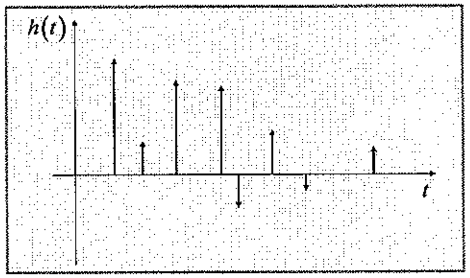
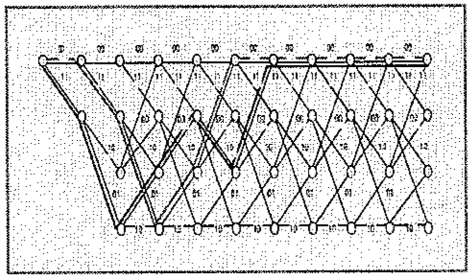
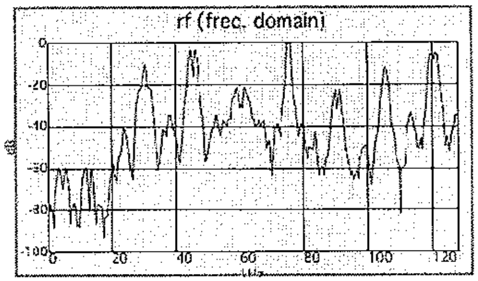
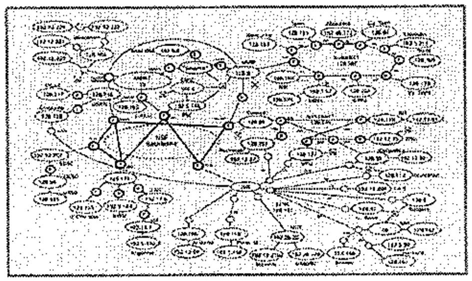
1









INTRODUCTION TO EECS II

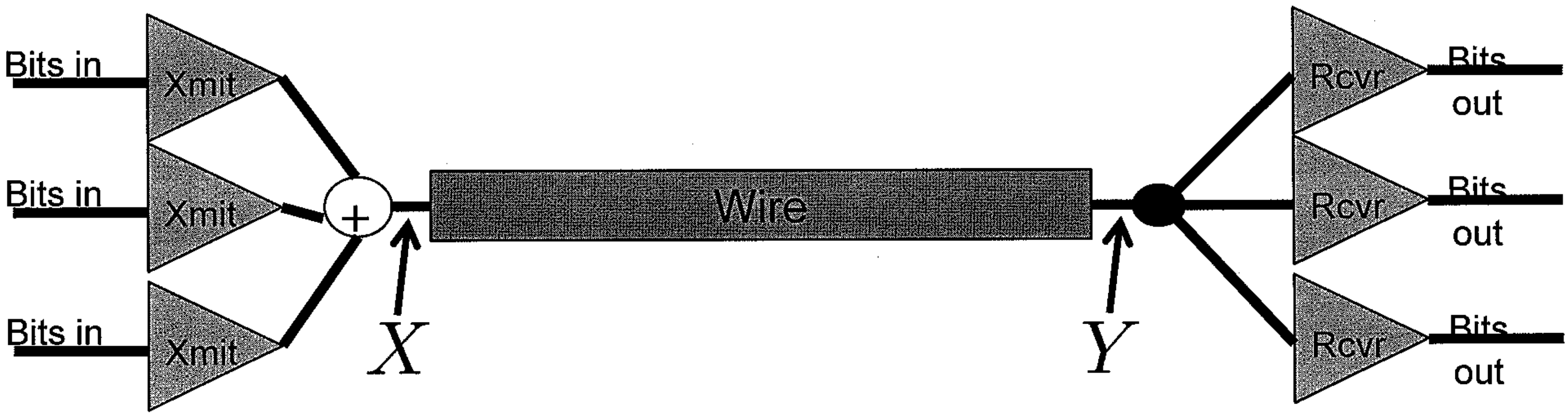
DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2009
Lecture #14

- FDM and Data
- Spectrum and Discrete Fourier Transform
- DFT Examples
- Rise Time and Spectrum

2

Simplified Big Picture



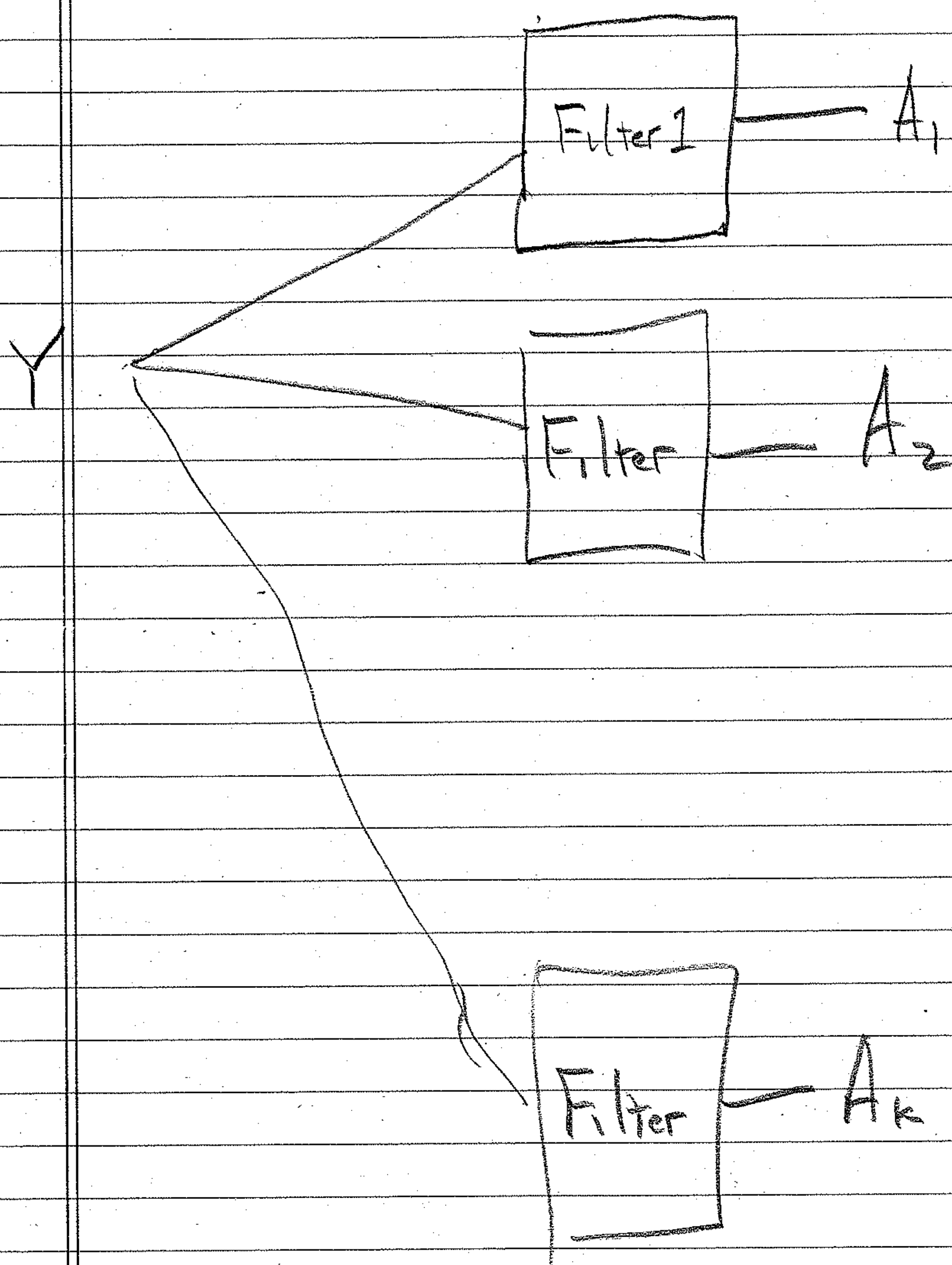
- Frequency Division Multiplexing Strategy
  - Represent each channel with a different frequency
    - For LTI systems, frequencies do not mix

$$x[n] = A_1 e^{j\Omega_1 n} + ... + A_K e^{j\Omega_K n}$$

$$\Downarrow$$

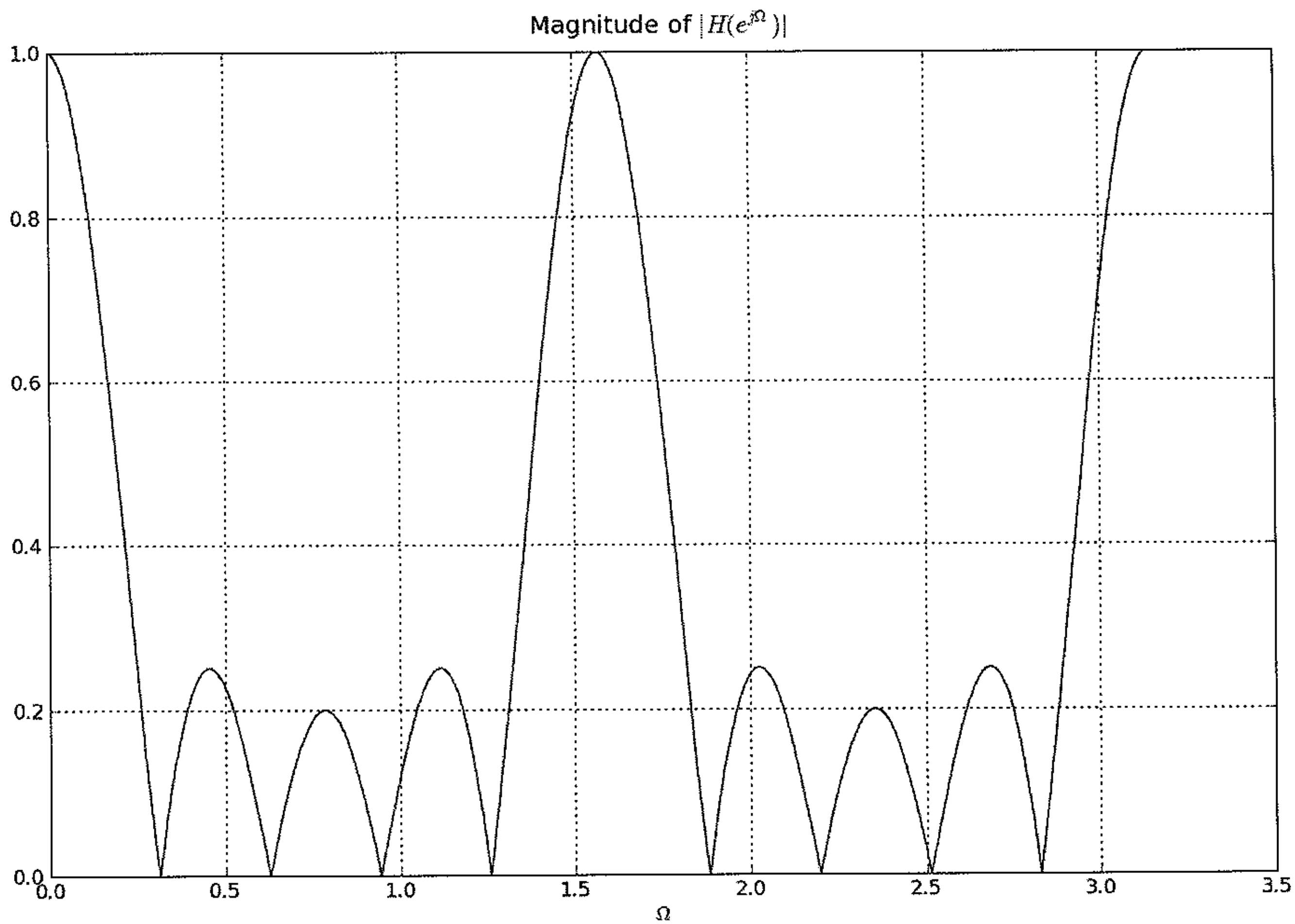
$$y[n] = H(e^{j\Omega_1}) A_1 e^{j\Omega_1 n} + ... + H(e^{j\Omega_K}) A_K e^{j\Omega_K n}$$

  - Use Filters to Separate Into different Channels


Y
Filter 1
A1
Filter
A2
Filter
Ak

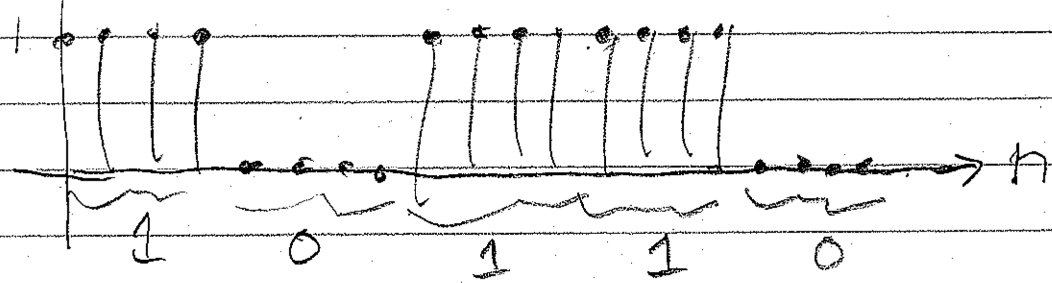
Channel 5, 8 Zeros Filter

4



Real Problem – Encoding Information

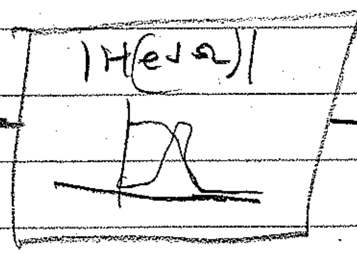
Samples[n] $\equiv s[n]$



$A_k e^{j\Omega_k n} \Rightarrow s[n]\, e^{j\Omega_k n}$

what happens when

$s[n]\, e^{j\Omega_k n}$ ——



—— ?

