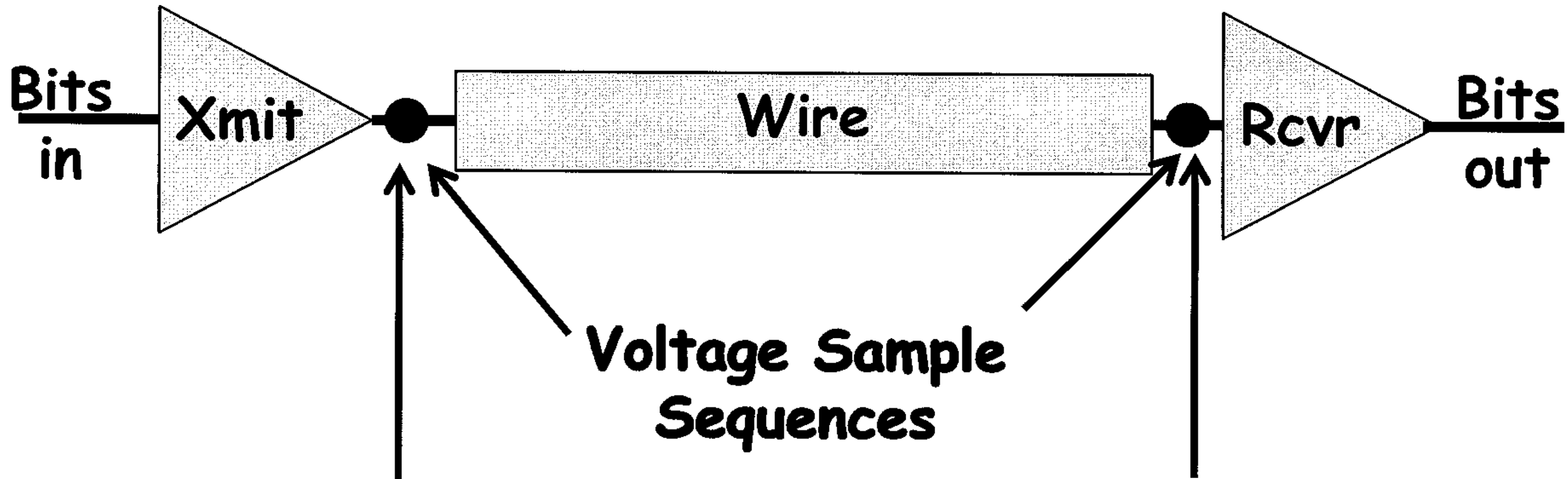
6.02 Lecture 3 - Unraveling Wires

(AfterNotes and Examples)

- Quick Reminder about Problem
  - Slow Wires and Eye diagrams
  - Wires model Casual and LTI
  - Improvements by processing
- Unit Sample Response
  - Convolution Sum
  - Connection to Difference Equations
  - Deconvolution
- Flip and slide convolution
  - Lends insight in to system response

