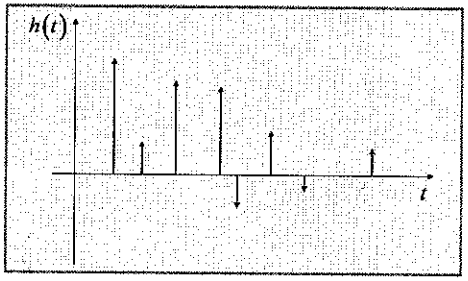
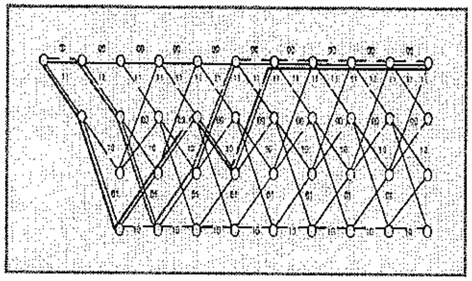
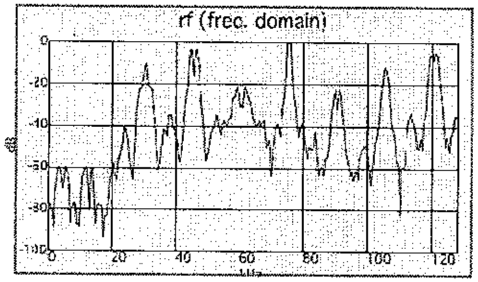
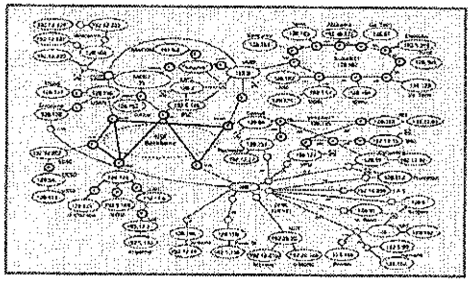
1









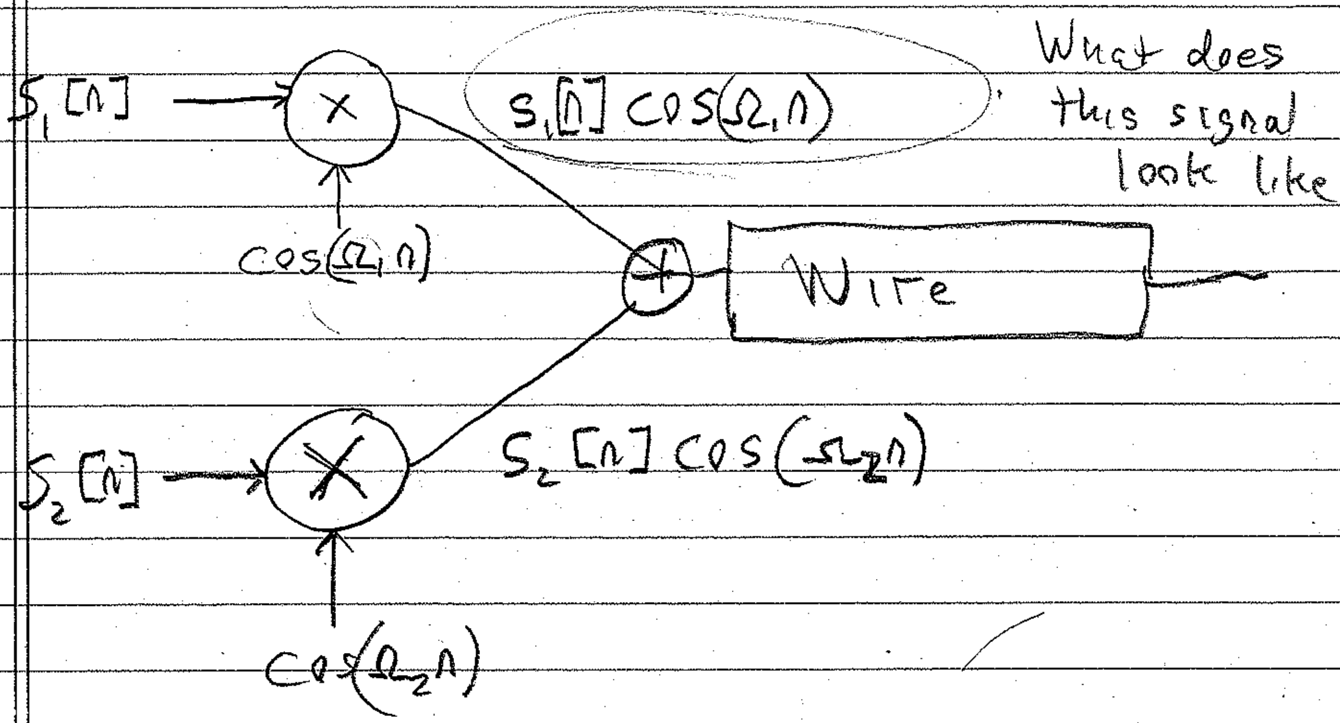
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2009
Lecture #15

- Examples of Plotting Spectrum
- Modulation
- Using DFT pictures for Modulation

Recall Problem



What does this signal look like

If Ω_1 >> bandwidth of s_1

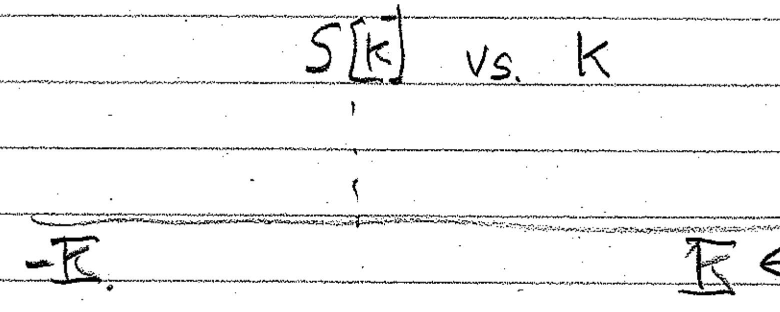
③

Reminder about DFT

$$s[n] = \frac{1}{N}\sum_{k=-K}^{K} S[k]e^{j\underbrace{\tfrac{2\pi}{N}k}_{\Omega_k}n} \qquad N = 2K+1$$

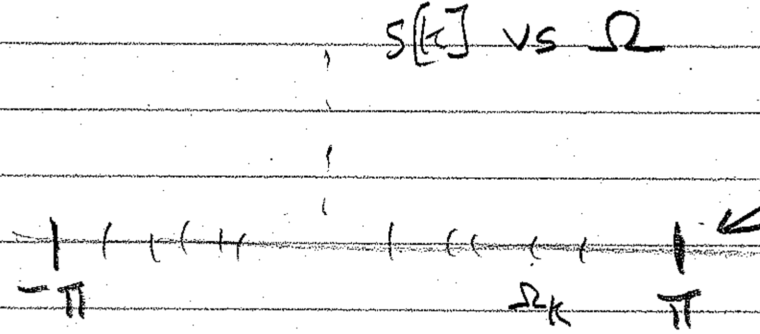
$$S[k] = \sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi}{N}kn}$$

Plot Many ways

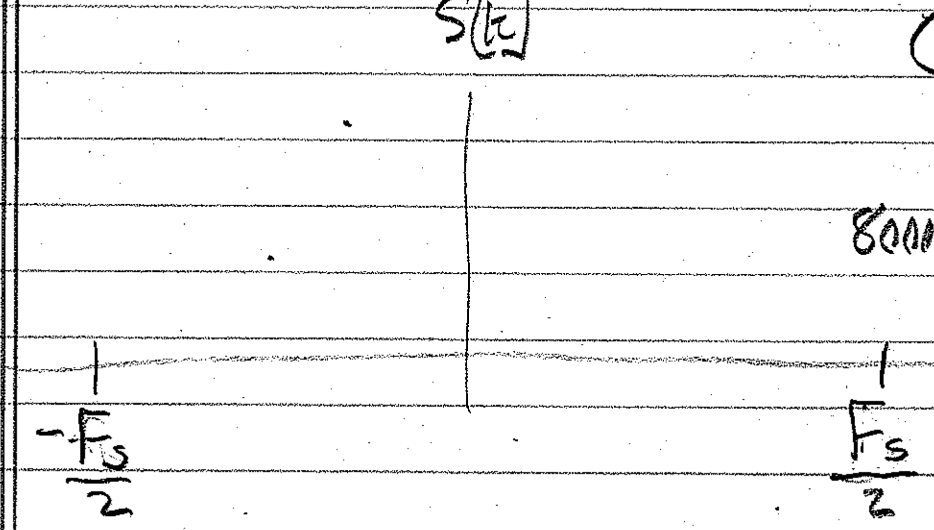


Dependes on # of samples $2\bar{K}+1 = N$

$$\Omega_K = \frac{2\pi}{N} \cdot K$$



$-\pi \leq \Omega_K \leq \pi$
Independent of N



Sampling frequency
(F_s = 8000 samples/sec)

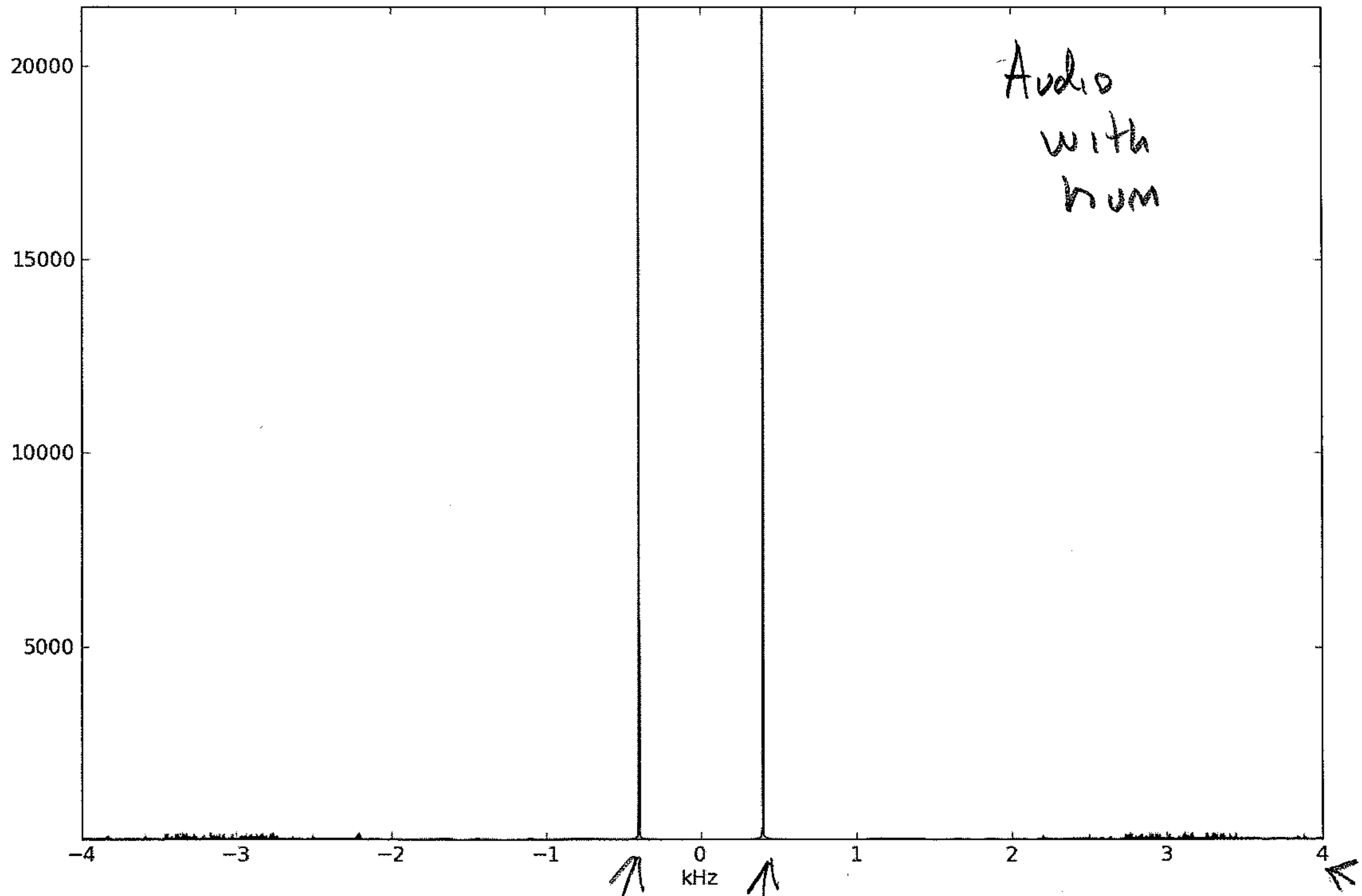
$$8000 \cdot \frac{\Omega_K}{2\pi} = f_K$$

$$K = \frac{N f_K}{F_s}$$

Relating K to f_K: $F_s \frac{2\pi}{N} \cdot K \Big/ 2\pi = F_s \frac{\Omega_K}{2\pi} = f_K$

|S[k]| vs frequency (8000 samples/sec)