How to Analyze?

Make a periodic Approximation

For some large value of N (e.g. $N = 100{,}001$)

Assume $s[n+N]\, e^{j\Omega_k (n+N)}$

$$=$$

$$s[n]\, e^{j\Omega_k n}$$

Why: Makes analysis tractable.

First Major ramification:

If $e^{j\Omega_k (n+N)} = e^{j\Omega_k (n)}$

Then only certain Ω_k's allowed

$\Omega_k (n+N)$ must equal $\Omega_k n + 2\pi L$ for some integer L

(Recall $-\pi \le \Omega_k \le \pi$)

Therefore

$$\Omega_k = \pm\frac{2\pi}{N}\cdot 0,\ \pm\frac{2\pi}{N}\cdot 1,\ \pm\frac{2\pi}{N}\cdot 2,\ \ldots\ \pm\frac{2\pi}{N} B$$

$B = \frac{N-1}{2}$ Assuming N odd

We will alway assume N is odd

Major Ramification 2

If $s[n]$ is real, periodic with period N, and N is odd:

$$s[n] = \sum_{k=1}^{\bar{K}} a_k \sin\frac{2\pi}{N}kn + \sum_{k=0}^{\bar{K}} b_k \cos\frac{2\pi}{N}kn$$

No $k=0$ term as $\sin(0)=0$

$\bar{K} = \frac{N-1}{2} \Rightarrow \{a_k's, b_k's\}$ has N elements

Using Complex Expenentials (And "Negative" Frequencies)

$$s[n] = \frac{1}{N}\sum_{k=-\bar{K}}^{\bar{K}} S[k]\, e^{j\frac{2\pi}{N}kn}$$

Complex Coefficients

$$N = 2\bar{K}+1$$

Note

$$\underbrace{s[n],\ n=0,\dots N-1}_{N \text{ unique Numbers}} \Rightarrow \underbrace{S[-k], \dots S[k]}_{N \text{ complex Numbers}}$$

But $S[k]$'s have certain properties

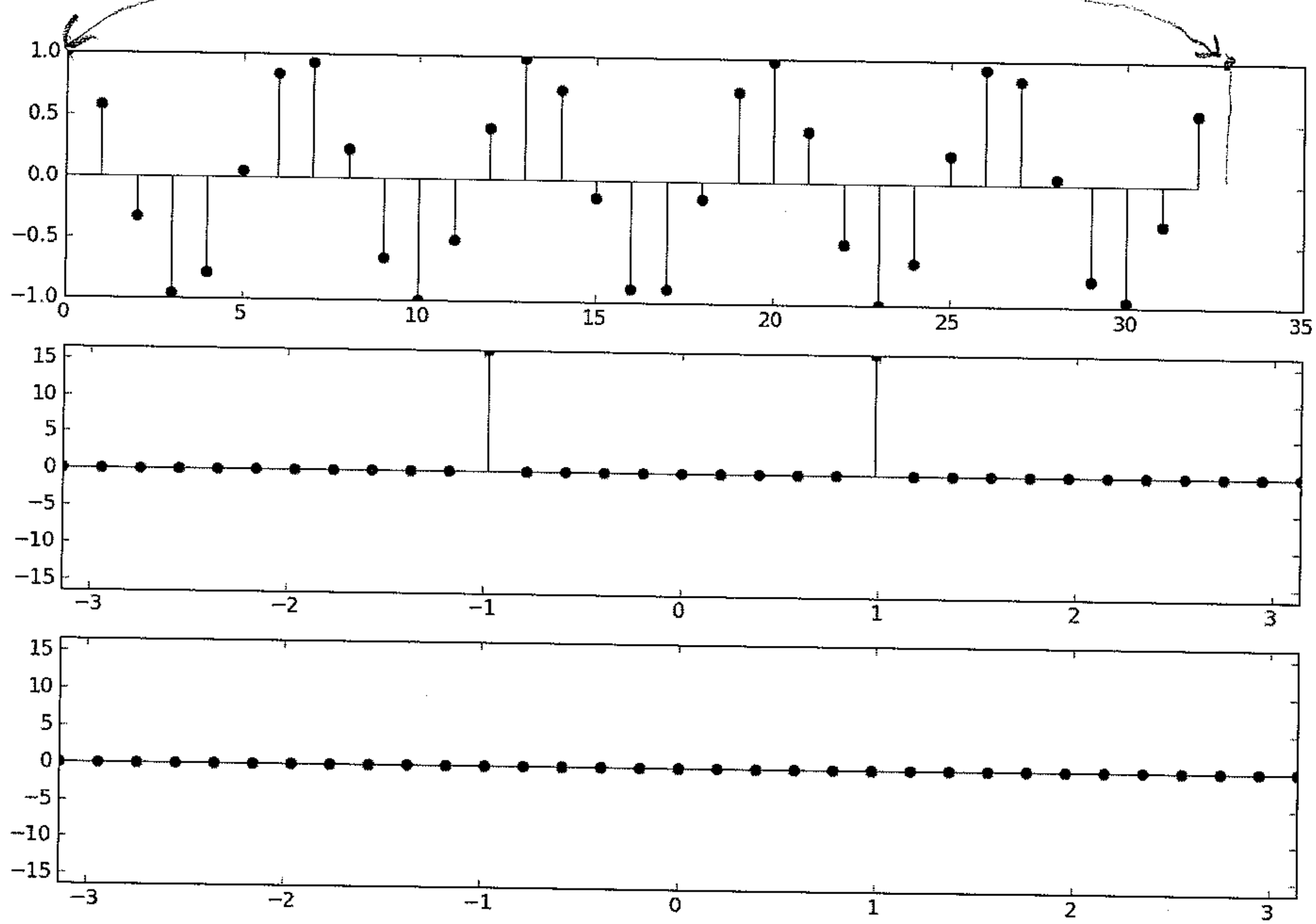
$$s[n] \text{ real} \Rightarrow \underbrace{Re(S[k]) = Re(S[-k])}_{\text{Real part even}}$$

$$\underbrace{Im(S[k]) = -Im(S[-k])}_{\text{Imag part odd}}$$

Cosine in Time, 2 real nonzero DFT values

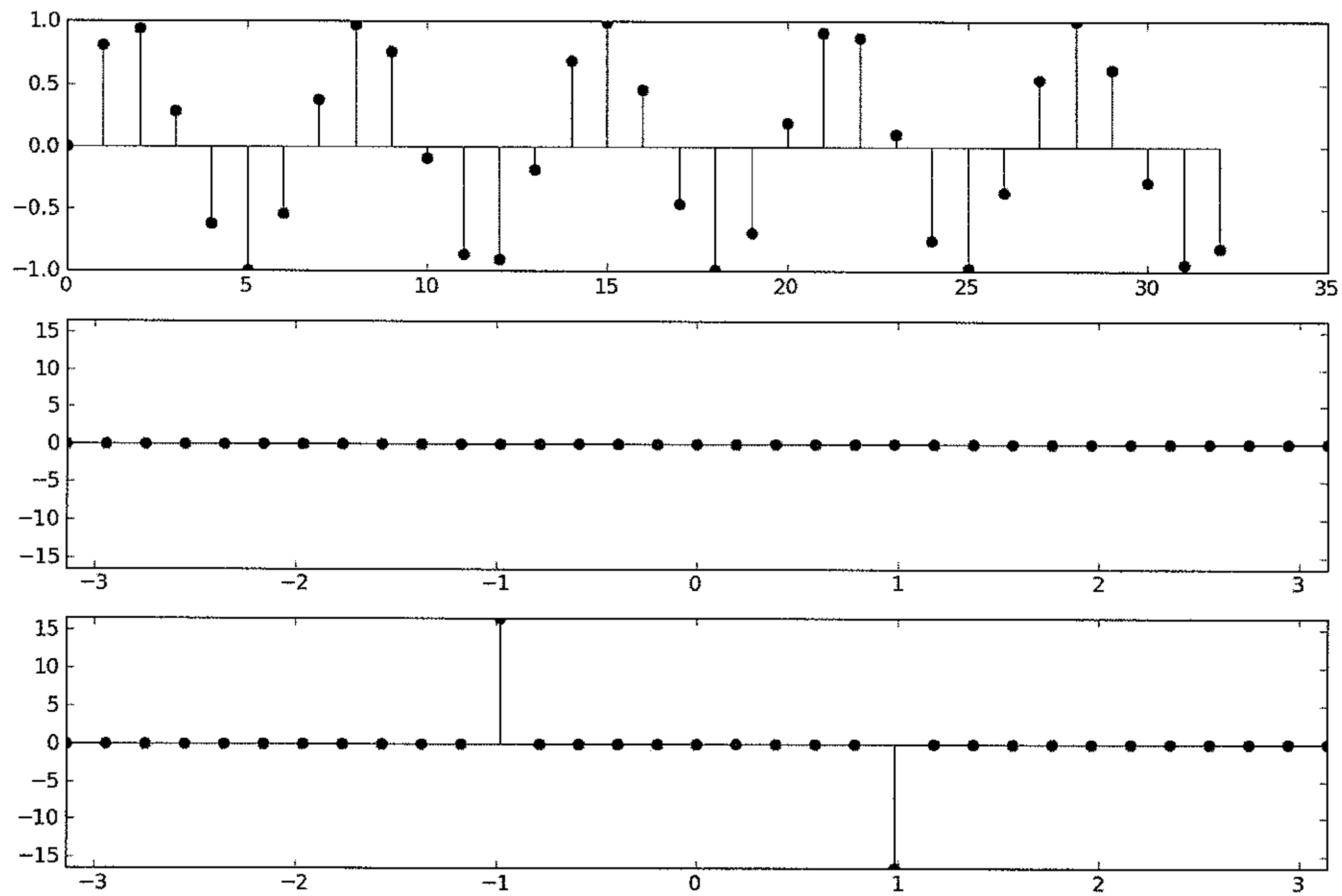
(8)

Must be identical



$\cos\left(\frac{2\pi}{33} \cdot 5n\right)$

Sine in Time, 2 Imaginary DFT values



$\sin\left(\frac{2\pi}{33} \cdot 5n\right)$

Computing Fourier Coefficients

$$e^{-j\frac{2\pi}{N}\ell n} s[n] = e^{-j\frac{2\pi}{N}\ell n}\left(\frac{1}{N}\sum_{k=-K}^{K} S[k]\, e^{+j\frac{2\pi}{N}kn}\right)$$

$$\sum_{n=0}^{N-1}\left(e^{-j\frac{2\pi}{N}\ell n}\, s[n]\right) = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=-K}^{K} S[k]\, e^{+j\frac{2\pi}{N}(k-\ell)n}\right)$$