2

Transmission Setup and Notation



$X \equiv$ entire sequence

$x[n] \equiv n^{th}$ sample value

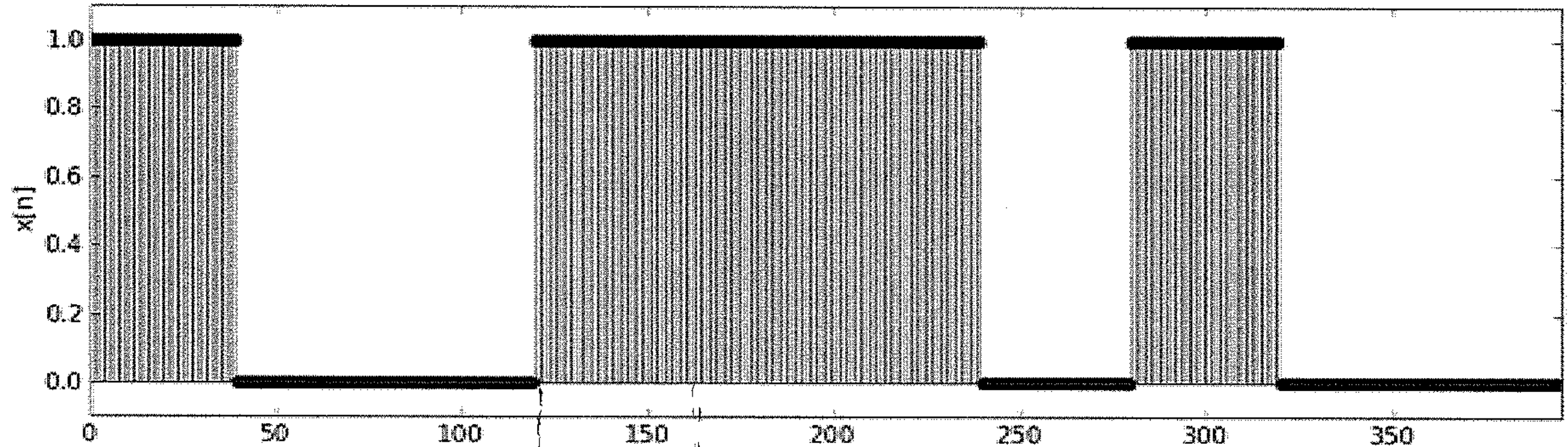
$Y \equiv$ entire sequence

$y[n] \equiv n^{th}$ sample value

Sample Rates:

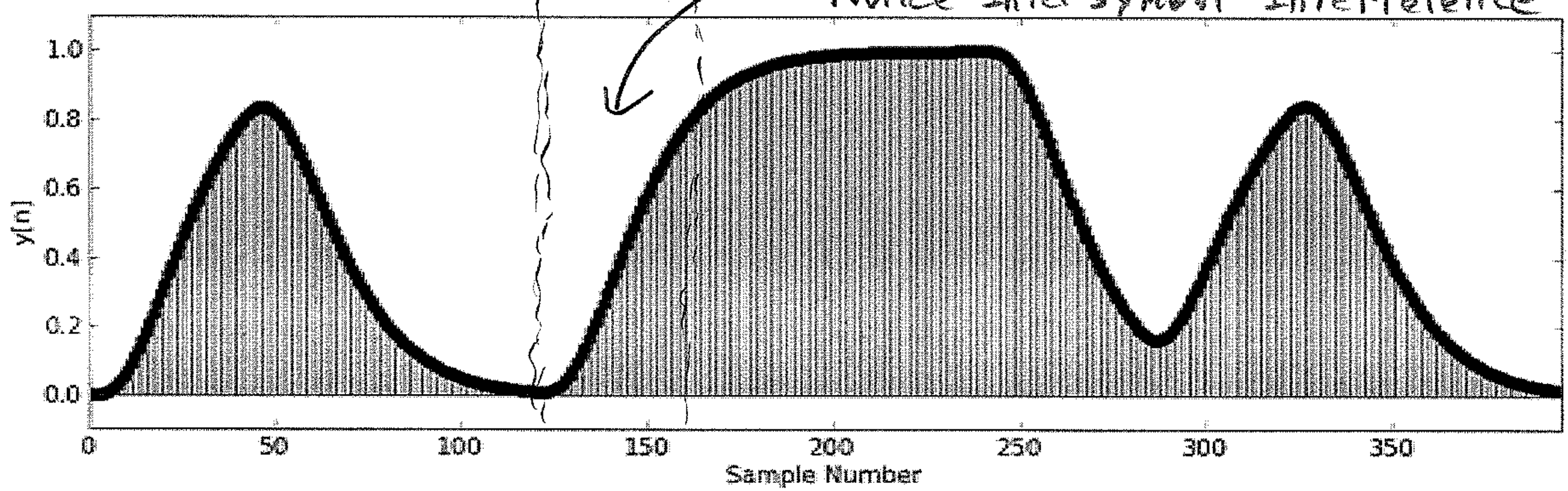
- **256,000 Samples/Second (lab)**
- **Up to gigaSamples/Second (Real World)**