73,313 Samples



Audio with hum

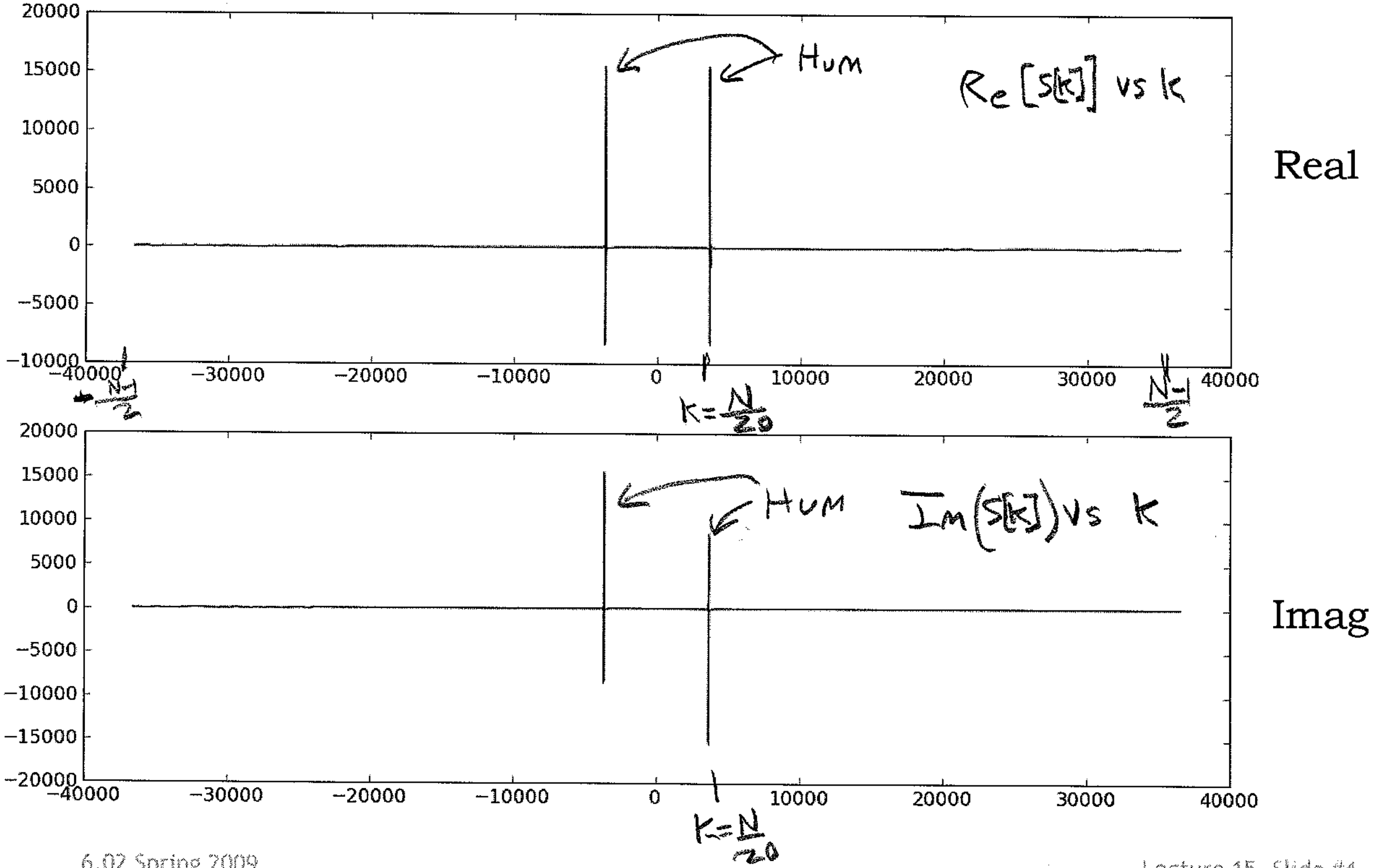
-400Hz

400 Hz

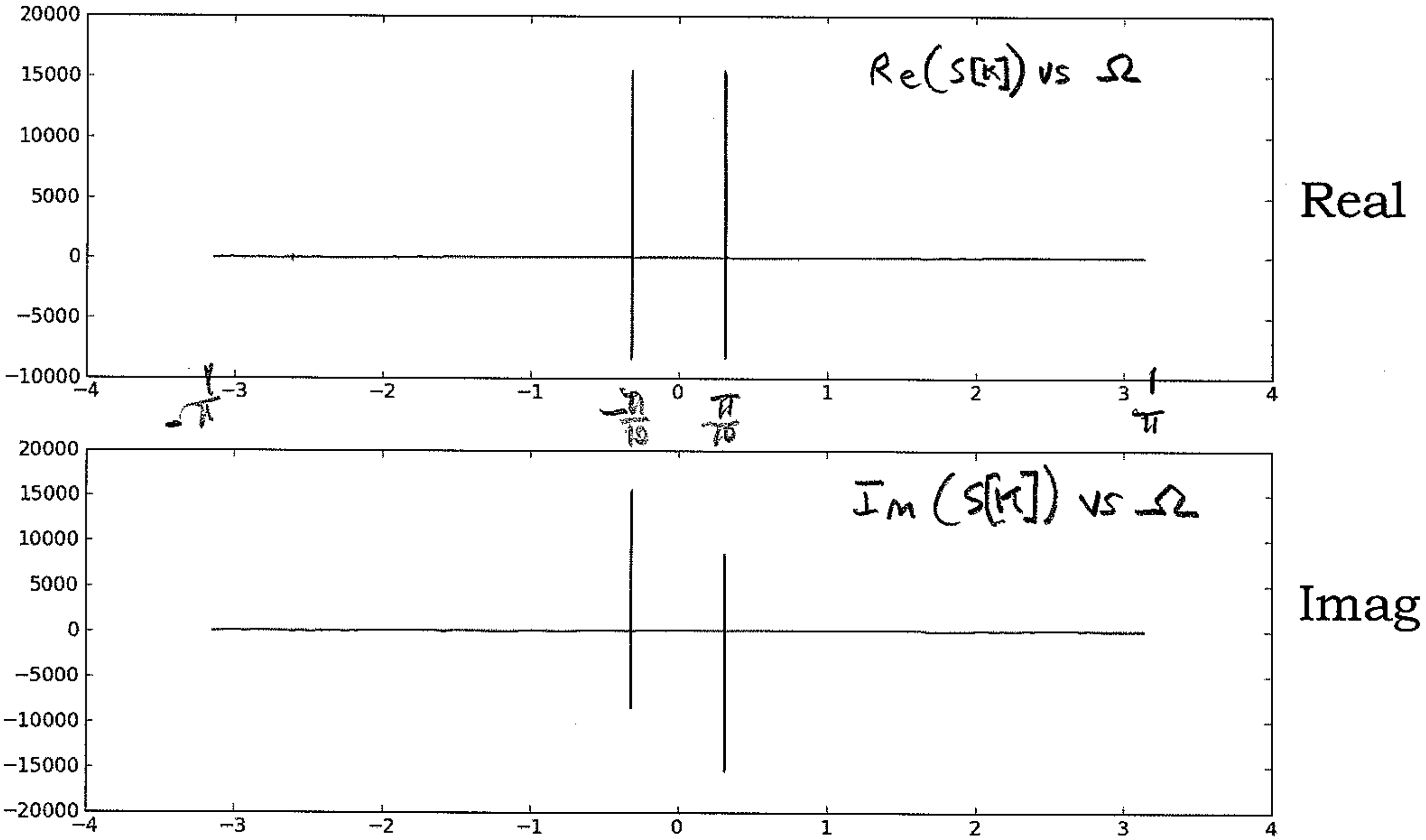
$K \approx \frac{73{,}313 \cdot 400}{8000} = \frac{73{,}313}{20}$

$\frac{8000}{2}$ Hz $= 4$kHz

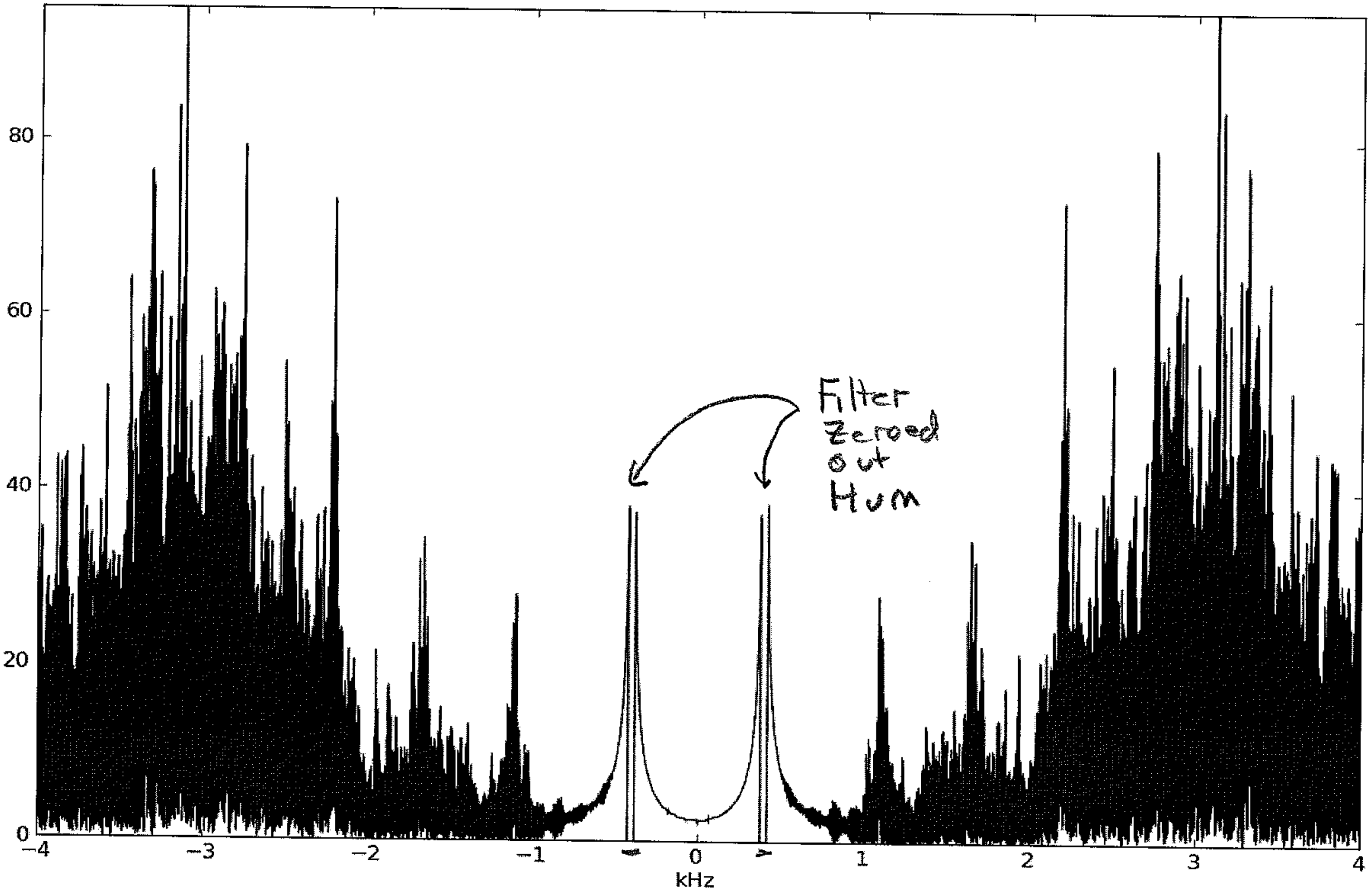
S[k] vs k (73,313 Samples)