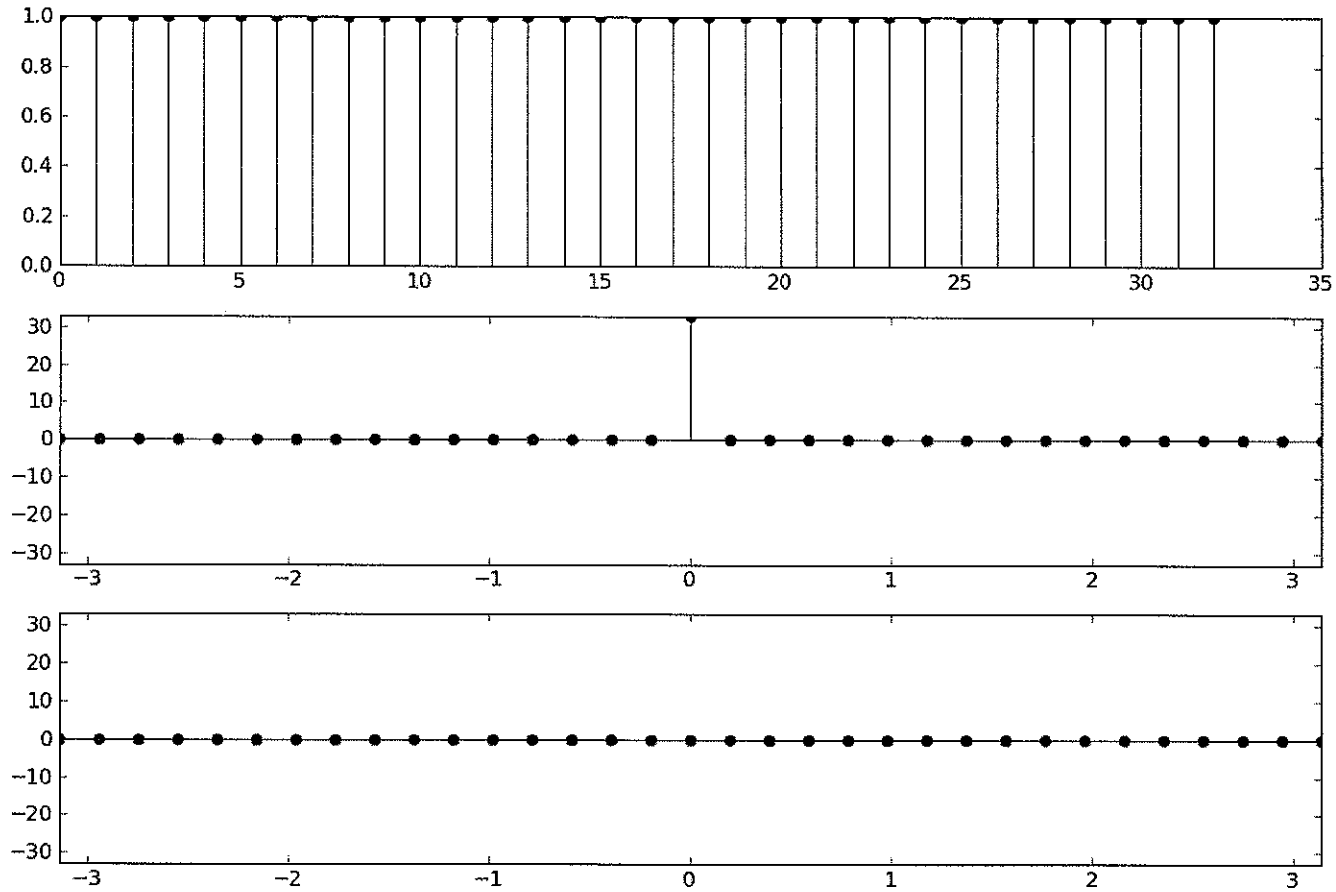
$$= \frac{1}{N}\sum_{k=-K}^{K} S[k] \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-\ell)n}}_{=\,0 \text{ unless } k=\ell}$$

$$= \frac{1}{N}\cdot S[\ell]\cdot N$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\, s[n] = S[k]$$

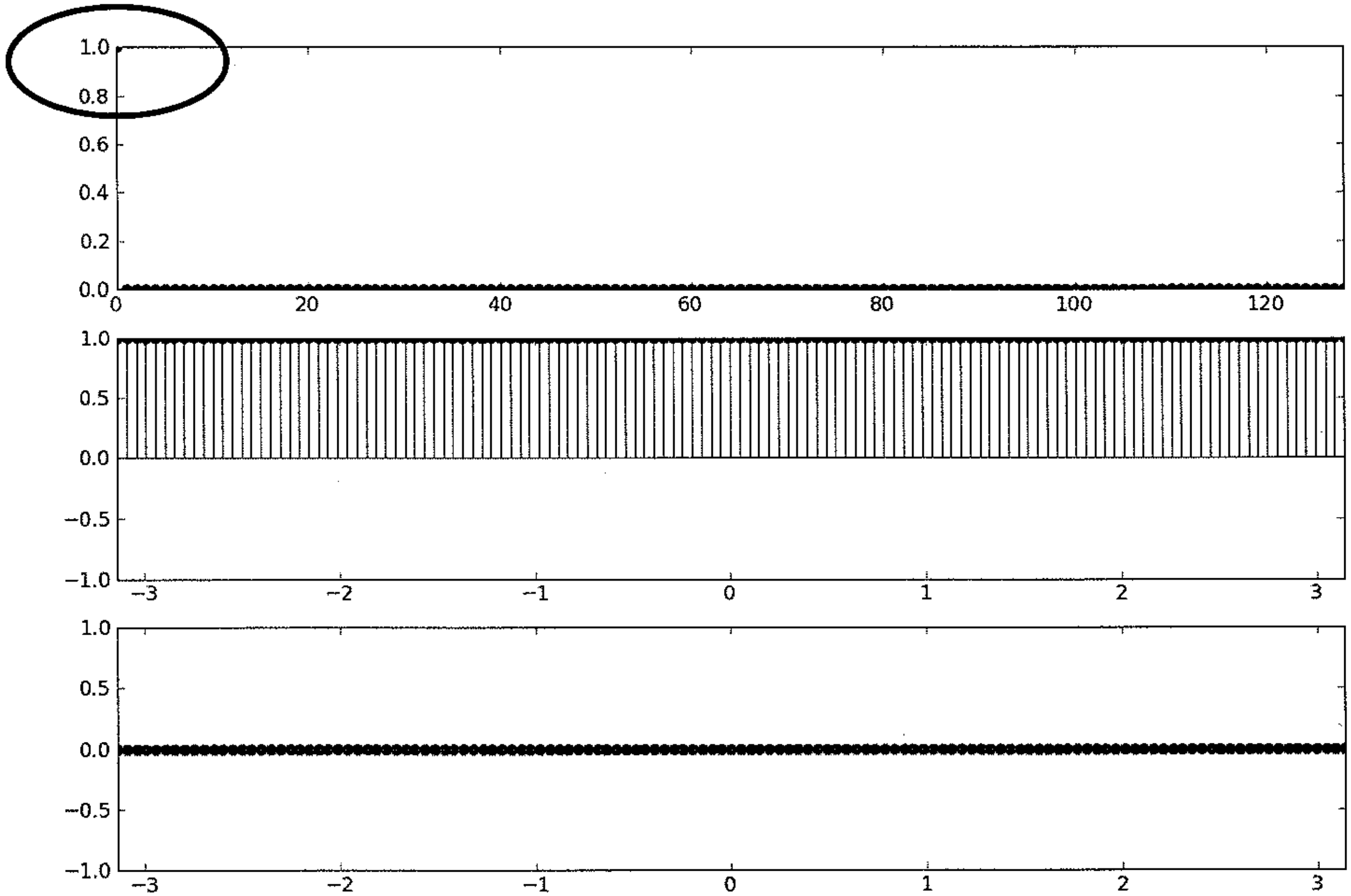
(11)

Constant in Time, one real nonzero DFT value



One Sample in Time, Constant DFT

(12)