Slow Wire and 40 Samples per bit



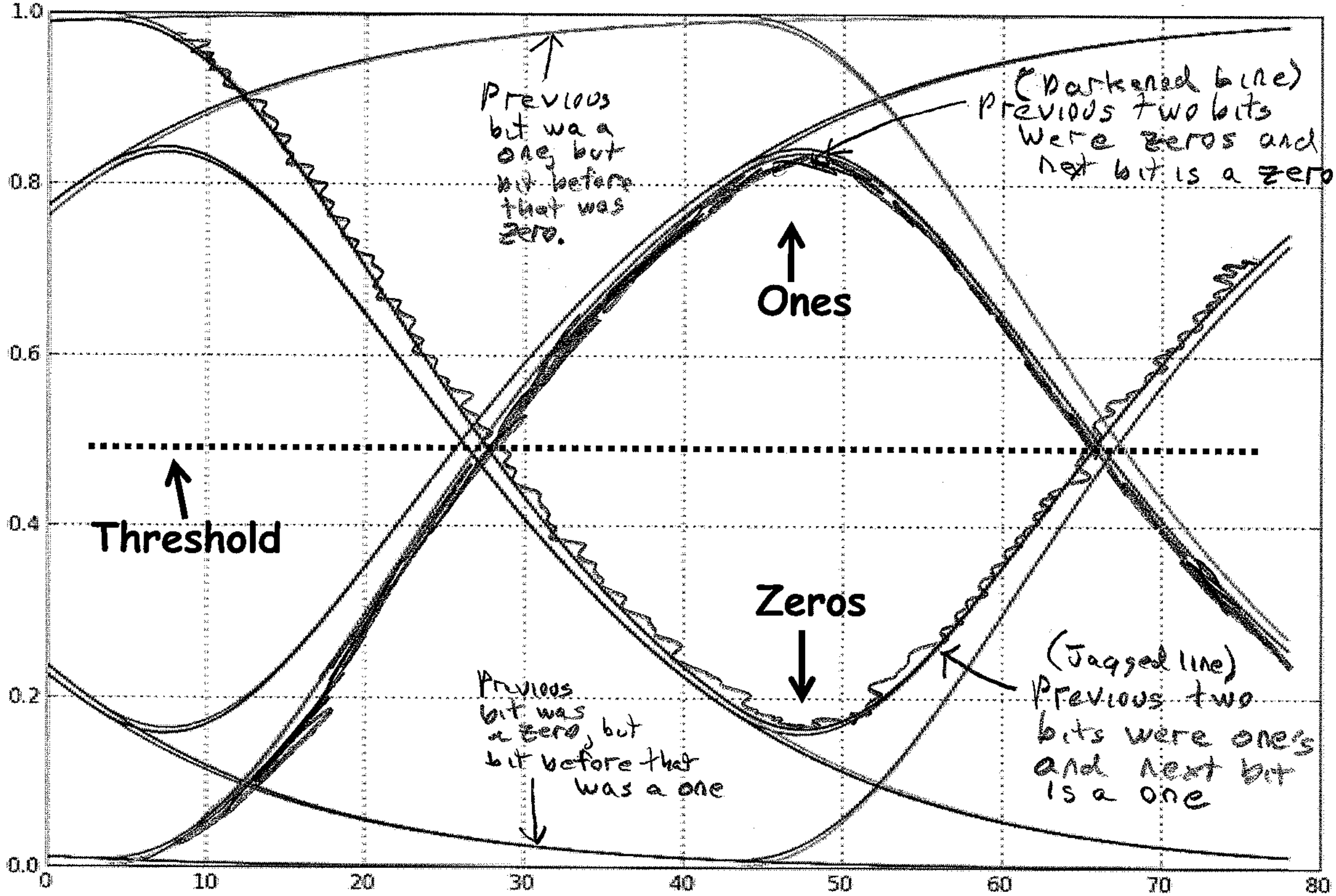
1 0 0 1 1 1 0 1 0 0

Notice Intersymbol Interference