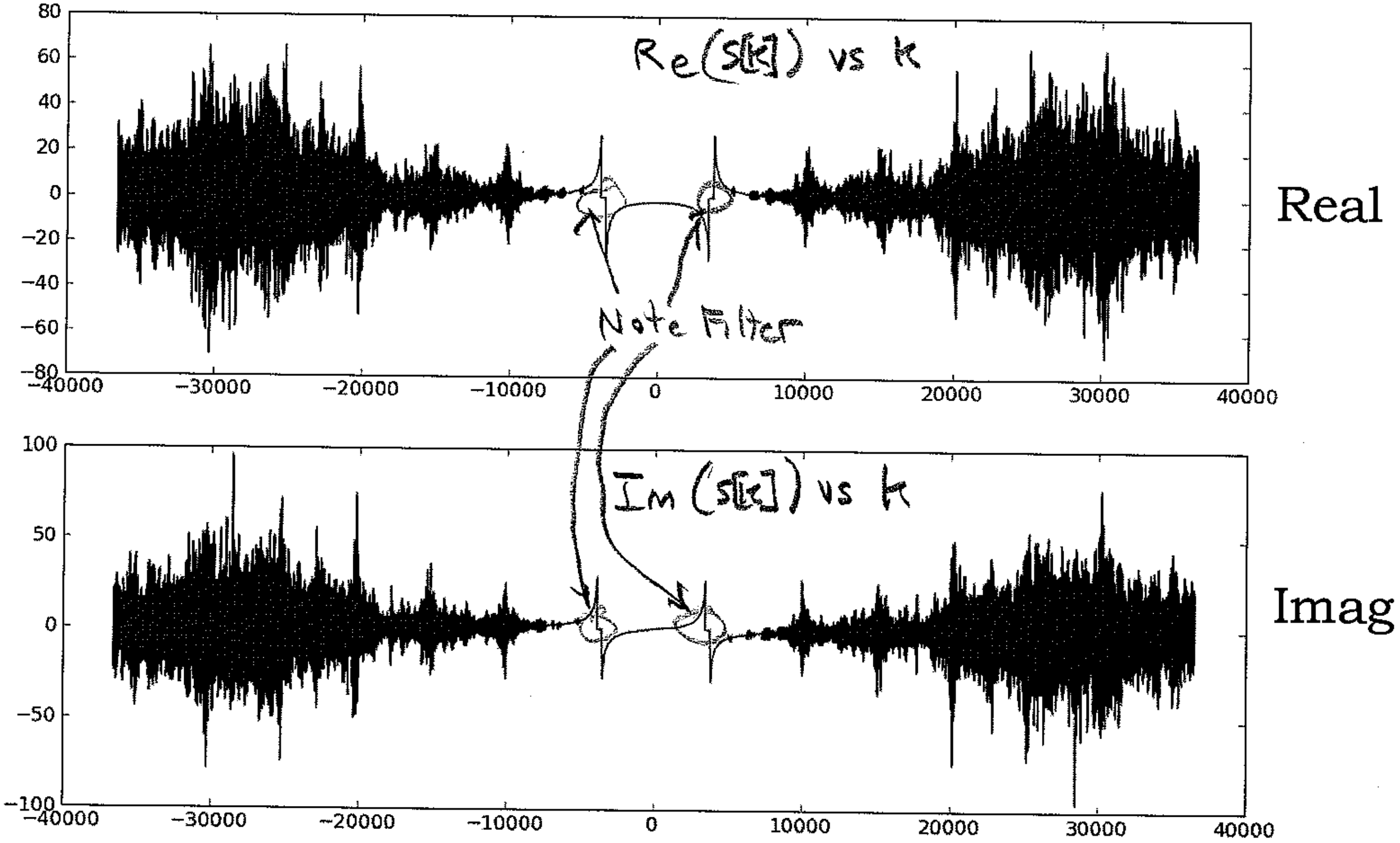
S[k] vs Omega


Re(S[k]) vs Ω
Real
Im(S[k]) vs Ω
Imag
20000
15000
10000
5000
0
−5000
−10000
−15000
−20000
−4
−3
−2
−1
0
1
2
3
4
−π
−π/10
π/10
π

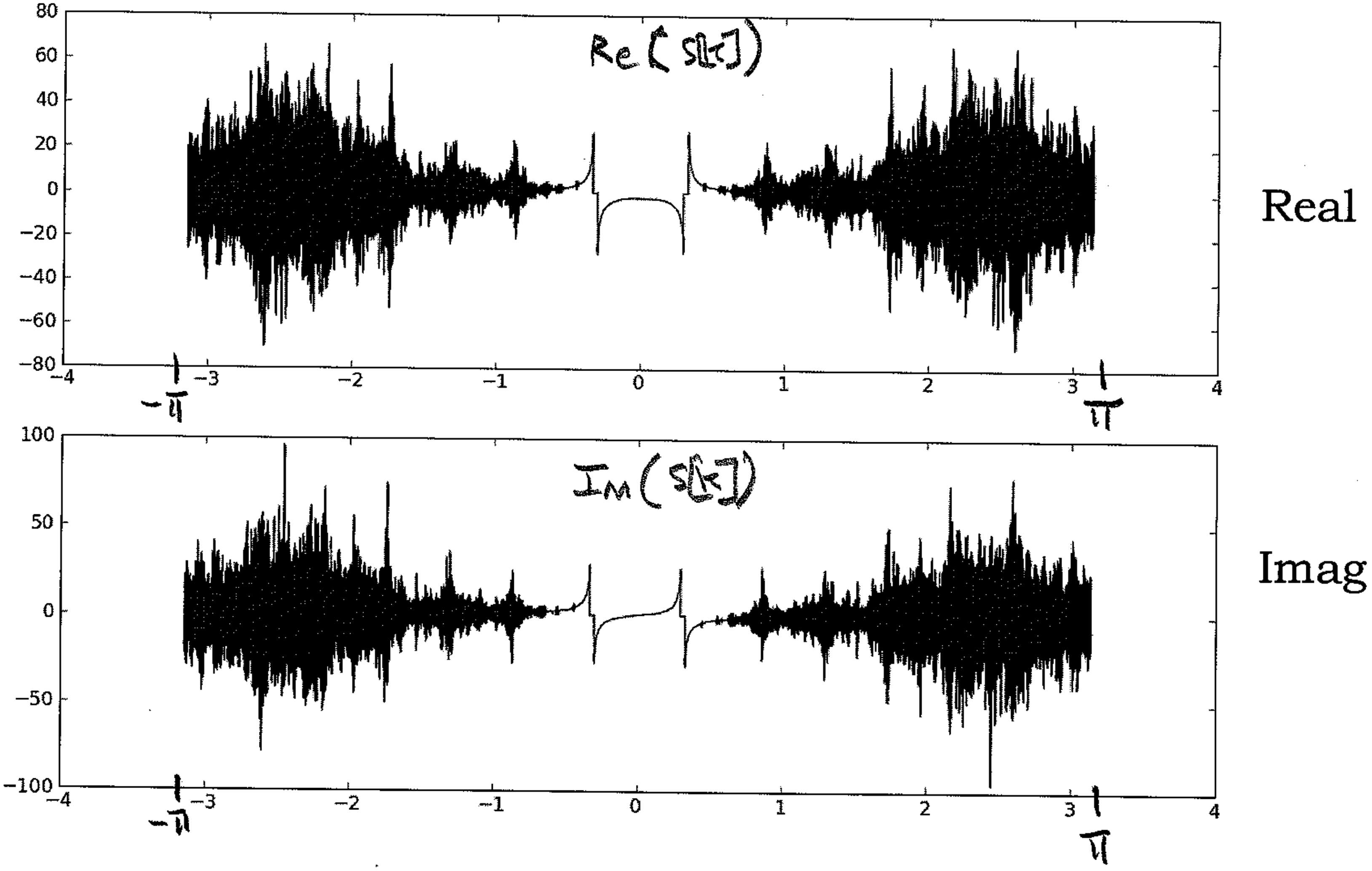
Filtered|S[k]|vs frequency (8K samp/sec)



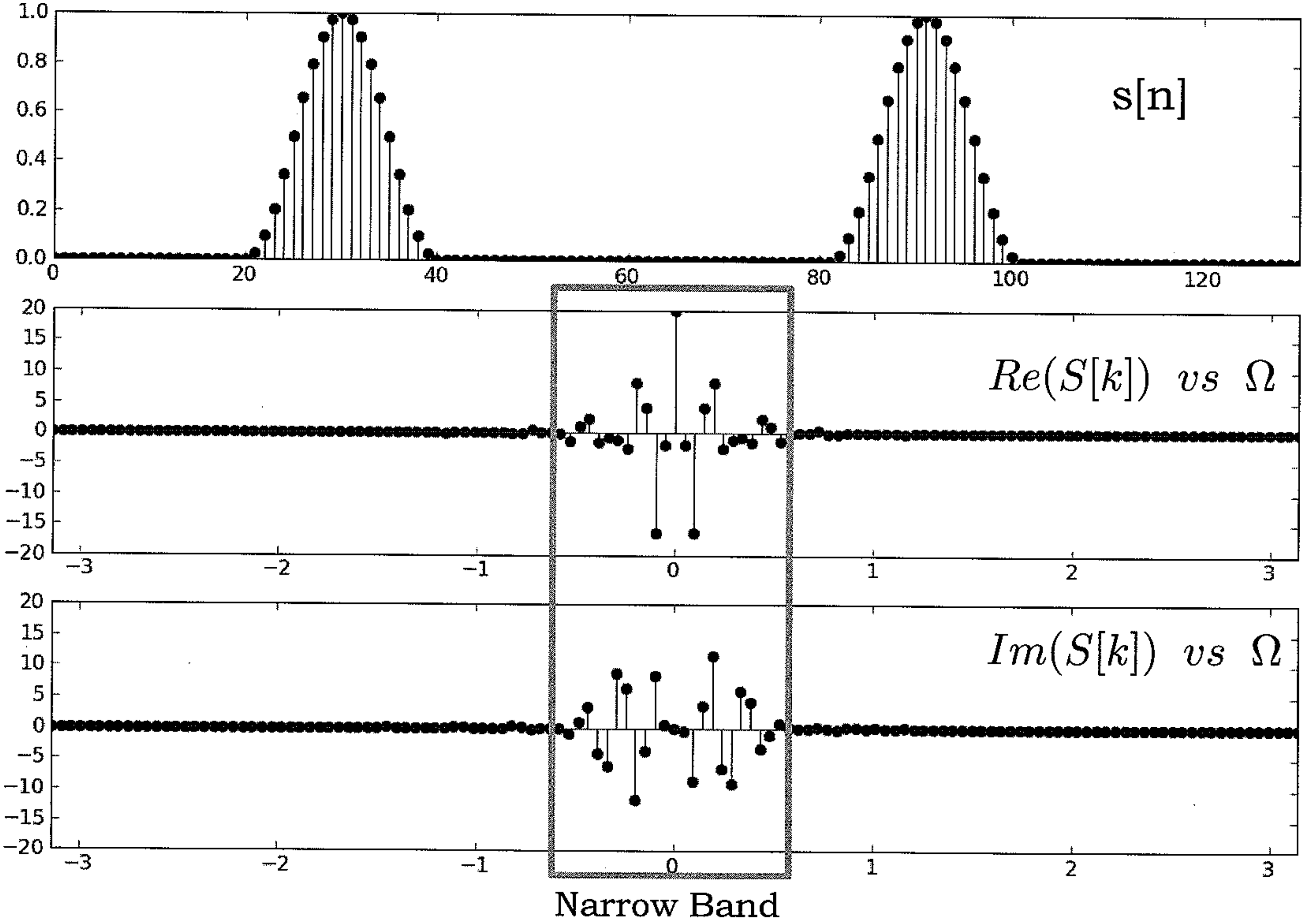
Filtered S[k] vs k (73,313 Samples)