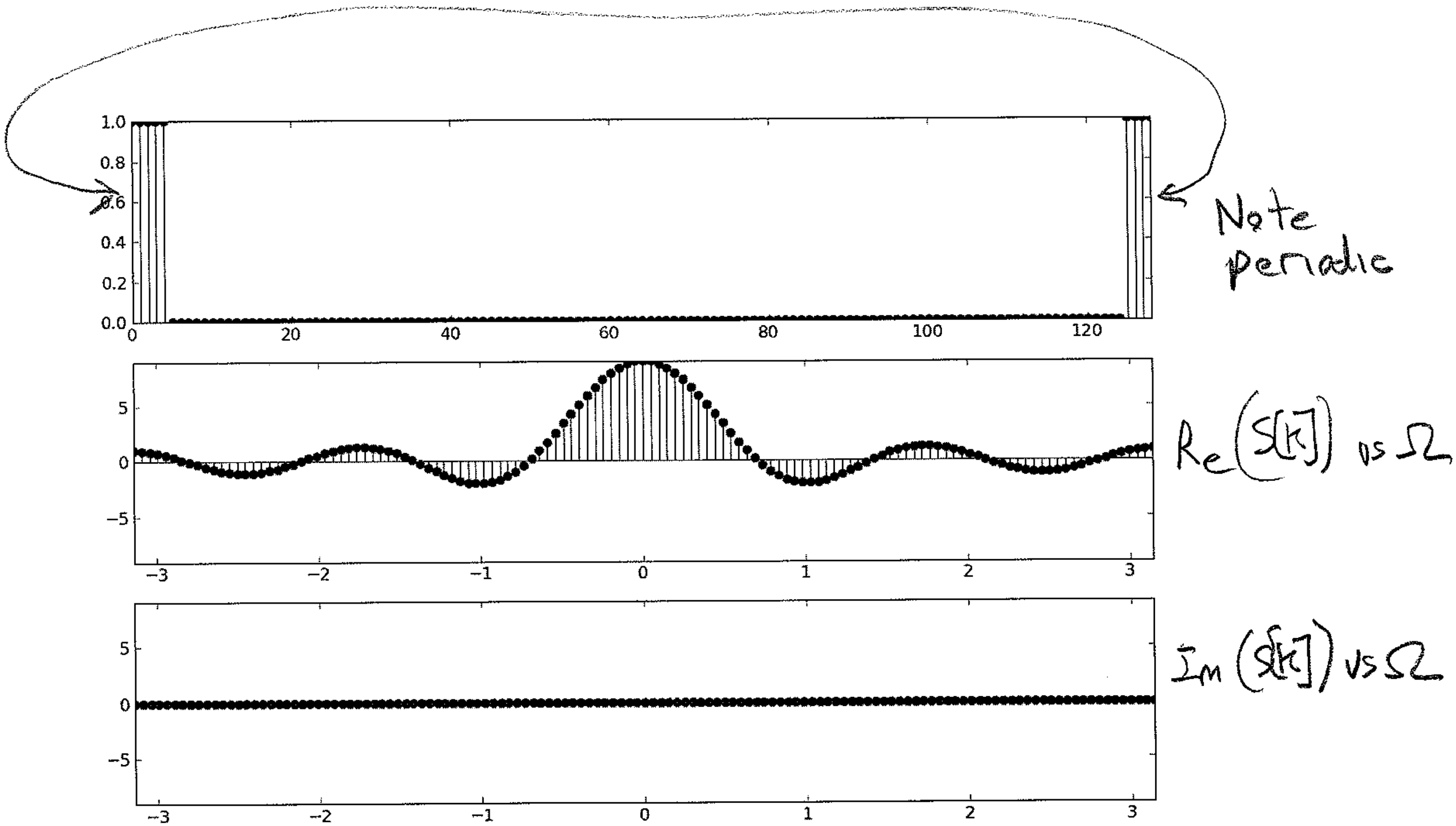
Re(S[k]) vs Ω

Im(S[k]) vs Ω

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k} s[n] = S[k]$$

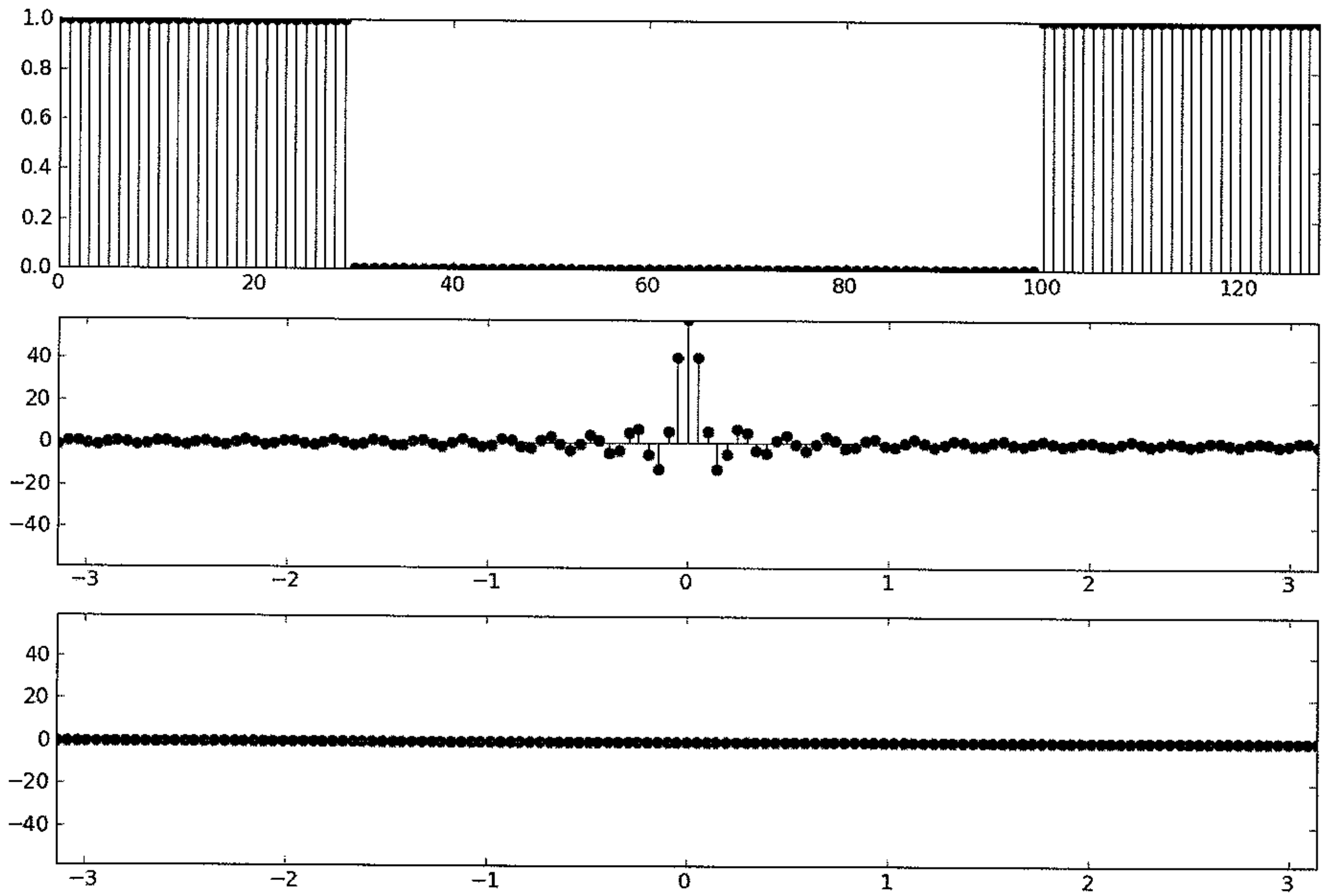
13

Medium Even Pulse, Real decaying DFT



14

Wider Even Pulse, Faster Decay

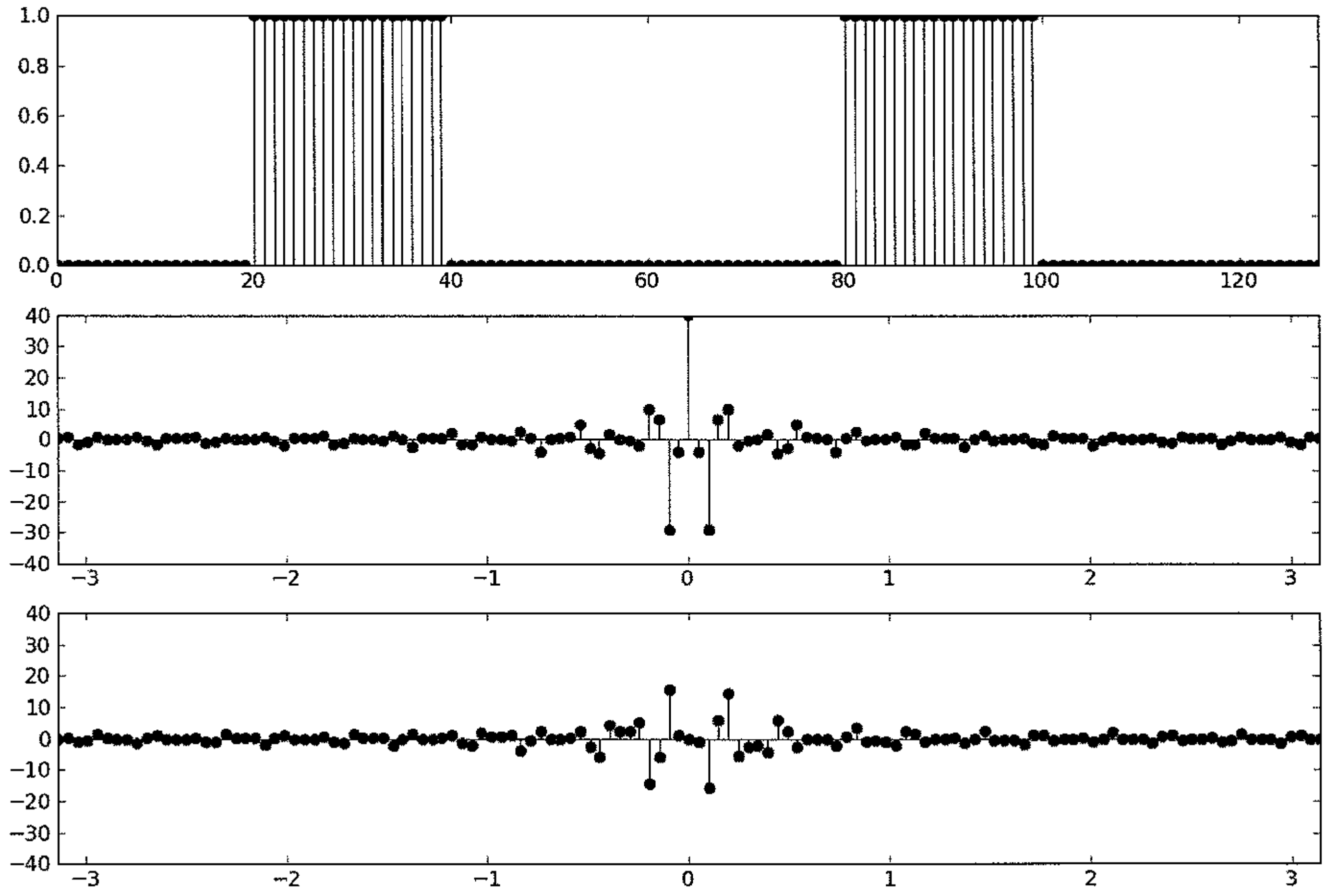


$Re(S[k])$ vs Ω

$Im(S[k])$ vs Ω

15

010010 Bit sequence with Fast Rise

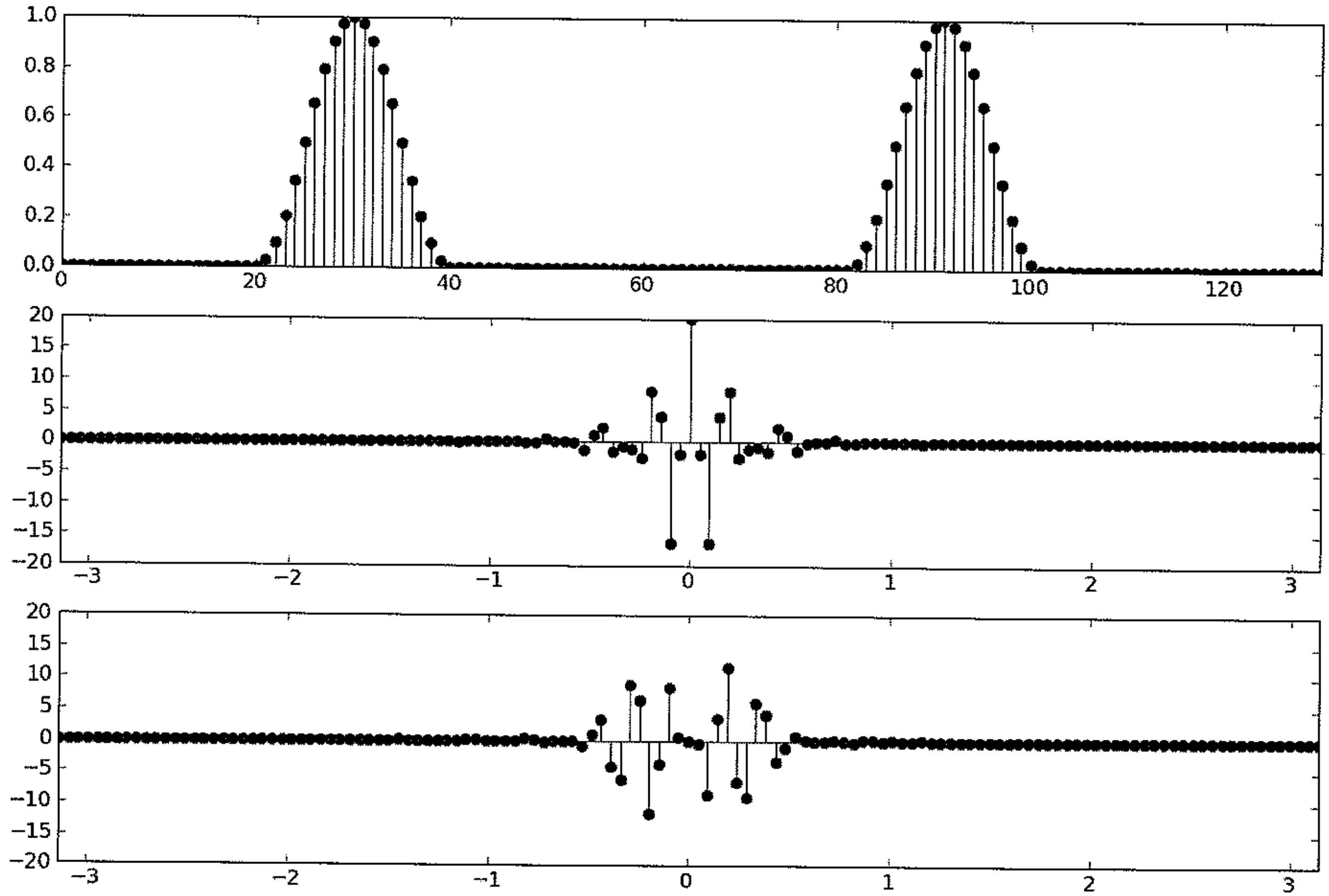


$Re(S[k])$ vs Ω

$Im(S[k])$ vs Ω

16

010010 Bit sequence with Slow Rise



Re (S[k]) vs Ω

Im(S[k]) vs Ω

Lec. 14 Recitation Notes

Recall the Basic Formulas

$$s[n] = \frac{1}{N}\sum_{k=-B}^{B} \mathcal{S}[k]\, e^{j\frac{2\pi}{N}kn} \qquad 2B+1=N \;(N \text{ odd})$$

$$\mathcal{S}[k] = \sum_{n=0}^{N-1} s[n]\, e^{-j\frac{2\pi}{N}kn}$$

__Suppose__

$$s[n] = \cos\left(\tfrac{2\pi}{7}n + \phi\right)$$

$$\Rightarrow \quad s[n] = \frac{1}{2}\left(e^{j\left(\frac{2\pi}{7}n+\phi\right)} + e^{-j\left(\frac{2\pi}{7}n+\phi\right)}\right)$$

$$= \frac{e^{j\phi}}{2}\, e^{j\frac{2\pi}{7}n} + \frac{e^{-j\phi}}{2}\, e^{j\frac{2\pi}{7}n}$$

Let $N = 21$, what is $\mathcal{S}[k]$, $-10 \le k \le 10$

$$\frac{2\pi}{21}k = \frac{2\pi}{7} \qquad k = 3$$

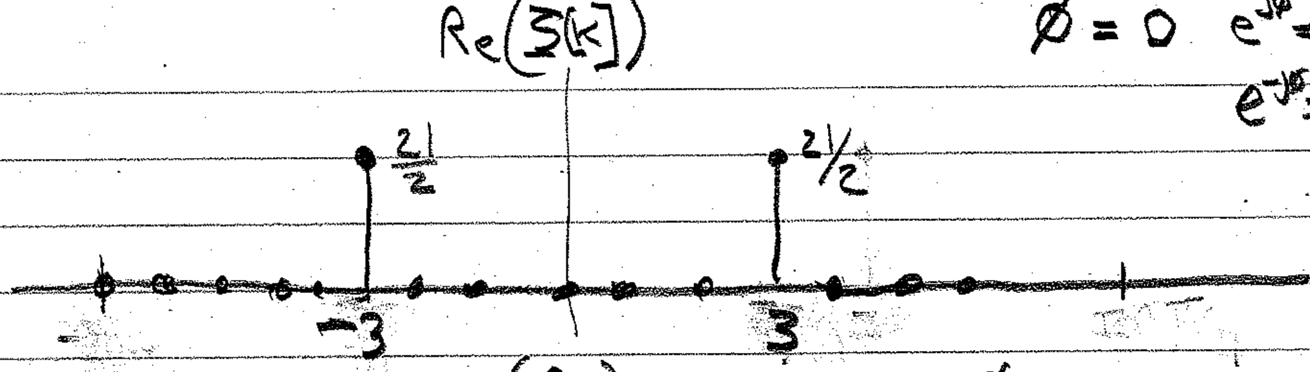
$$\Rightarrow \quad \mathcal{S}[k] = 0 \quad k \ne 3, -3$$

$$\mathcal{S}[3] = \frac{e^{j\phi}}{2}\cdot N \qquad \mathcal{S}[-3] = \frac{e^{-j\phi}}{2}\cdot N$$

