\Omega} \left| 1 - (e^{-j\Omega})^2 + (e^{-j\Omega})^4 \right|$$

$$= 3$$

<u>Scaling</u> $h^{\text{unscaled}}[n]$

$$h[n] = \frac{h^{\text{unscaled}}[n]}{3}$$





Consider

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n]$$

If $x[n] = e^{j\Omega n}$ then $y[n] = H(e^{j\Omega}) e^{j\Omega n}$

$$a_0 H(e^{j\Omega}) e^{j\Omega n} + a_1 H(e^{j\Omega}) e^{j\Omega(n-1)} + a_2 H(e^{j\Omega}) e^{j\Omega(n-2)} = e^{j\Omega n}$$

Dividing by $e^{j\Omega n}$

$$a_0 H(e^{j\Omega}) + a_1 H(e^{j\Omega}) e^{-j\Omega} + a_2 H(e^{j\Omega}) (e^{-j\Omega})^2 = 1$$

$$H(e^{j\Omega}) = \frac{1}{a_0 + a_1 e^{-j\Omega} + a_2 e^{-2j\Omega}}$$

$$= \frac{1}{(z - \frac{1}{p_0})(z - \frac{1}{p_1})}$$

poles

Pick poles to be

$$r e^{j\frac{\pi}{2}}, \quad r e^{-j\frac{\pi}{2}}$$

$$H(e^{j\Omega}) = \frac{1}{\left(z - \frac{1}{r}e^{j\pi/2}\right)\left(z - \frac{1}{r}e^{-j\pi/2}\right)}\Bigg|_{z=e^{j\Omega}}$$

$$H(e^{j\Omega}) = \frac{1}{\left(z^2 - 2\frac{1}{r}\cos\frac{\pi}{2}z + \frac{1}{r}^2\right)}$$

$$a_2^{\text{unscaled}} = 1 \qquad a_1 = 0 \qquad a_0^{\text{unscaled}} = \frac{1}{r}^2$$

$$\left(\frac{1}{r}\right)^2 y[n] + y[n-2] = x[n]$$

At $\Omega = \pi/2$, $H^{\text{unscaled}}(e^{j\frac{\pi}{2}}) =$

$$\frac{1}{\left(1-\frac{1}{r}\right)e^{-j\pi/2}\cdot\left(1-\frac{1}{r}\underbrace{e^{j\pi}}_{=-1}\right)e^{-j\pi/2}}$$

$$= \frac{1}{\left(1-\frac{1}{r}\right)\underbrace{e^{-j\pi}}_{=-1}\left(1+\frac{1}{r}\right)} = -\frac{r^2}{(r-1)(r+1)}$$

$$= \frac{r^2}{1-r^2}$$

$r = 0.9$

$$H^{\text{unscaled}}(e^{j\pi/2}) = \frac{0.81}{1-0.81} \simeq 4.3$$

$r = 0.99$

$$H^{\text{unscaled}}(e^{j\pi/2}) = \frac{0.99^2}{1-0.99^2} \simeq 50$$

If $1-r$ is small

$$\max_{\Omega} |H(e^{j\Omega})| = |H(e^{j\pi/2})| = \frac{r^2}{1-r^2}$$

Scaling

Note Multiply not divide

$$a_i = a_i^{\text{unscaled}} \cdot \max_{\Omega} |H(e^{j\Omega})|$$

$$a_0 = \frac{1}{1-r^2}$$

$$a_1 = 0$$

$$a_2 = \frac{r^2}{1-r^2}$$

Finally put both together

$$W[n] = \frac{1}{3}x[n] - \frac{1}{3}x[n-2] + \frac{1}{3}x[n-4]$$

Note
Unscaled

$$y[n] + r^2 y[n-2] = W[n]$$

$$X \rightarrow \underbrace{\boxed{H^{zeros}(e^{j\Omega})} \xrightarrow{W} \boxed{H^{poles}(e^{j\Omega})}}_{H^{full}(e^{j\Omega})} \rightarrow Y$$

or

$$y[n] - r^2 y[n-2] = \frac{1}{3}x[n] - \frac{1}{3}x[n-2] + \frac{1}{3}x[n-4]$$

$$H^{full}(e^{j\Omega}) = \left.\frac{\frac{1}{3}z^4 - \frac{1}{3}z^2 + \frac{1}{3}}{1 + r^2 z^2}\right|_{z = e^{j\Omega}}$$

$r = 0.99$