Filtered S[k] vs Omega

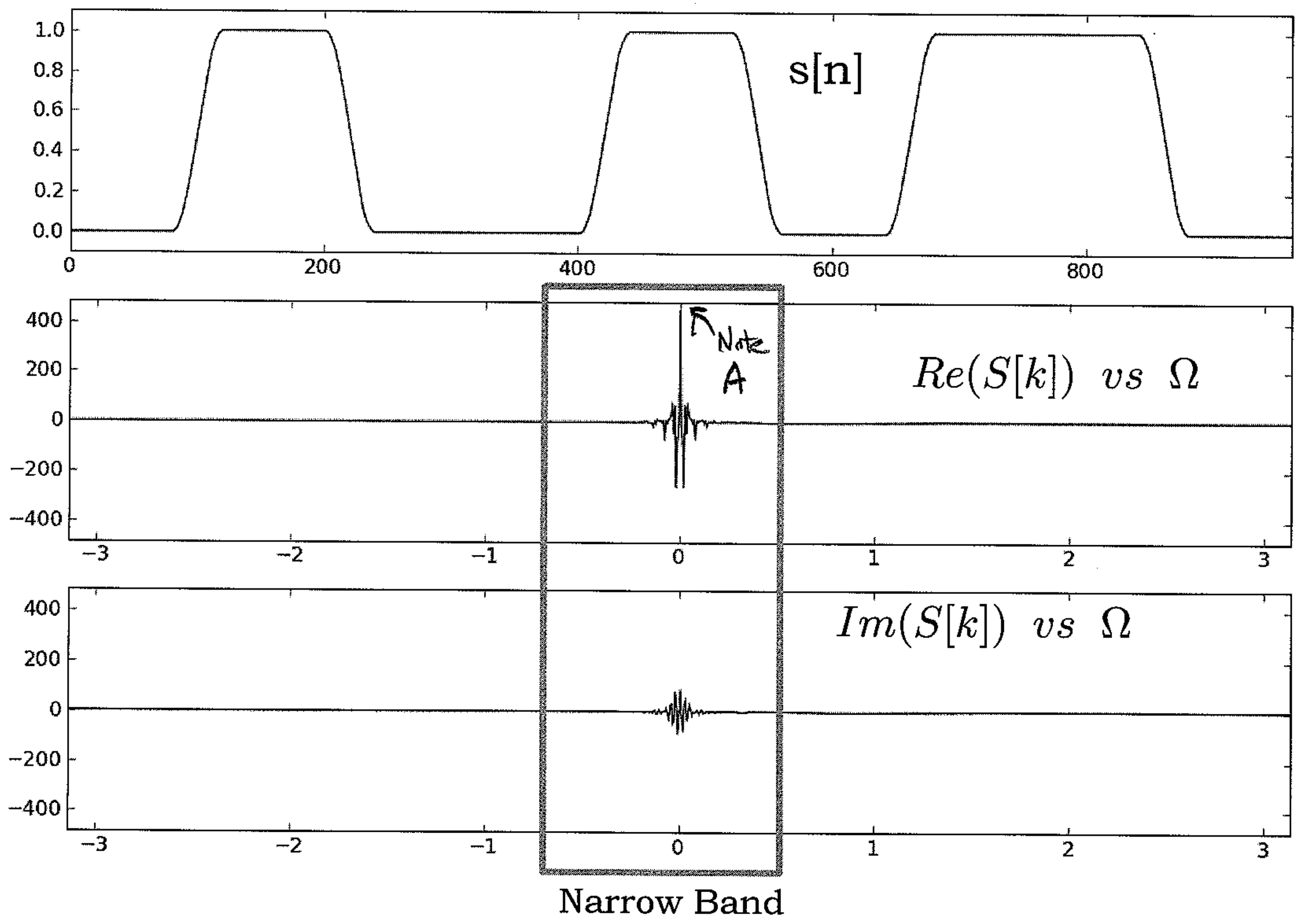


010010 Bit sequence with Slow Rise

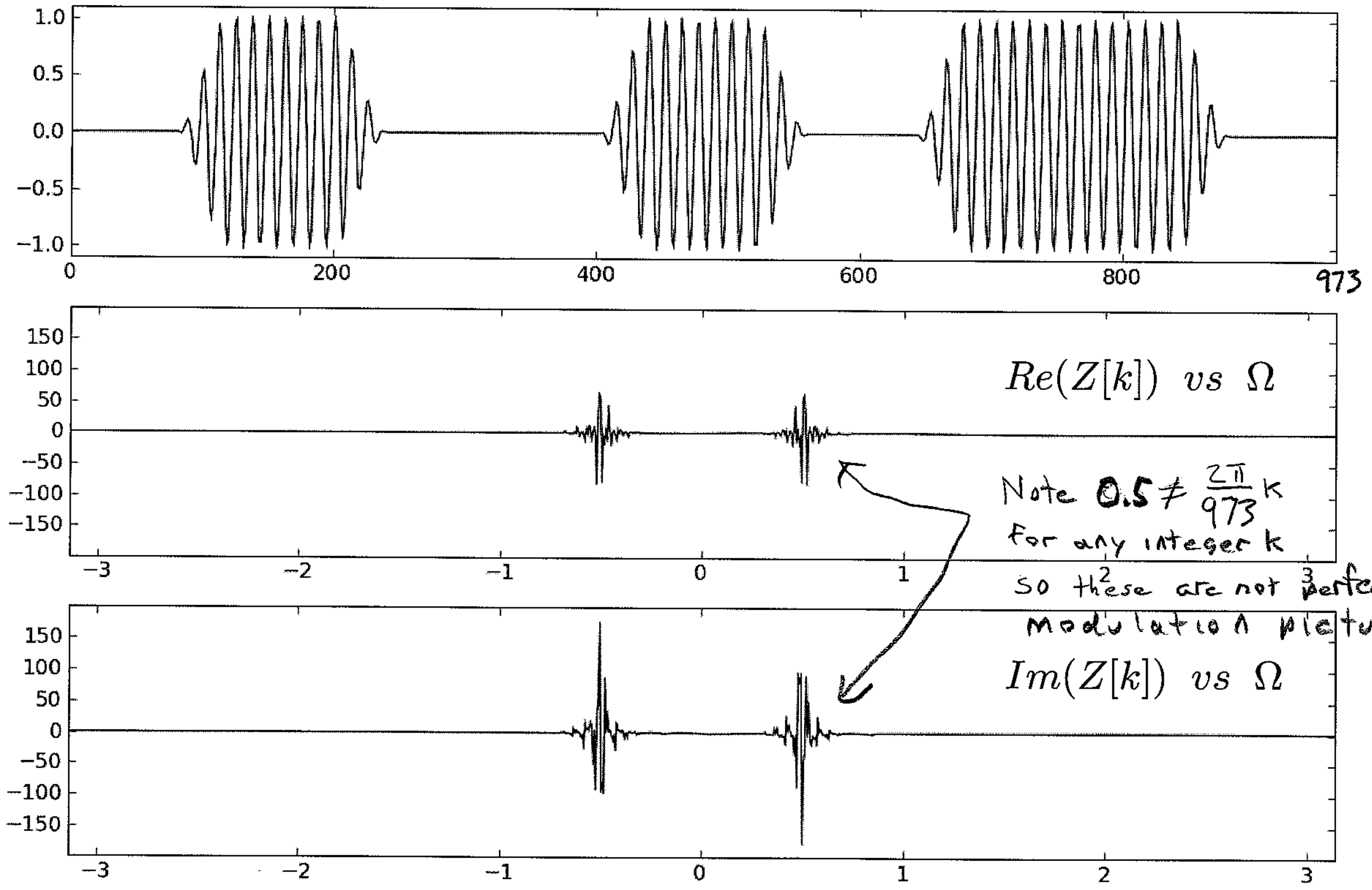


12

010010110 Slow Bit sequence (No Stems)



$z[n] = s[n] \cos 0.5n$



$Re(Z[k])$ vs Ω

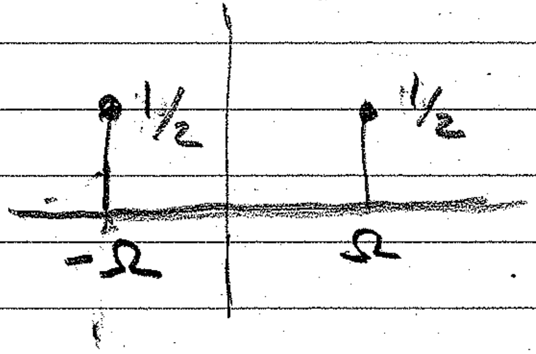
Note $0.5 \neq \frac{2\pi}{973}k$ for any integer k

so these are not perfect modulation pictures!

$Im(Z[k])$ vs Ω

$$s[n] = \underbrace{\frac{1}{N}\sum_{k=-K}^{K} \underline{S}[k]\, e^{j\frac{2\pi}{N}kn}}_{\underline{S}}$$