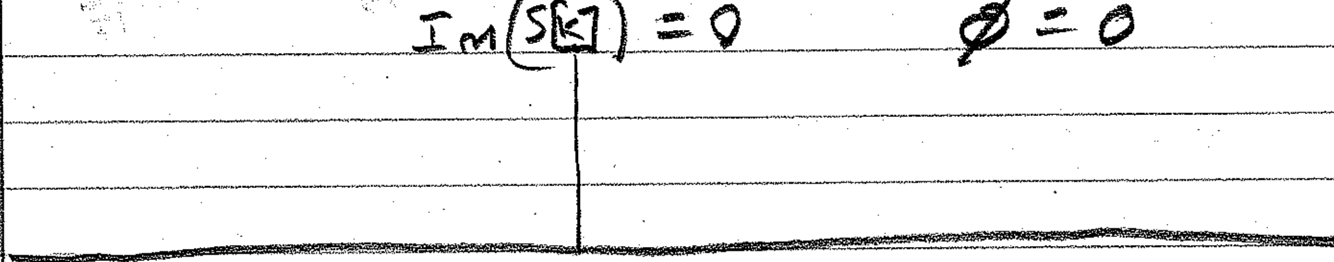
2

$\phi = 0 \quad e^{j\phi} = 1$

$e^{-j\phi} = 1$



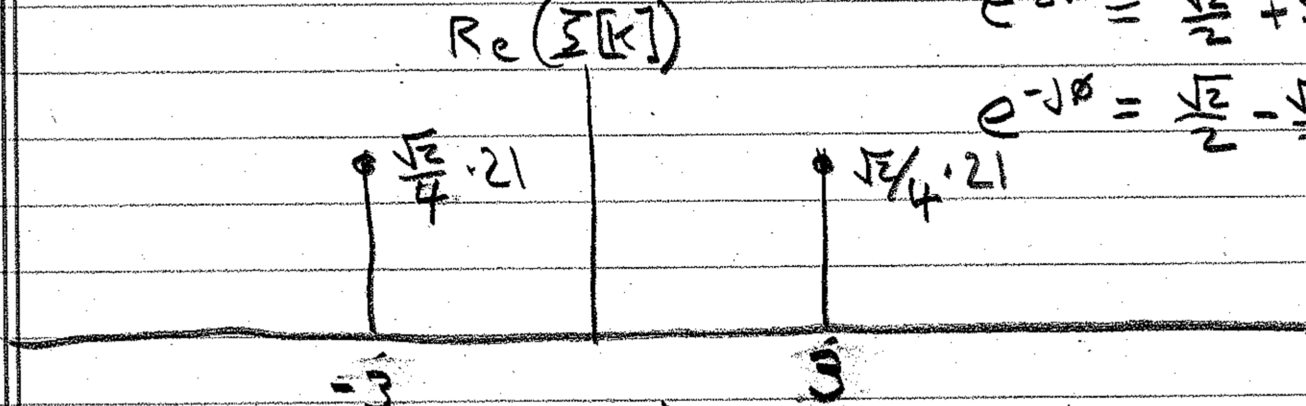
$\phi = 0$

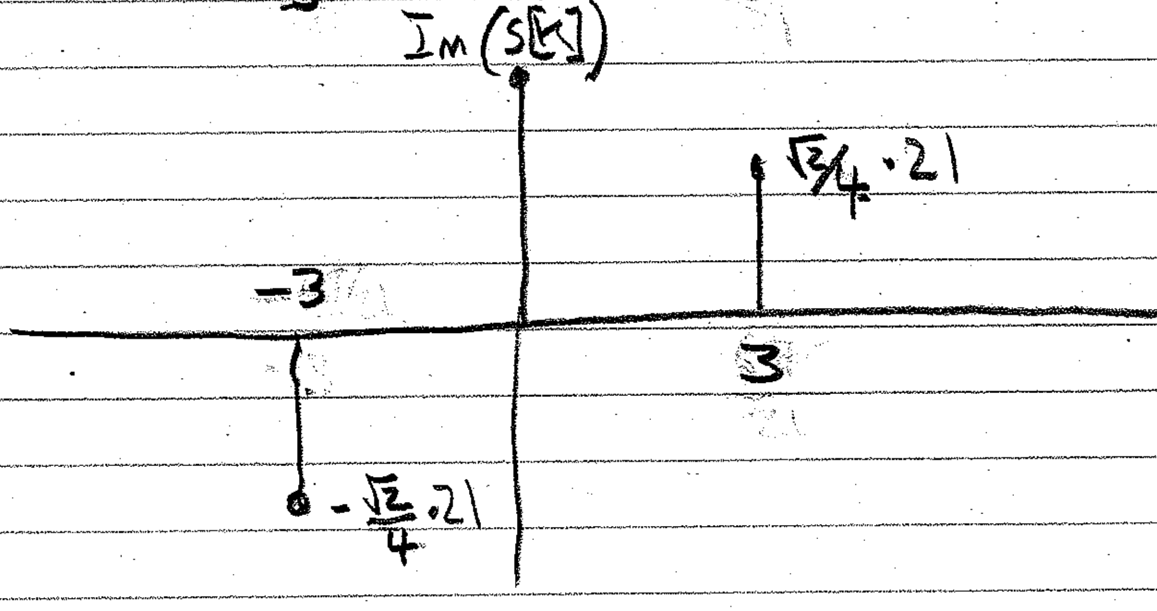


$\phi = \frac{\pi}{4}$

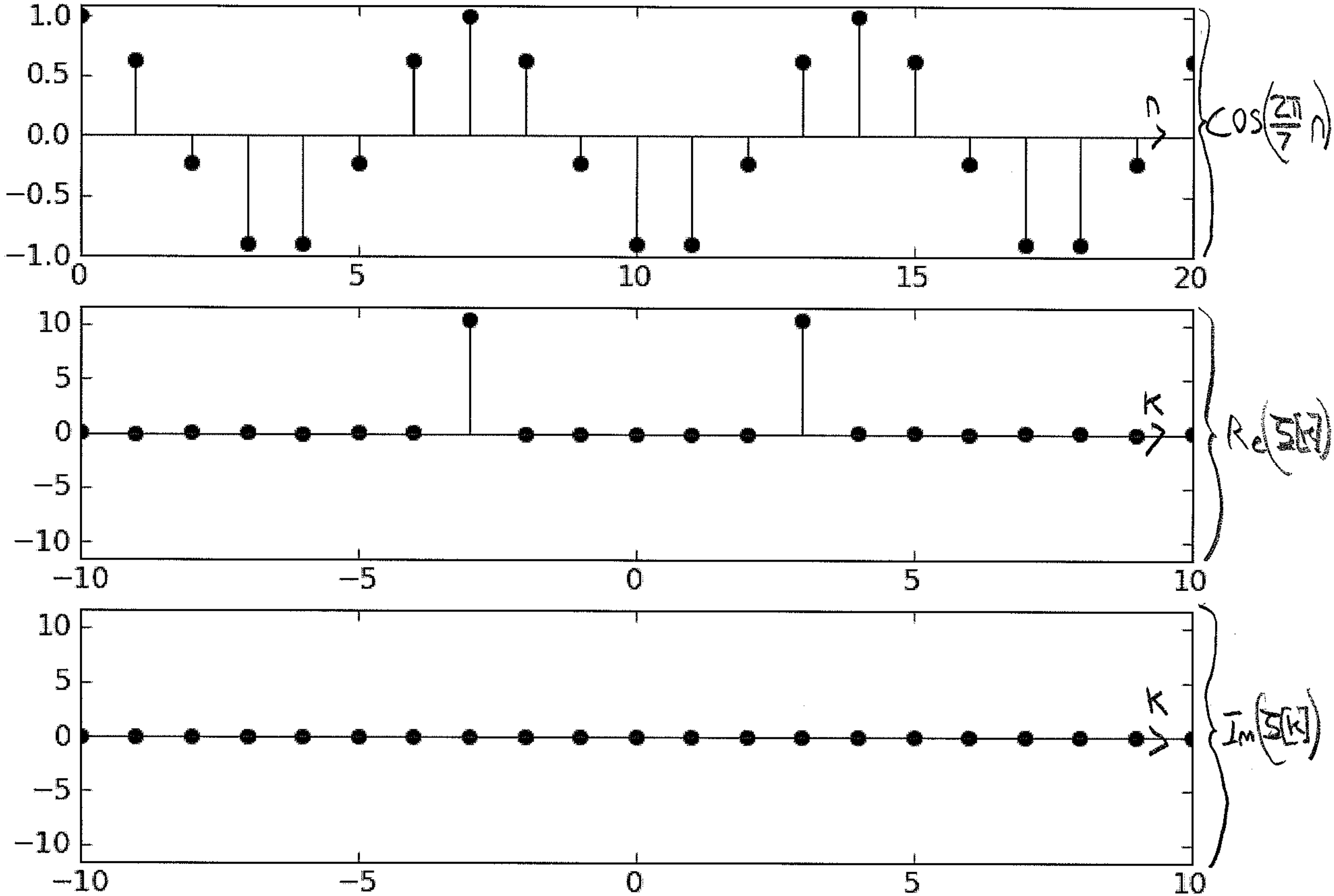
$e^{j\phi} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$

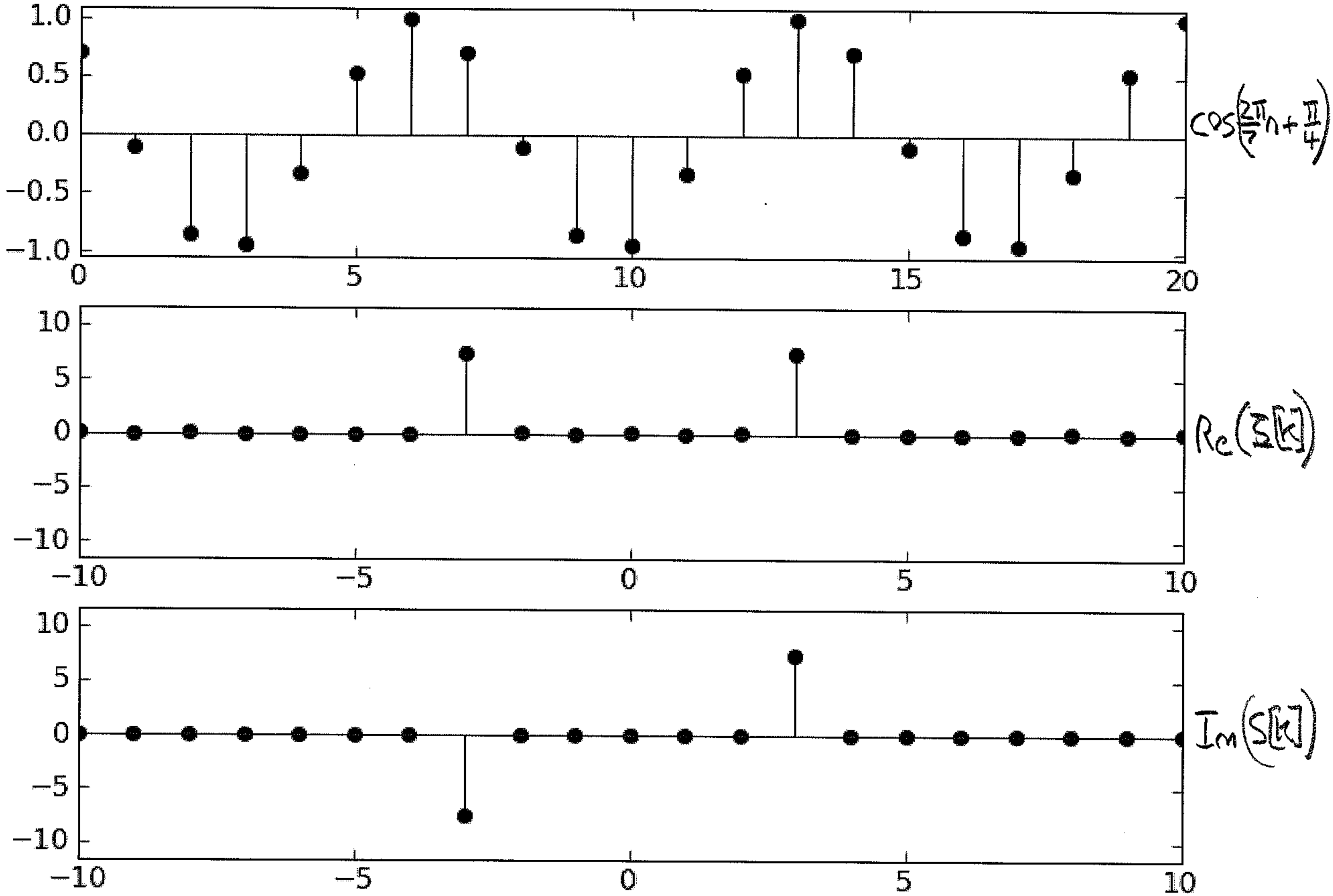
$e^{-j\phi} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$





(2)