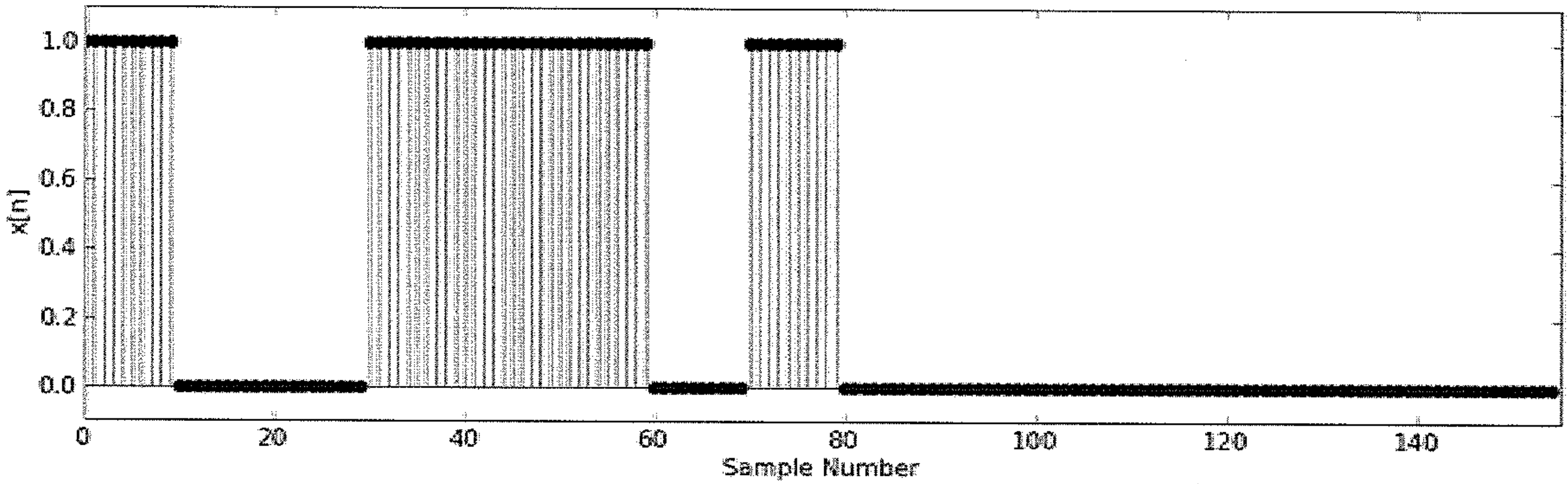


40 Samples per bit Eye diagram

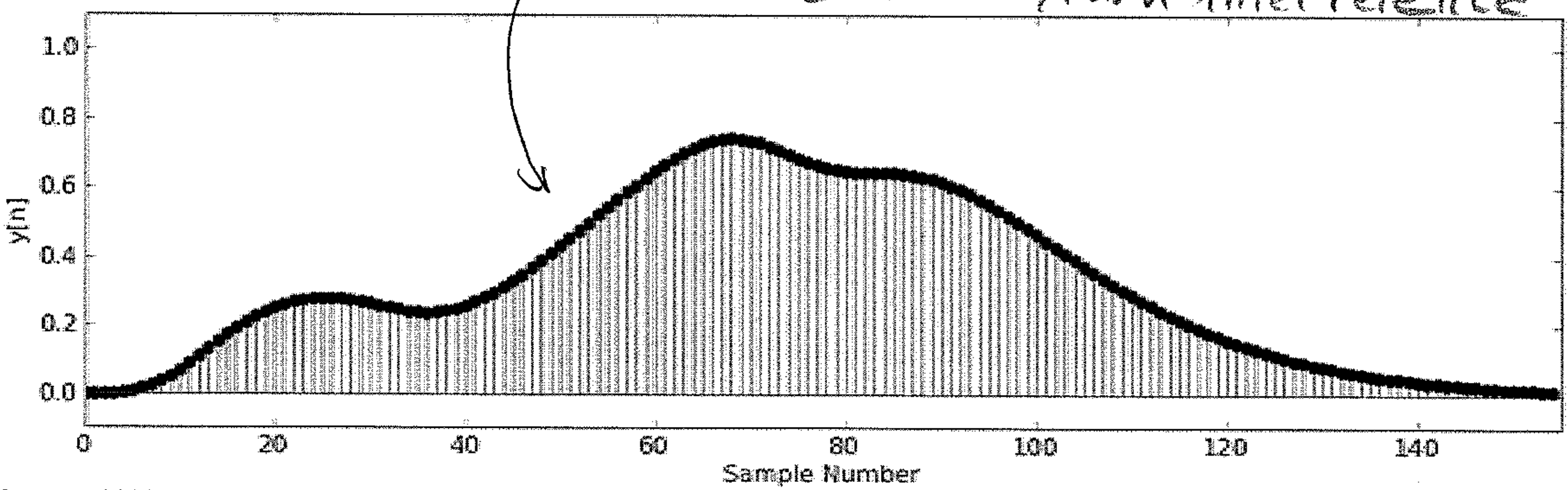
(4)