$$\cos\Omega n = \tfrac{1}{2}e^{j\Omega n} + \tfrac{1}{2}e^{-j\Omega n}$$



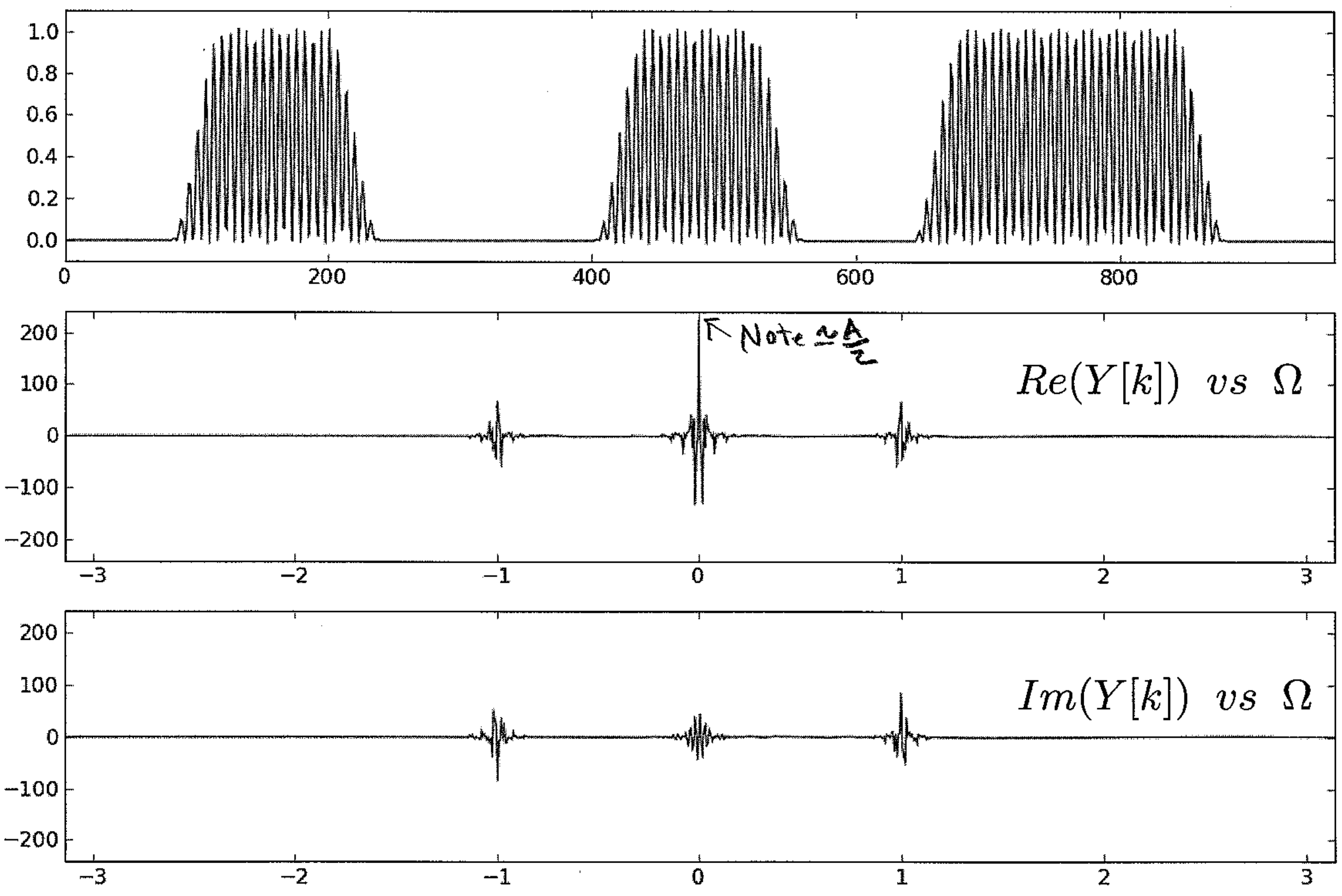
$$s[n]\cos(\Omega n)$$

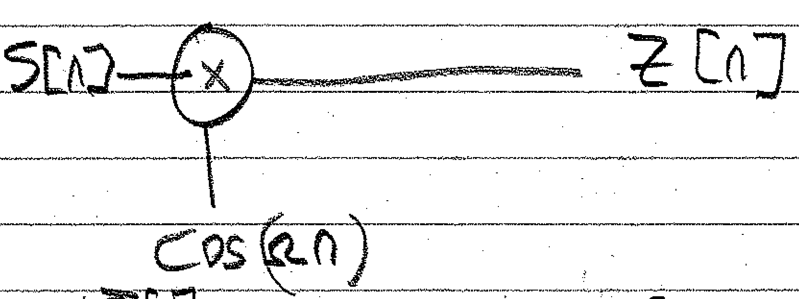
$\Omega = \frac{2\pi}{N}k$ for some k

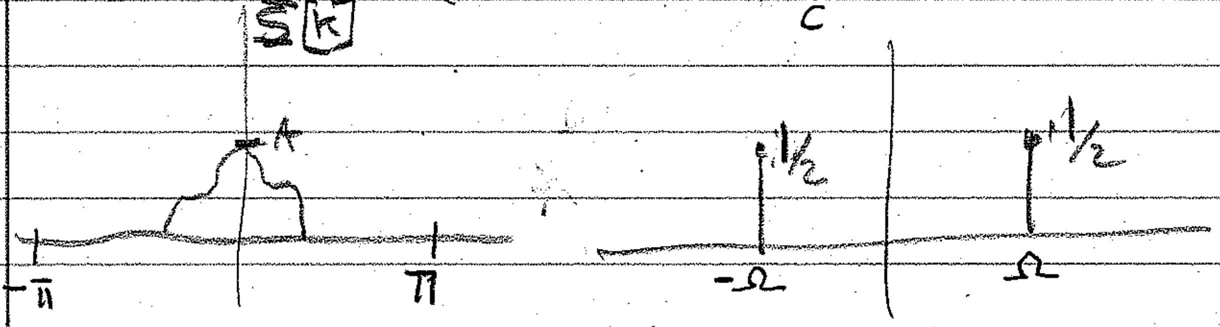
$$\cos(\Omega n) = \left(\tfrac{1}{2}e^{j\Omega n} + \tfrac{1}{2}e^{-j\Omega n}\right)$$

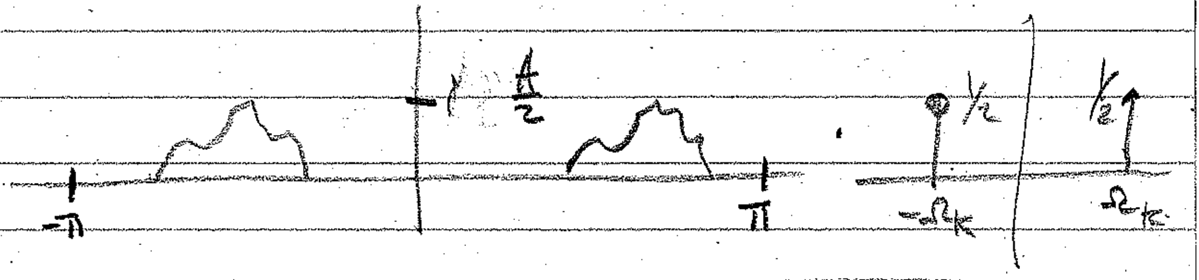
$$\left(\frac{1}{N}\sum_{k=-K}^{K} \underline{S}[k]\, e^{j\frac{2\pi}{N}kn}\right)\left(\tfrac{1}{2}e^{j\Omega n} + \tfrac{1}{2}e^{-j\Omega n}\right)$$

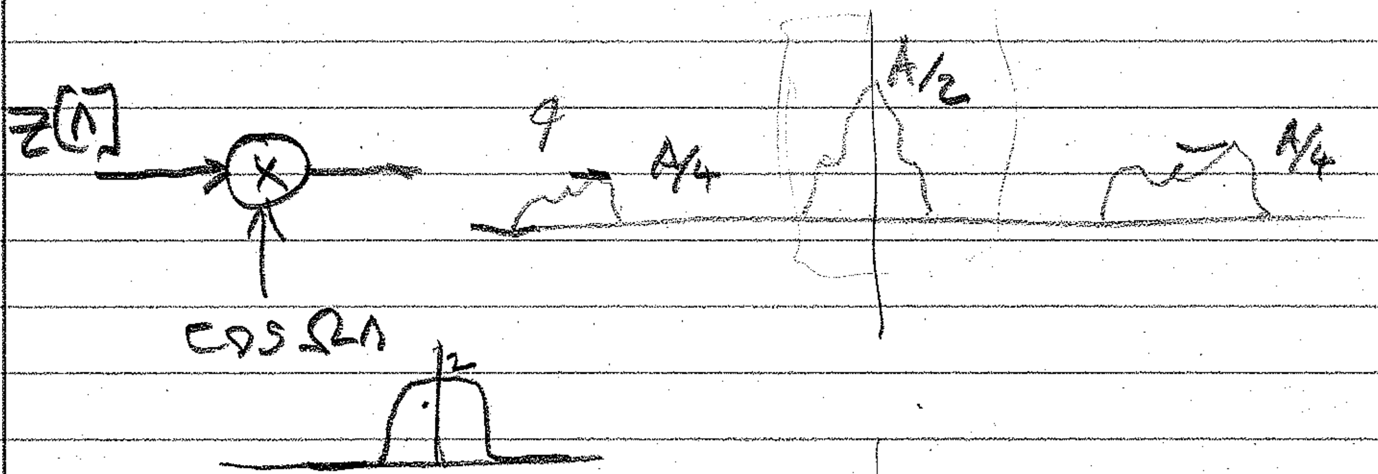
$y[n] = (s[n] \cos 0.5n) \cos 0.5n$

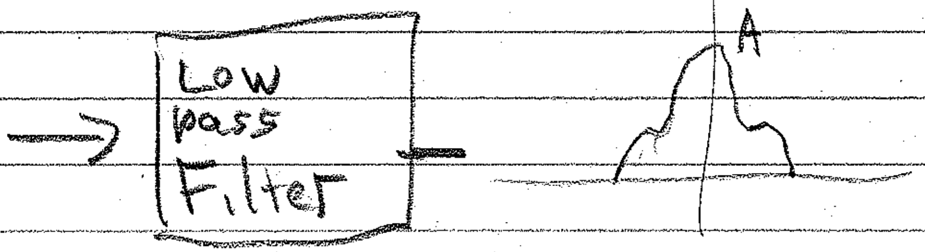


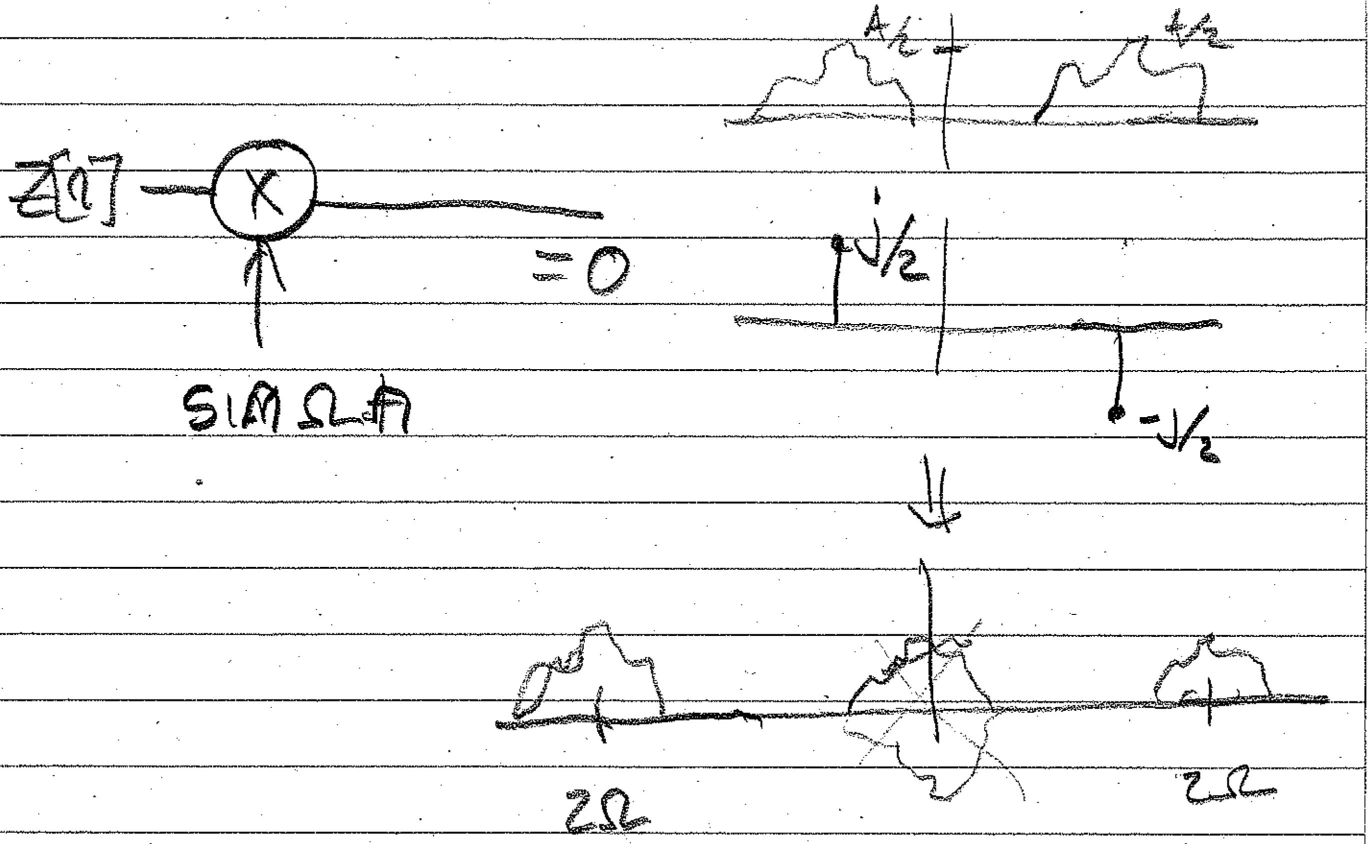

S[n]
x
Z[n]
cos(Ωn)