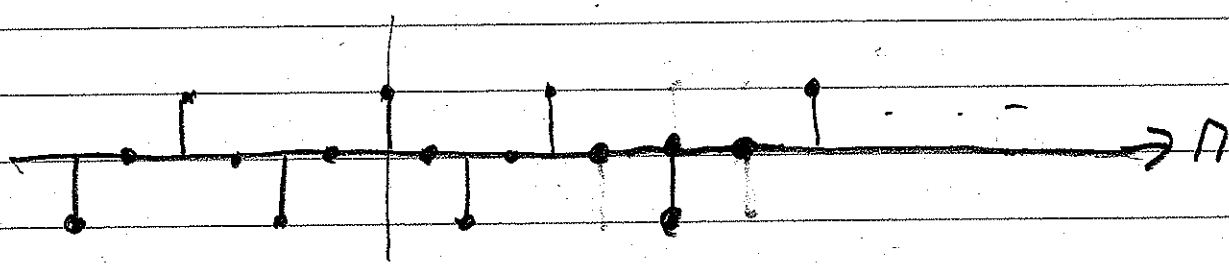




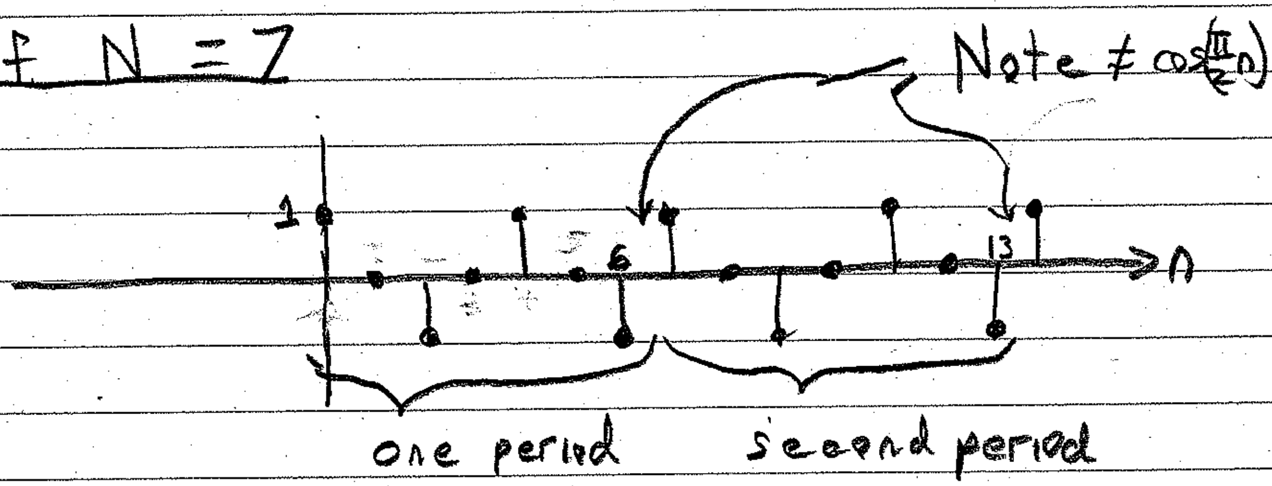
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
$\cos\left(\frac{2\pi}{7}n+\frac{\pi}{4}\right)$
10
5
0
−5
−10
−10
−5
0
5
10
Re(S[k])
10
5
0
−5
−10
−10
−5
0
5
10
Im(S[k])

Periodicity Issue

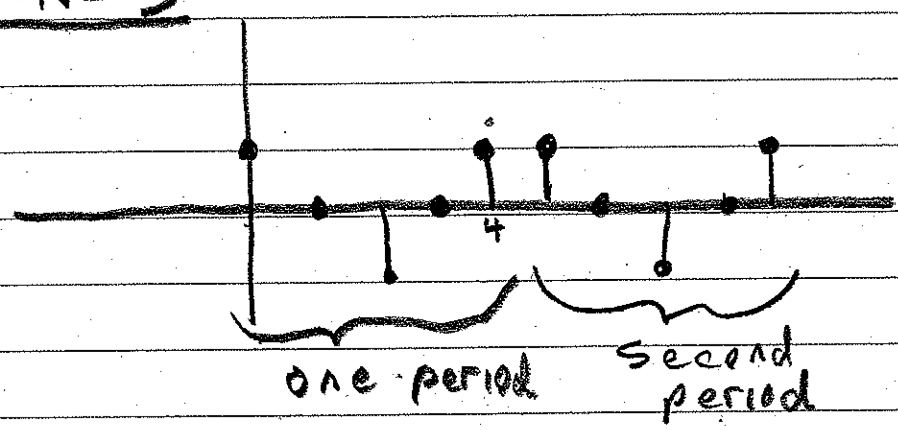
Suppose $s[n] = \cos\left(\frac{\pi}{2} n\right)$



If $N = 7$



If $N = 5$



If $N=5$

$$\underline{S}[k] = \sum_{n=0}^{4} s[n]\, e^{-j\frac{2\pi}{5}kn} =$$

$$= 1\cdot e^{-j\frac{2\pi}{5}k0} - 1\cdot e^{-j\frac{2\pi}{5}k\cdot 2} + 1 e^{-j\frac{2\pi}{5}k\cdot 4}$$

$$\Rightarrow \underline{S}[0] = 1 - 1 + 1 = 1$$

$$\underline{S}[1] = 1 - 1e^{-j\frac{4\pi}{5}} + 1e^{-j\frac{8\pi}{5}}$$

$$\underline{S}[-1] = 1 - 1e^{+j\frac{4\pi}{5}} + 1e^{+j\frac{8\pi}{5}}$$

Note for $N=5$ $\sum_{n=0}^{4} s[n] \neq 0$

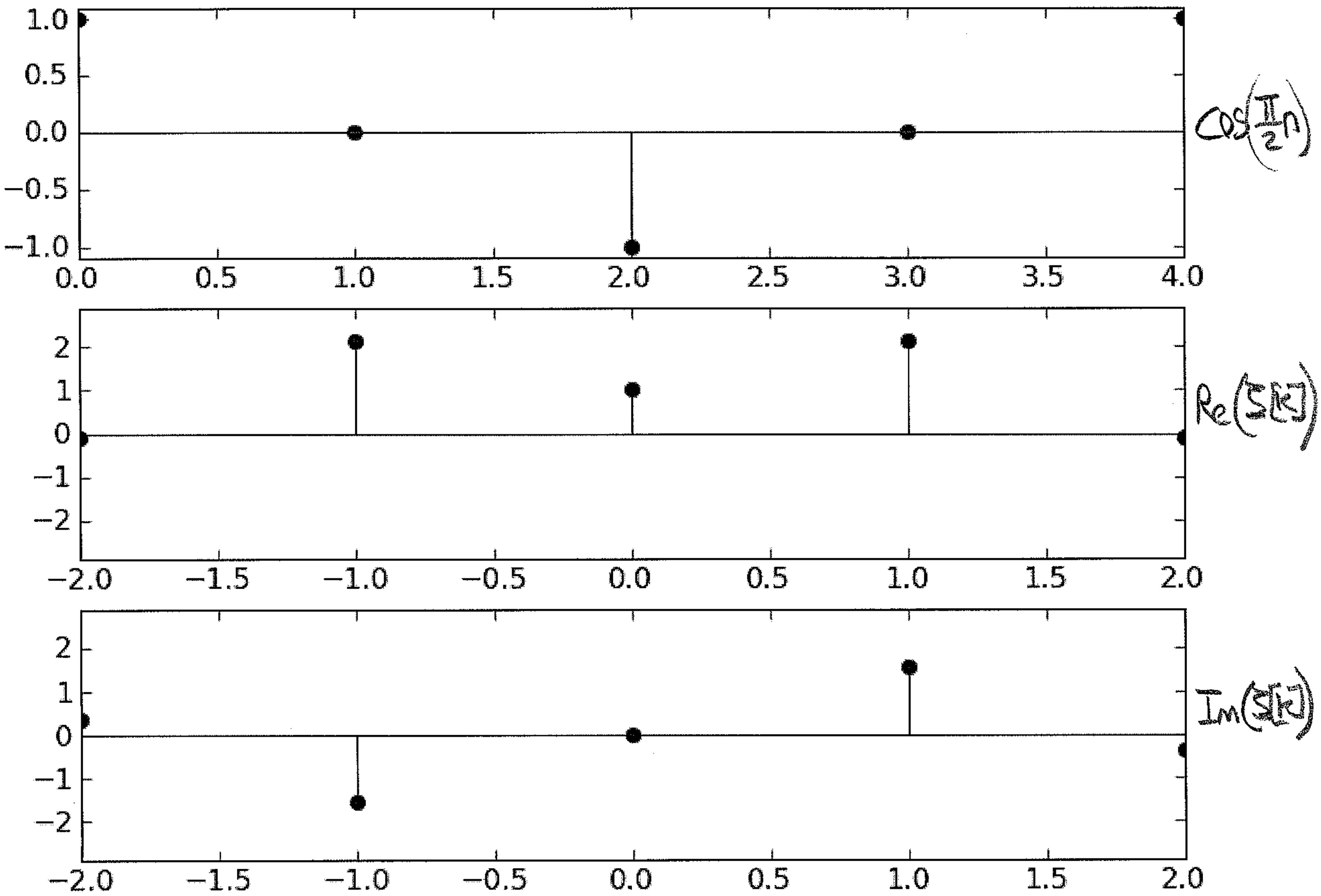
so $\underline{S}[0] \neq 0$

If $N=7$

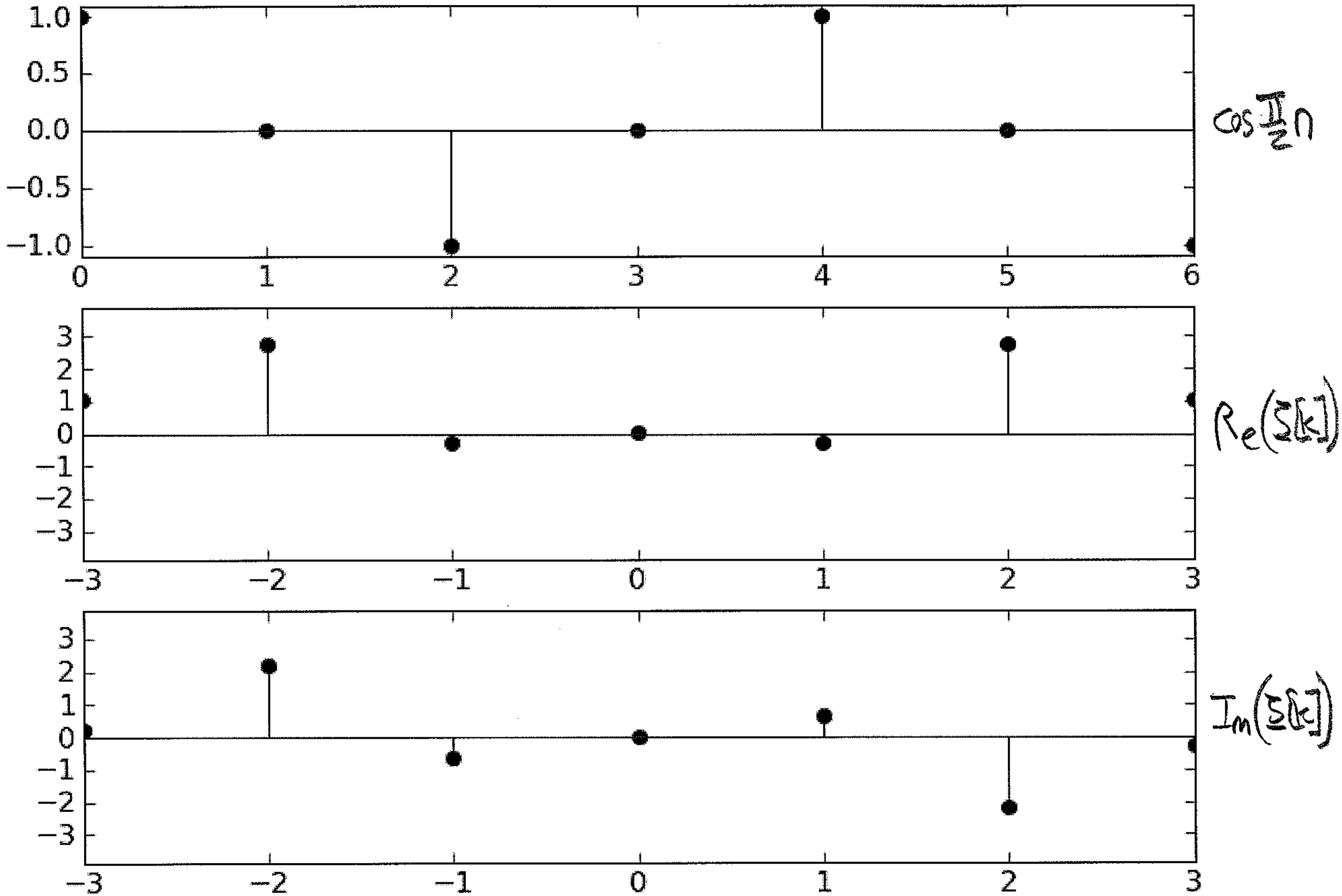
$$\underline{S}[0] = \sum_{n=0}^{6} s[n] = 0$$