Slow Wire and 10 Samples per bit

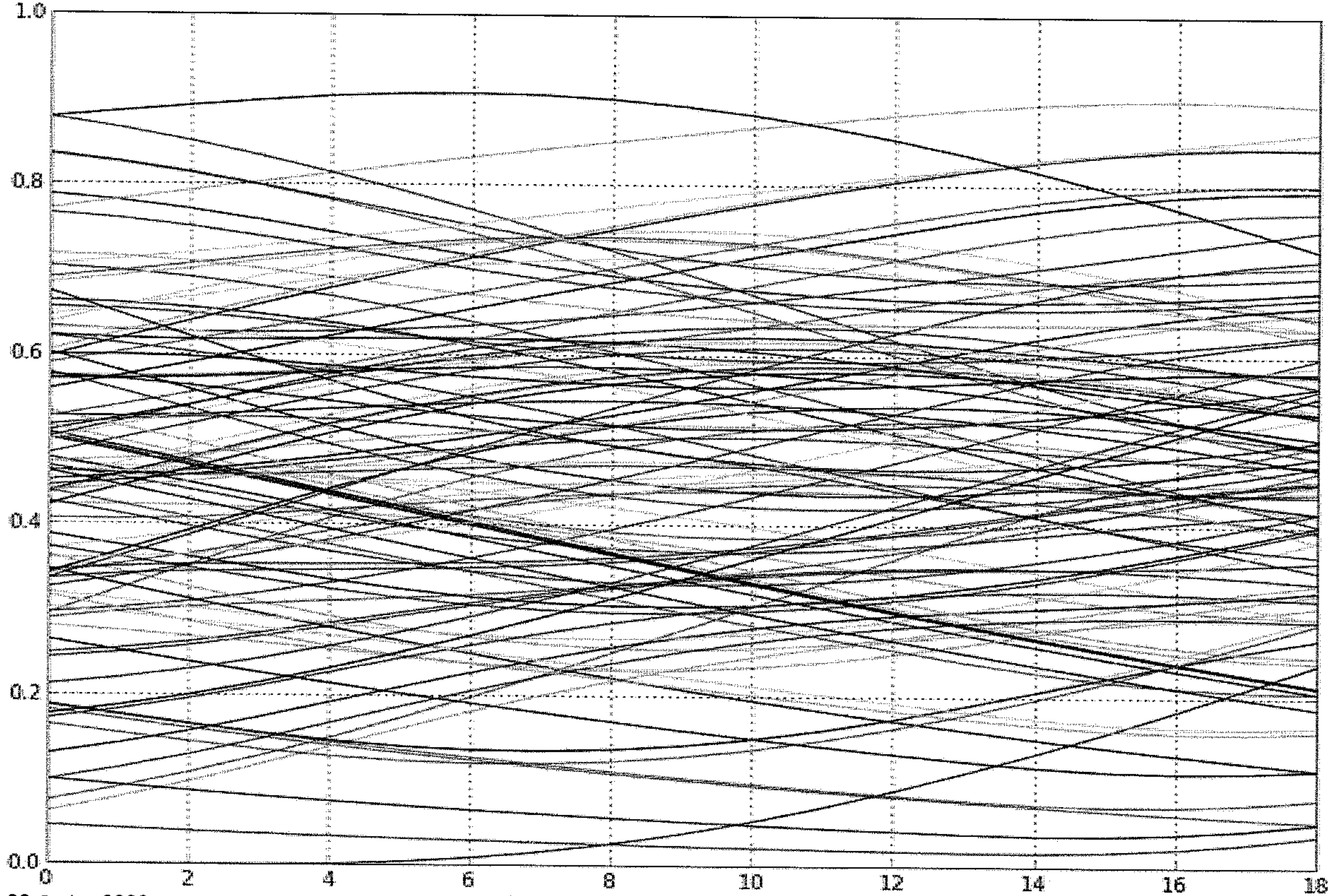


Much worse intersymbol interference



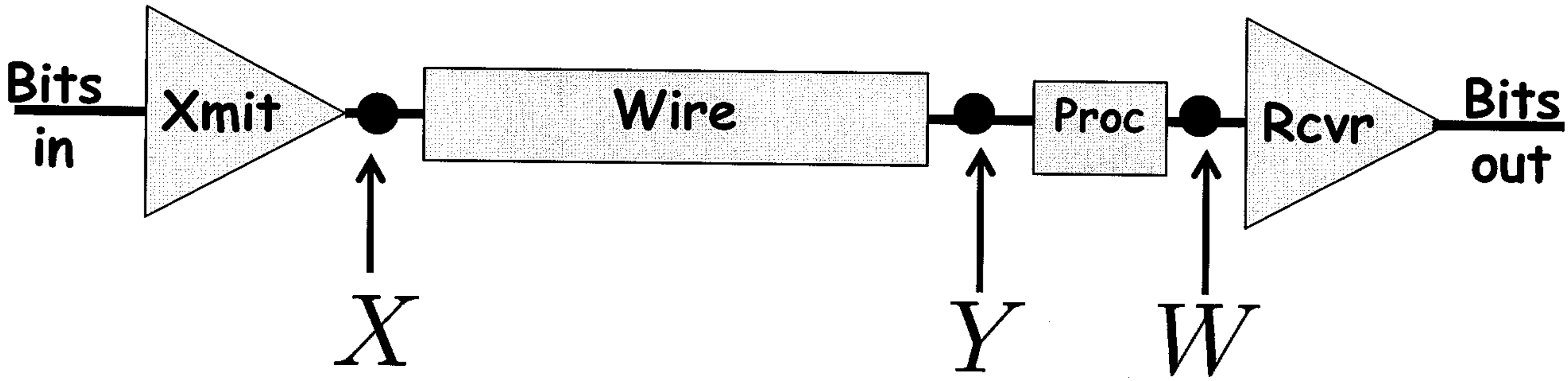