S[k]
A
-π
π
C
1/2
1/2
-Ω
Ω


A/2
-π
π
1/2
-Ωk
1/2
Ωk


Z[n]
x
cos Ωn
A/4
A/2
A/4
2


Low pass Filter
A



Problem Reciever does not Know the phase

Addendum to Lec 15

①

The Algebra of Modulation
N assumed odd

Suppose $s[n] = \sum_{k=L}^{L} \bar{S}[k] e^{j\frac{2\pi}{N}kn}$ $\quad L << \bar{K}$

$\bar{K} = \frac{N-1}{2}$

(L for Band - Limited)

Modulate by $\cos(\underbrace{\boxed{\frac{2\pi}{N}M}}_{\Omega_M} n)$

Assume $M + L < \bar{K}$

$$\underbrace{w[n]}_{\uparrow} = s[n] \cos\left(\frac{2\pi}{N} M n\right)$$

Modulated Signal

$$\underbrace{\sum_{k=-\bar{K}}^{\bar{K}} \bar{W}[k] e^{-j\frac{2\pi}{N}kn}}_{w[n]} = \overbrace{\left(\sum_{k=L}^{L} \bar{S}[k] e^{j\frac{2\pi}{N}kn}\right)}^{s[n]} \overbrace{\left(\frac{1}{2}e^{j\frac{2\pi}{N}Mn} + \frac{1}{2}e^{-j\frac{2\pi}{N}Mn}\right)}^{\cos(\frac{2\pi}{N}Mn)}$$

Determining $\bar{W}[k]$

$$\sum_{k=-\bar{K}}^{\bar{K}} W[k] e^{-j\frac{2\pi}{N}kn} =$$

$$\sum_{k=-L}^{L} \frac{1}{2}\bar{S}[k] e^{j\frac{2\pi}{N}(k+M)n} + \sum_{k=-L}^{L} \frac{1}{2}\bar{S}[k] e^{j\frac{2\pi}{N}(k-M)n}$$

$$\sum_{k=-B}^{B} W[k]\, e^{-j\frac{2\pi}{N}kn} =$$

$$\sum_{k=-L+M}^{L+M} \frac{1}{2} S[k-M]\, e^{-j\frac{2\pi}{N}kn} +$$

$$\sum_{k=-L-M}^{L-M} \frac{1}{2} S[k+M]\, e^{-j\frac{2\pi}{N}kn}$$

Now Assume $M > L$

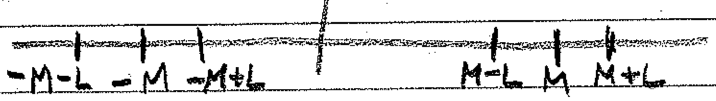
$\underline{M-L>0}$ $\underline{L-M<0}$

$$W[k] = S[k-M] \qquad k>0$$

$$W[k] = S[k+M] \qquad k<0$$

$W[k]$ is nonzero if

$M-L \leq k \leq M+L$ or $-L-M<k \leq L-M$



Computing $S[k]$ from $s[n] = e^{j\frac{2\pi}{N}\ell n}$

$$S[k] = \sum_{n=0}^{N-1} s[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}\ell n}\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(\ell-k)n}$$

If $\ell - k \neq 0$ $\quad \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(\ell-k)n} = 0$

(Summing exactly ℓ-k periods of a sine and a cosine $= 0$)

If $\ell - k = 0$ $\quad \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(\ell-k)n} = \sum_{n=0}^{N-1} 1 = N$

$\Rightarrow$ $S[k] = N \quad k = \ell$

$S[k] = 0 \quad$ otherwise

①

Post Lec15 Recitation Notes: Reminder about DFT (Watch for N)

$$s[n] = \frac{1}{N}\sum_{k=-K}^{K} S[k]e^{j\overset{\Omega_k}{\boxed{\frac{2\pi}{N}}}kn} \quad N = 2K+1$$

$$S[k] = \sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi}{N}kn}$$

Use this formula and complex exponential form of s[n] to derive S[k] from s[n], shows how N arises.

②

In the following modulation example, a signal with a simple DFT is modulated with a cosine at 10 times the lowest frequency and then demodulated, once with a cosine, then with a sine. Then the signal is modulated with a cosine shifted by pi/4 and then demodulated.

$$N = 101$$

$$\Omega_1 = \frac{2 * \pi}{101}$$

$$s[n] = 2.0 + 0.5 \cos \Omega_1 n + 2.0 \sin 2\Omega_1 n$$

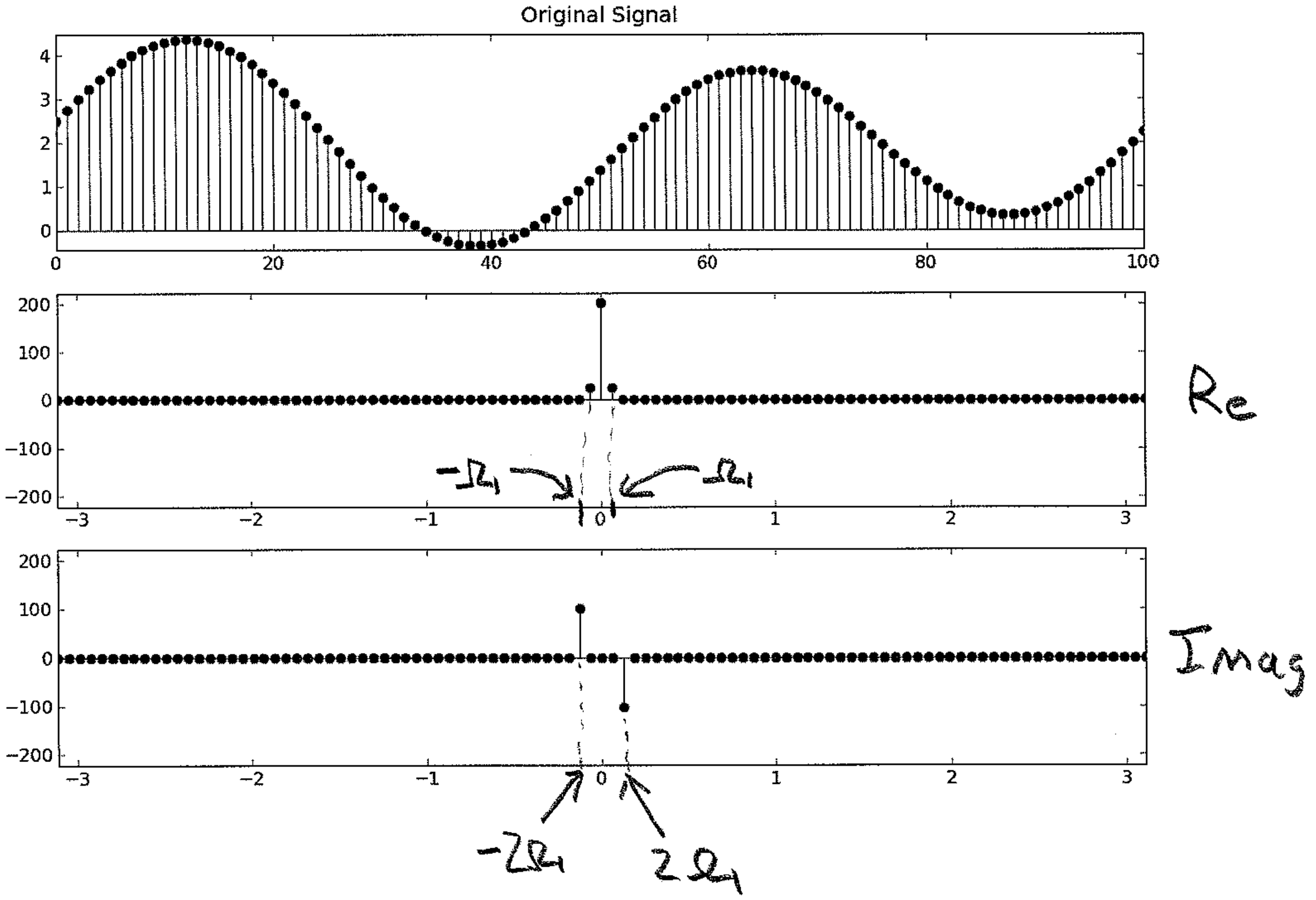
There is a relation between Fourier series index k, the associated Fourier Frequency, Omega_k, and frequency f_k in hertz associated with a sample rate F_s:

Omega = (2*pi*frequency in hertz)/sample rate
k = Omega_k*N/(2*pi)
k = (N * f_k)/F_s (If ratio is an integer)

③

Plot of s[n], Re(S[k]), Im(S[k])

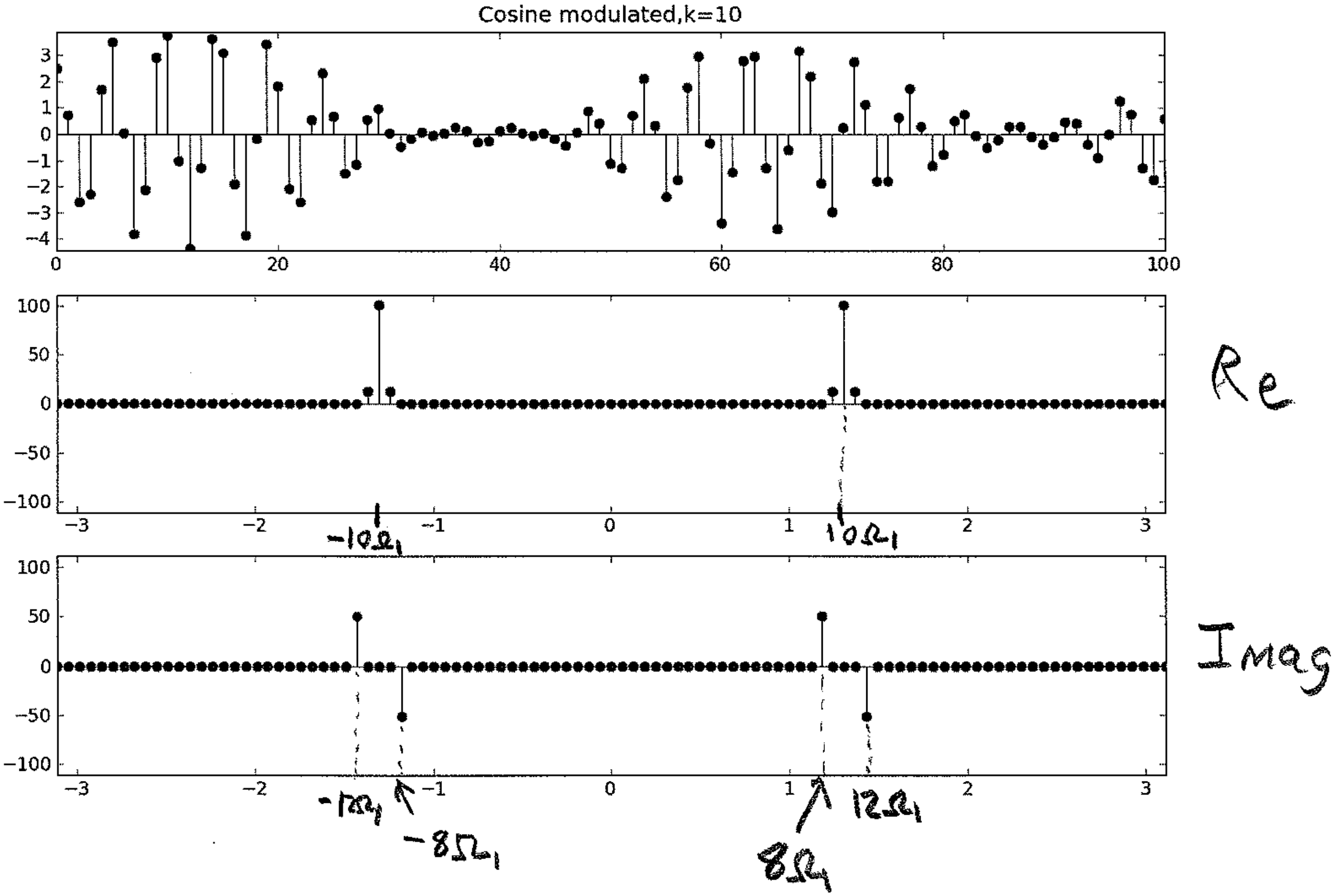
$$s[n] = 2.0 + 0.5\cos\Omega_1 n + 2.0\sin 2\Omega_1 n$$