7

N=5

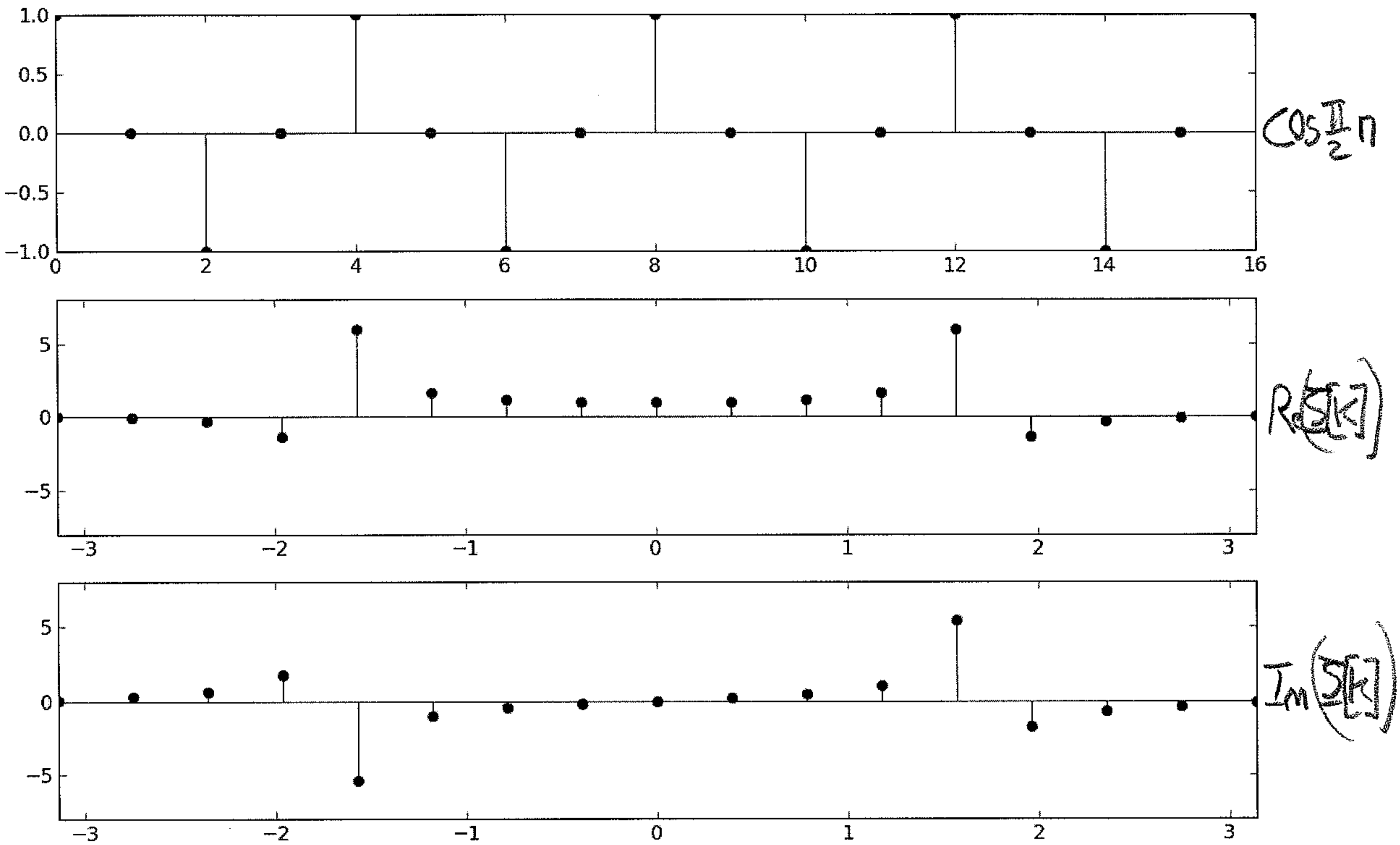

1.0
0.5
0.0
−0.5
−1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$\cos\left(\frac{\pi}{2}n\right)$
2
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
$Re(S[k])$
2
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
$Im(S[k])$

$N = 7$


1.0
0.5
0.0
−0.5
−1.0
0
1
2
3
4
5
6
$\cos \frac{\pi}{2} n$
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
$Re(S[k])$
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
$Im(S[k])$

(9)

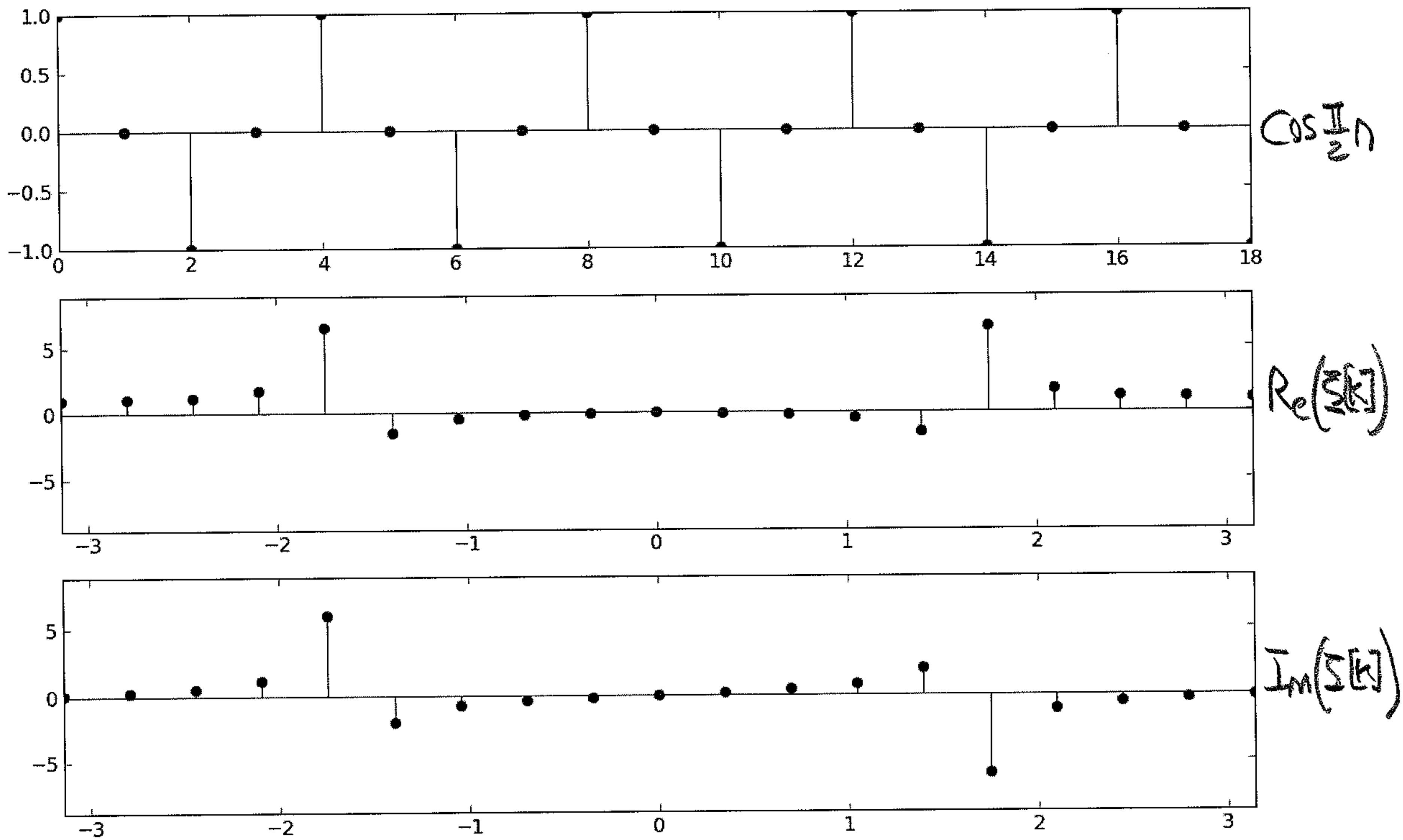
N=17



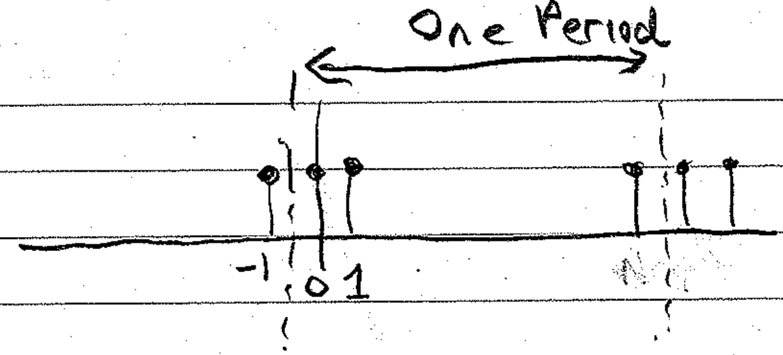
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
14
16
$\cos \frac{\pi}{2} n$
5
0
−5
−3
−2
−1
0
1
2
3
$Re(S[k])$
5
0
−5
−3
−2
−1
0
1
2
3
$Im(S[k])$

10

$N = 19$


1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
14
16
18
$\cos \frac{\pi}{2} n$
5
0
−5
−3
−2
−1
0
1
2
3
$Re(S[k])$
5
0
−5
−3
−2
−1
0
1
2
3
$Im(S[k])$

Square Pulse Case



Suppose N = 11

$$\underline{S}[k] = \overbrace{s[0]}^{=1} \cdot 1 + \overbrace{s[1]}^{=1} \cdot e^{-j\frac{2\pi}{11}k\cdot 1} + \overbrace{s[11-1]}^{=1} \cdot e^{-j\frac{2\pi}{11}k(11-1)}$$

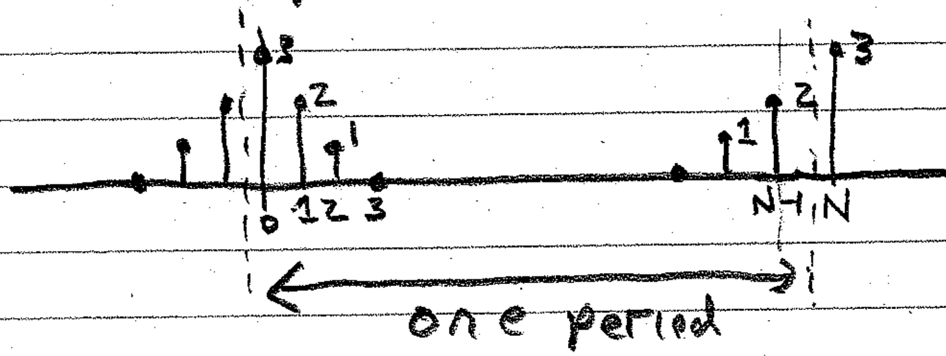
$$= 1 + e^{-j\frac{2\pi}{11}k} + \underbrace{e^{-j\frac{2\pi\cdot 10}{11}k}}_{=e^{+j\frac{2\pi}{11}k}}$$

$$= 1 + 2\cos\left(\frac{2\pi}{11}k\right)$$

For general N

$$\underline{S}[k] = 1 + 2\cos\left(\frac{2\pi}{N}k\right)$$

Triangular Pulse Case



N = 11

$$S[k] = 3\cdot 1 + 2e^{-j\frac{2\pi}{11}k} + 1e^{-j\frac{4\pi}{11}k} + 1e^{-j\frac{2\pi 9}{11}k} + 2e^{-j\frac{2\pi 10}{11}k}$$

$$= 3 + 4\cos\left(\frac{2\pi}{11}k\right) + 2\cos\left(\frac{4\pi}{11}k\right)$$

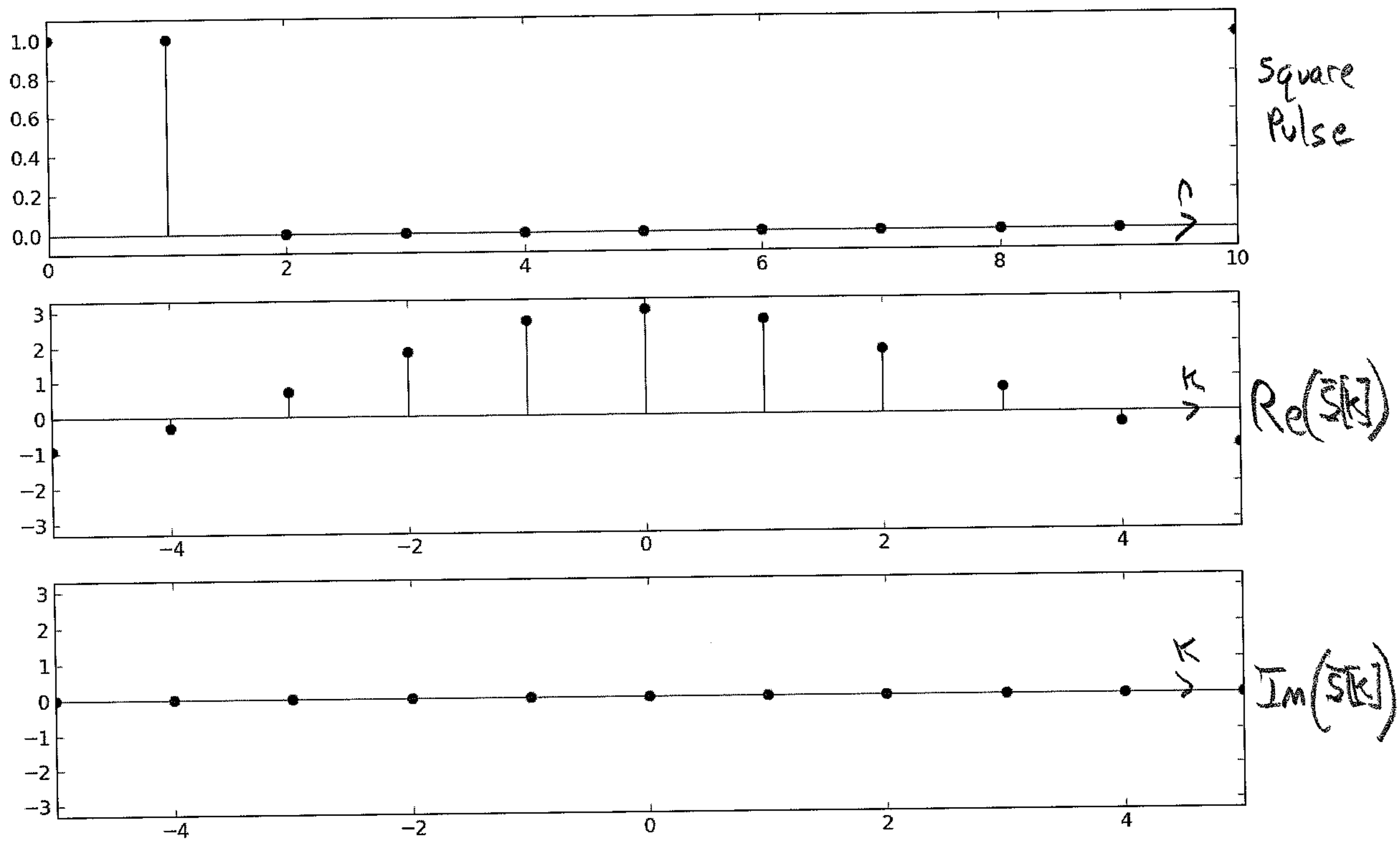
Note: Compare to Product of Square Pulse $S[k]$'s

$$\left(1 + e^{-j\frac{2\pi}{11}k} + e^{+j\frac{2\pi}{11}k}\right)\left(1 + e^{-j\frac{2\pi}{11}k} + e^{+j\frac{2\pi}{11}k}\right)$$

$$= 3 + 2e^{-j\frac{2\pi}{11}k} + e^{-j\frac{4\pi}{11}k} + e^{j\frac{4\pi}{11}k} + 2e^{j\frac{2\pi}{11}k}$$

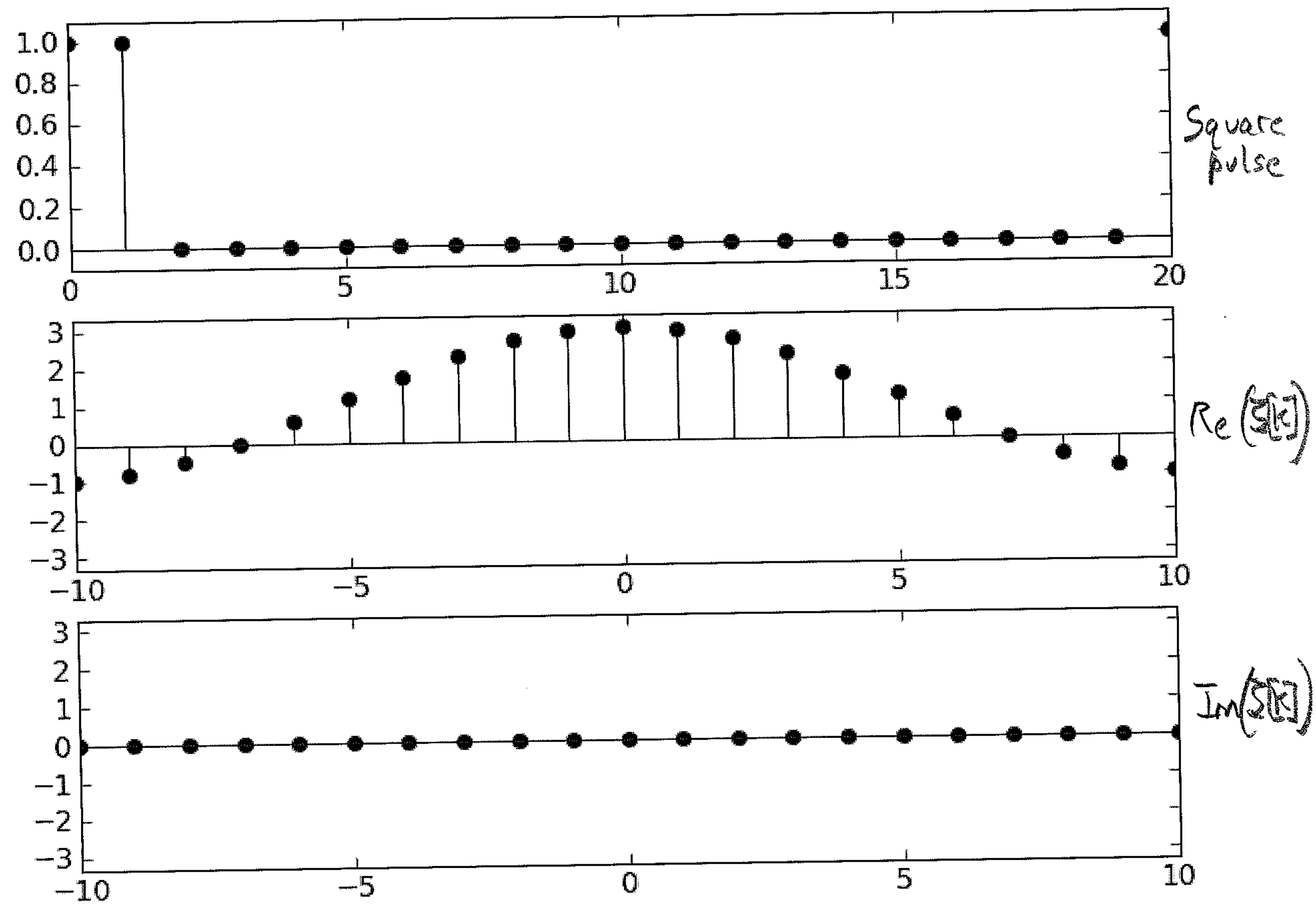
12

N = 11



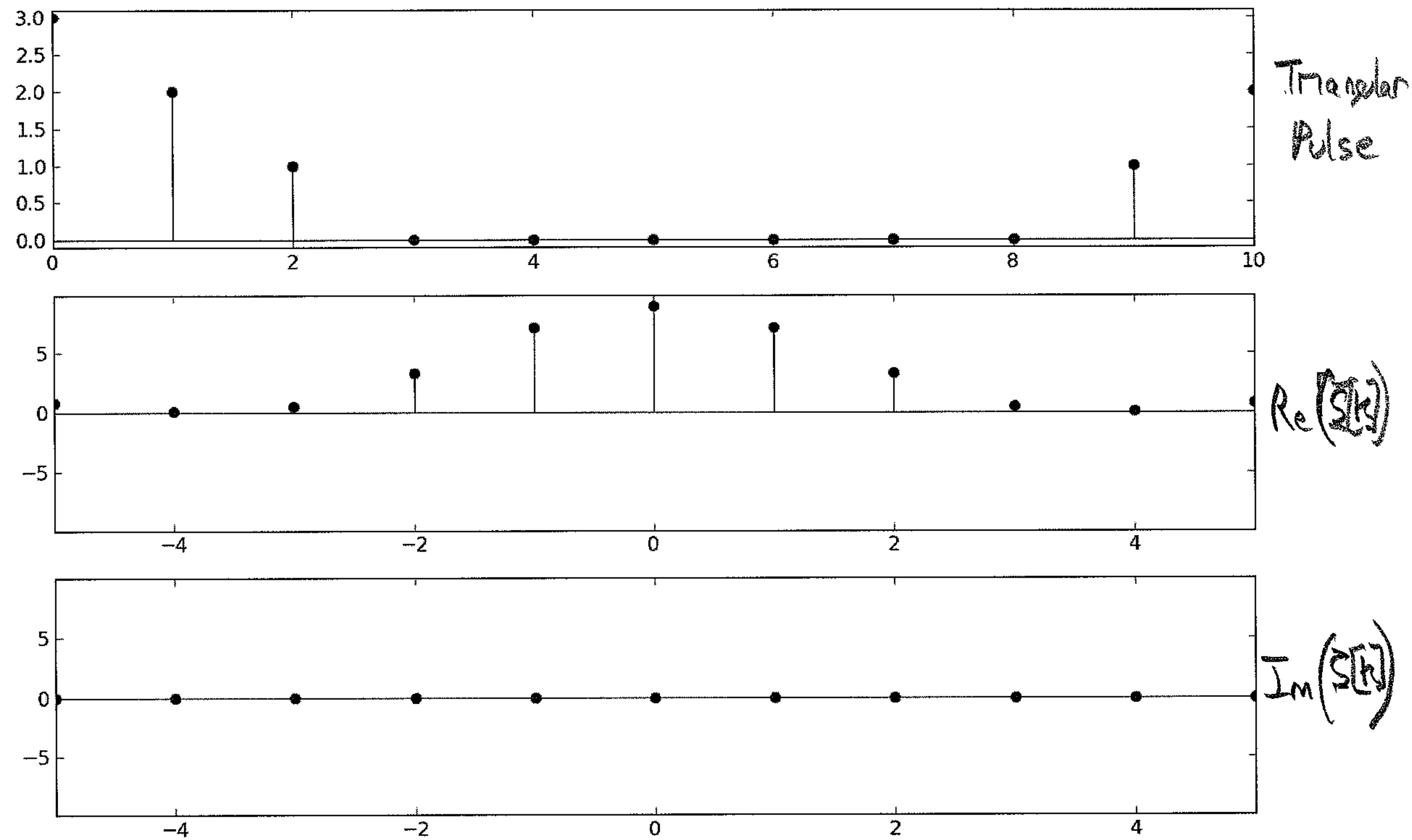
14

N = 21



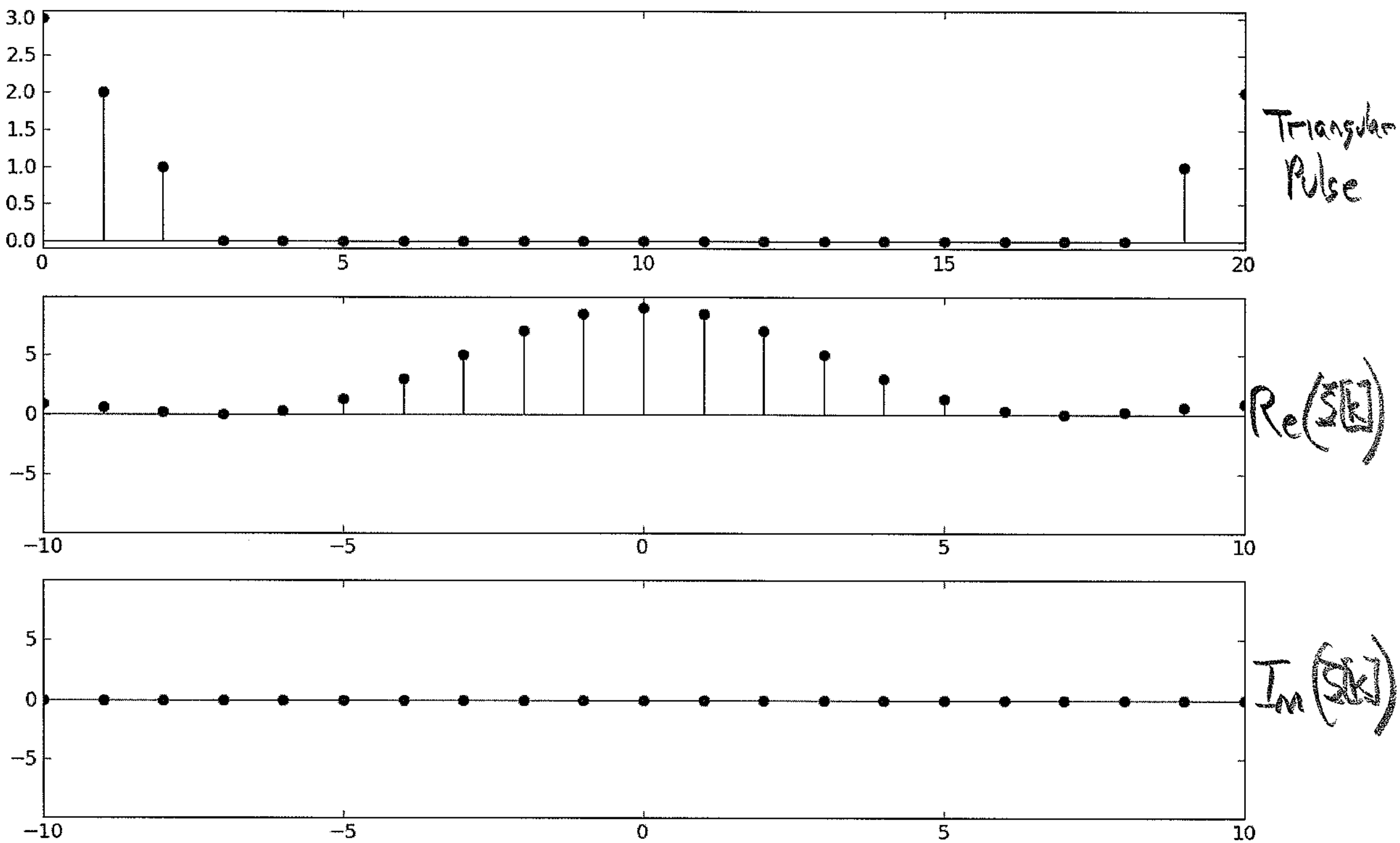
15

$N = 11$


3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
Triangular Pulse
5
0
−5
−4
−2
0
2
4
Re(S[k])
5
0
−5
−4
−2
0
2
4
Im(S[k])

16

N=21


3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
Triangular Pulse
5
0
−5
−10
−5
0
5
10
Re(S[k])
5
0
−5
−10
−5
0
5
10
Im(S[k])