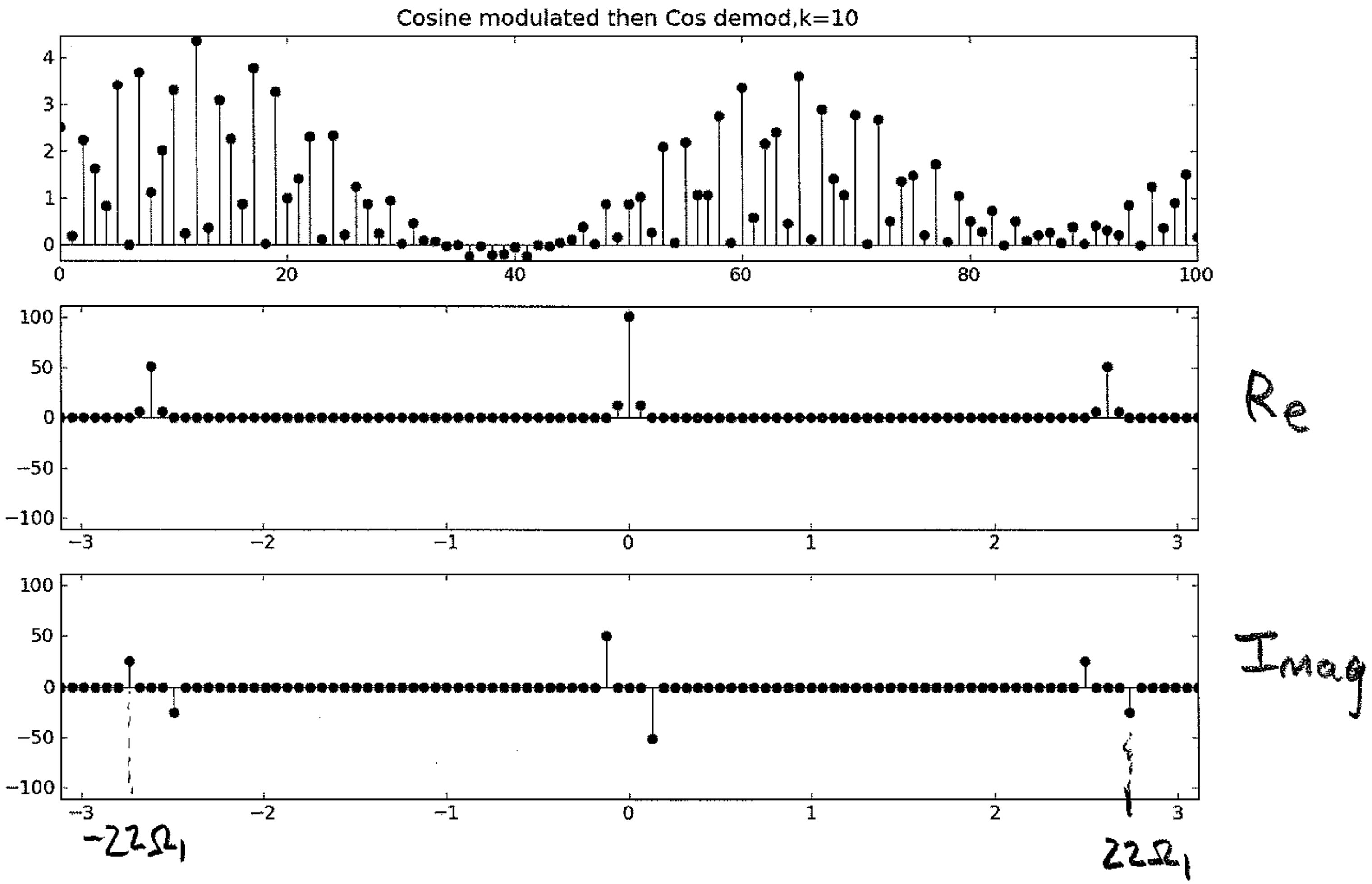
Note that
since
N =101
the peaks
have the
values
2*101,
101/4,
-101j,
101j

④

Plot of s[n]cos10wn

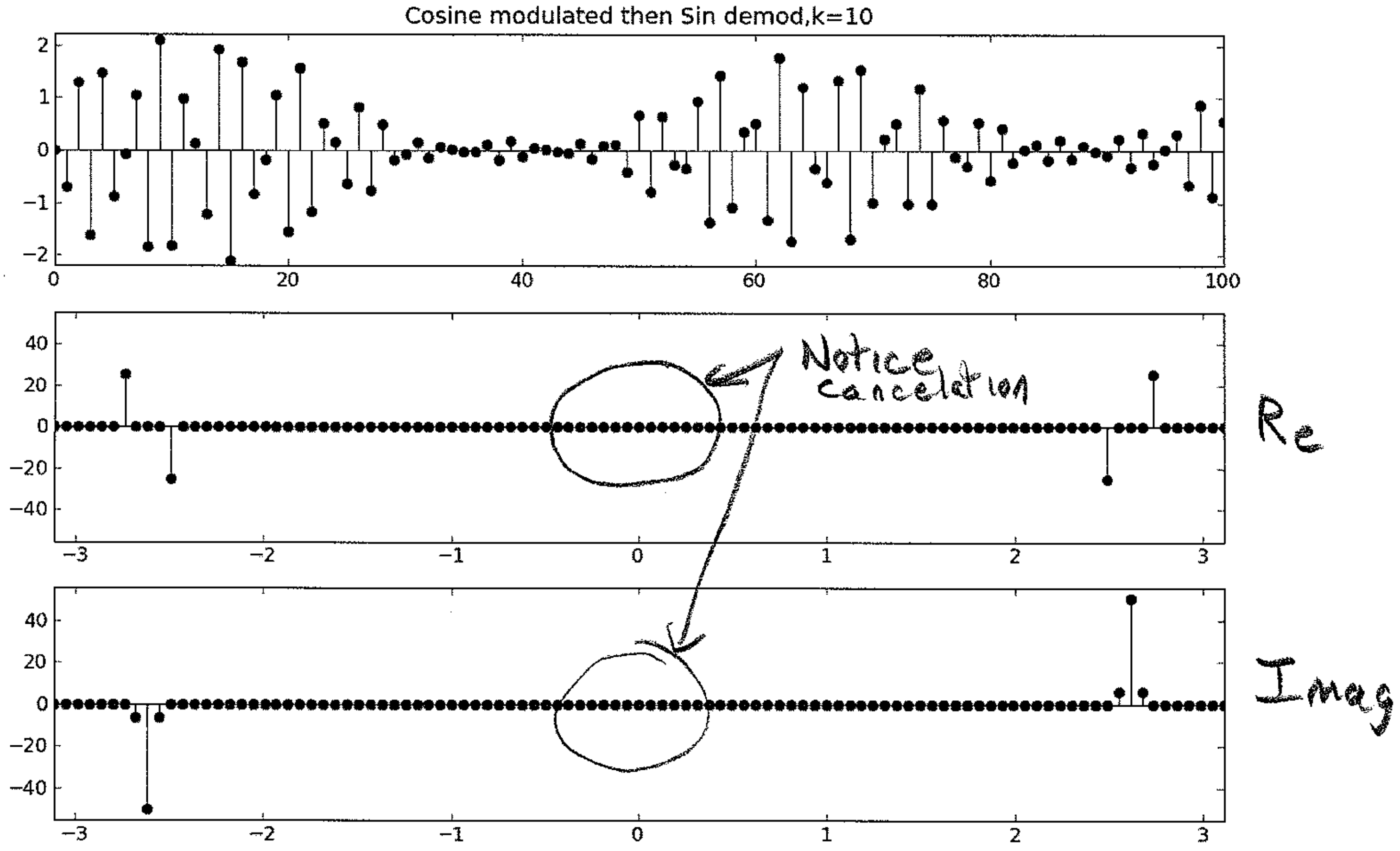


Plot of (s[n]cos10wn)cos10wn



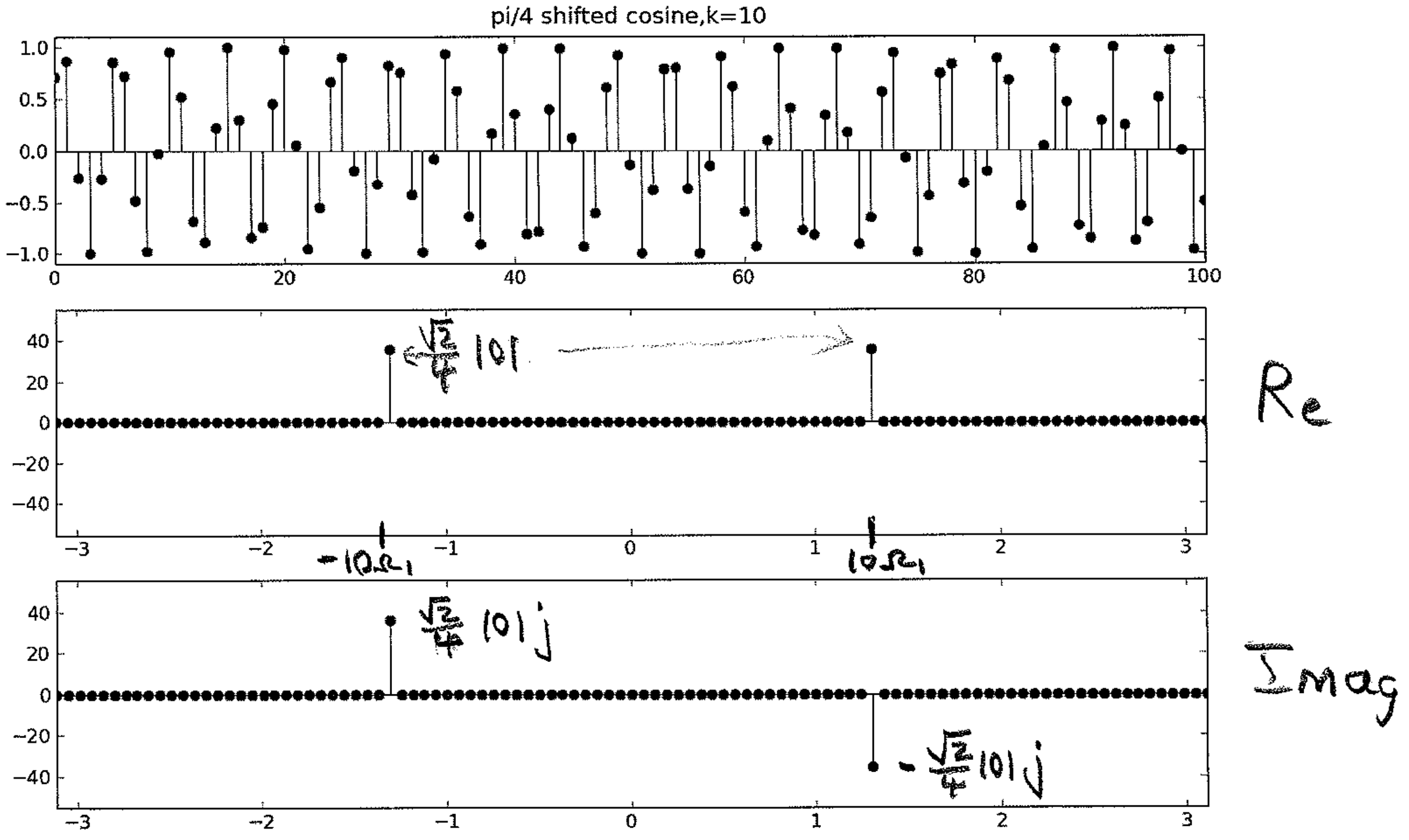
⑥

Plot of (s[n]cos10wn)sin10wn



⑦

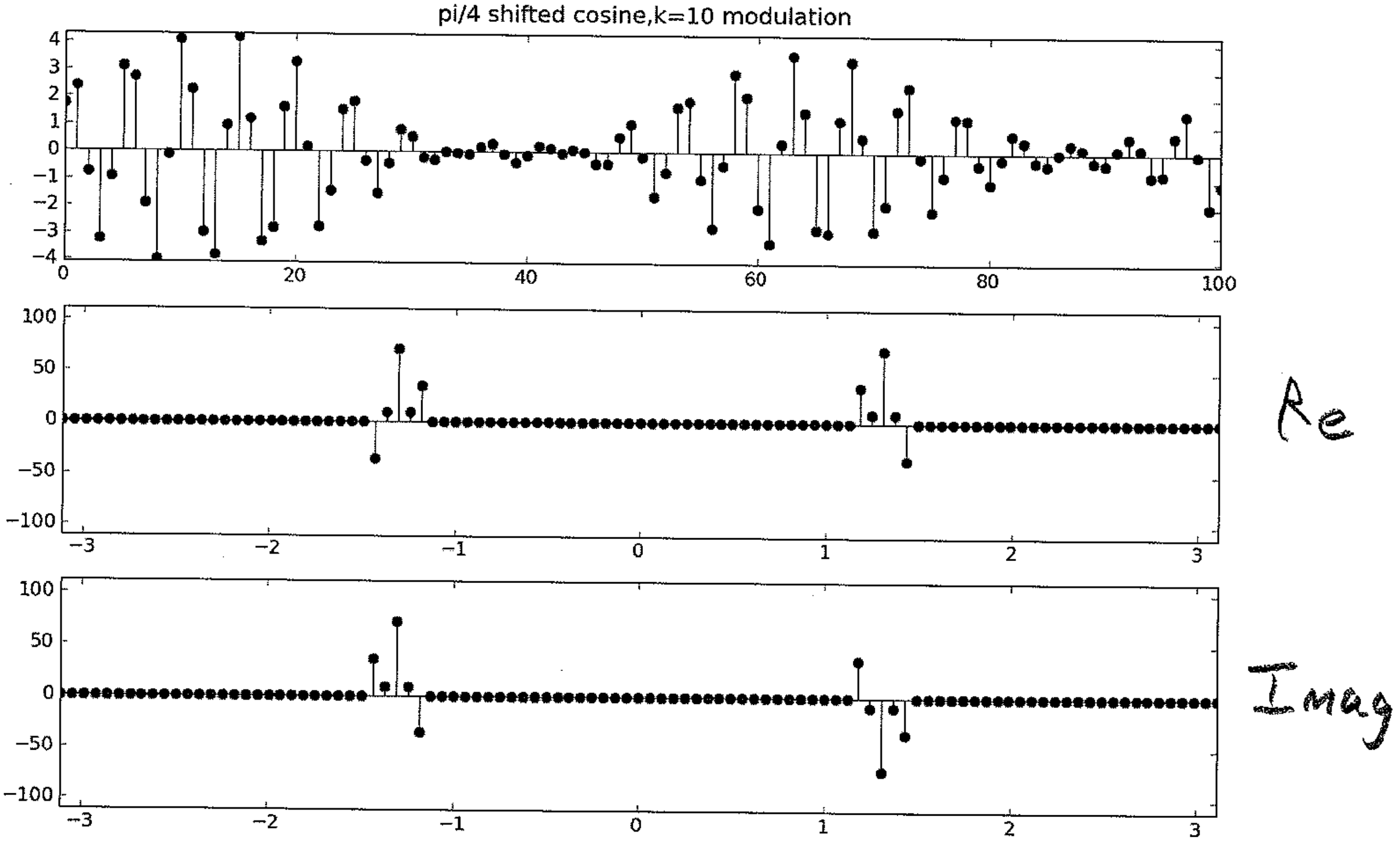
Plot of cos(10wn + pi/4)



Note the sqrt(2)* 101/4 height of peaks (due to cos(pi/ 4))

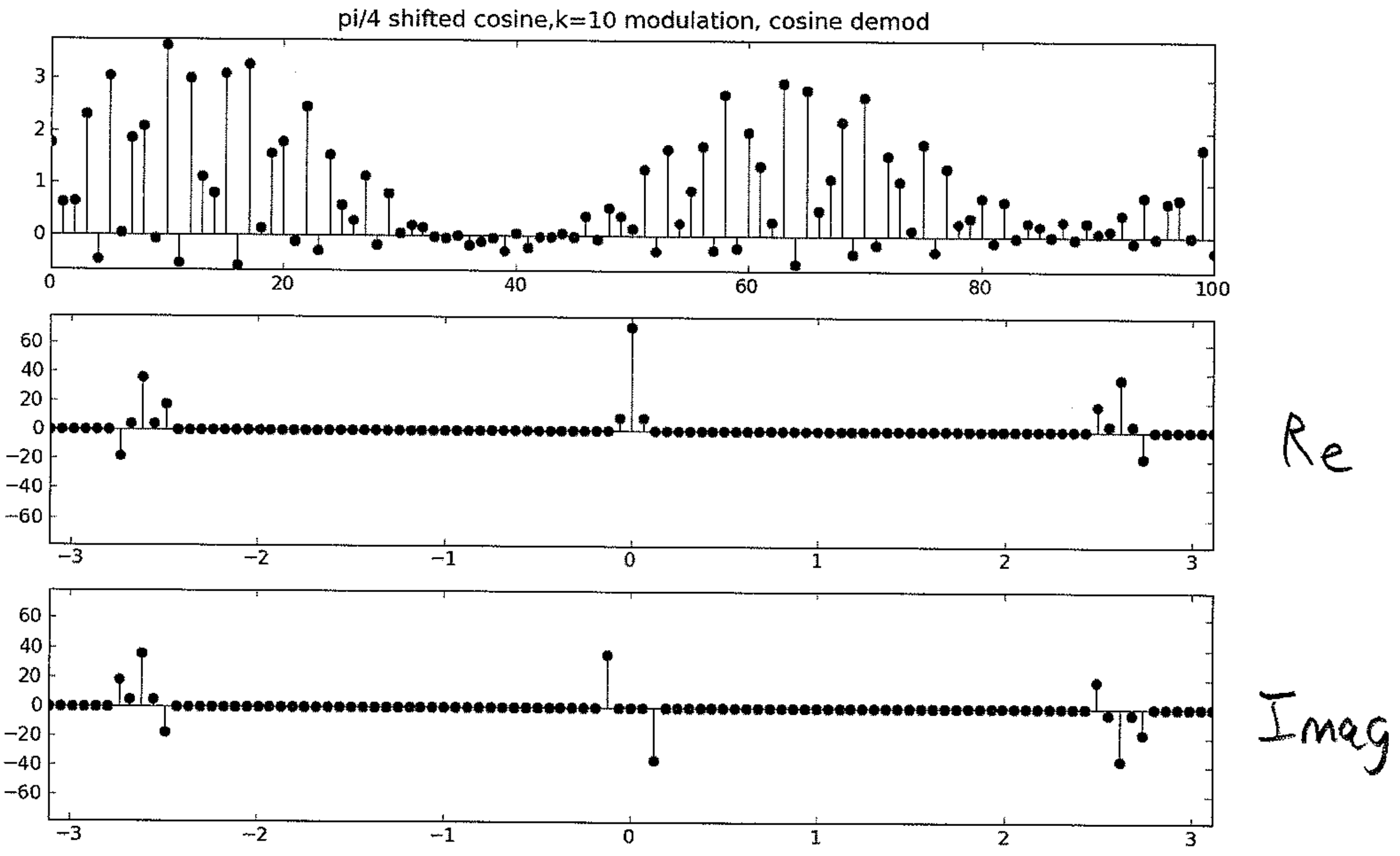
⑧

Plot of s[n]cos(10wn+pi/4)



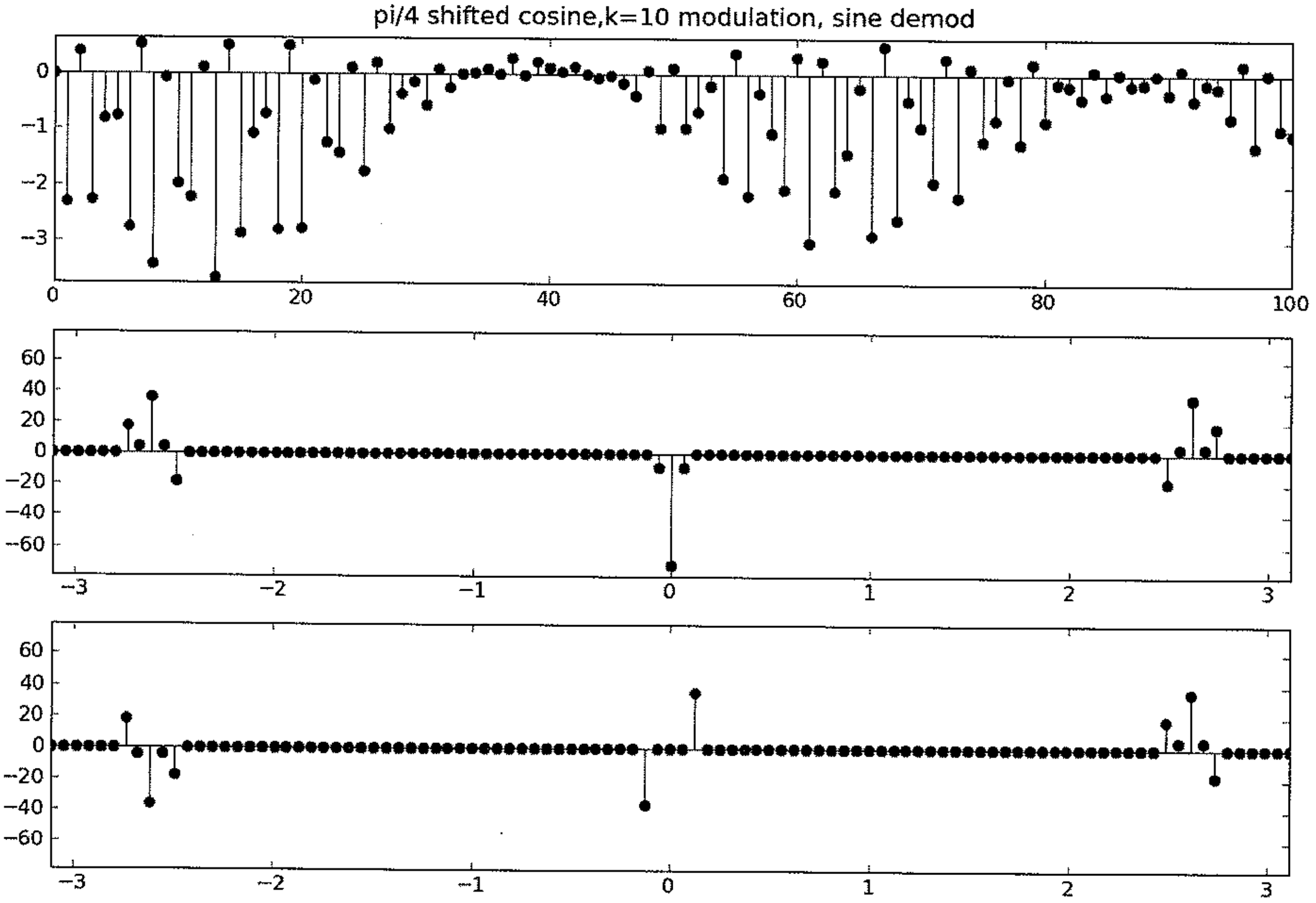
⑨

Plot of (s[n]cos(10wn+pi/4))cos(10wn)



⑩

Plot of (s[n]cos(10wn+pi/4))sin(10wn)



pi/4 shifted cosine,k=10 modulation, sine demod