6

10 Samples per bit Eye diagram



No "open" eye

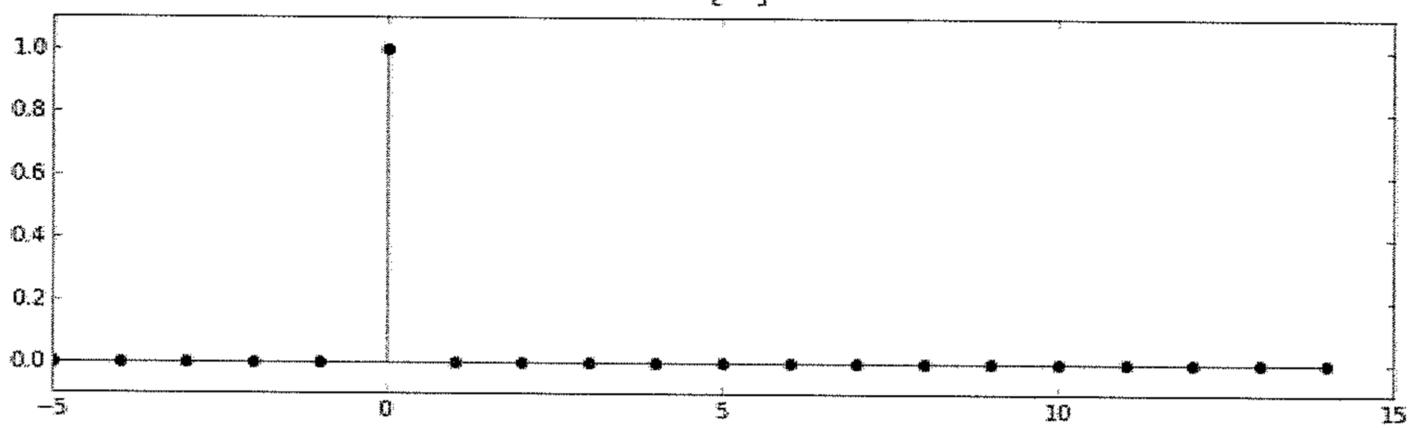
Can Signal Processing Help?



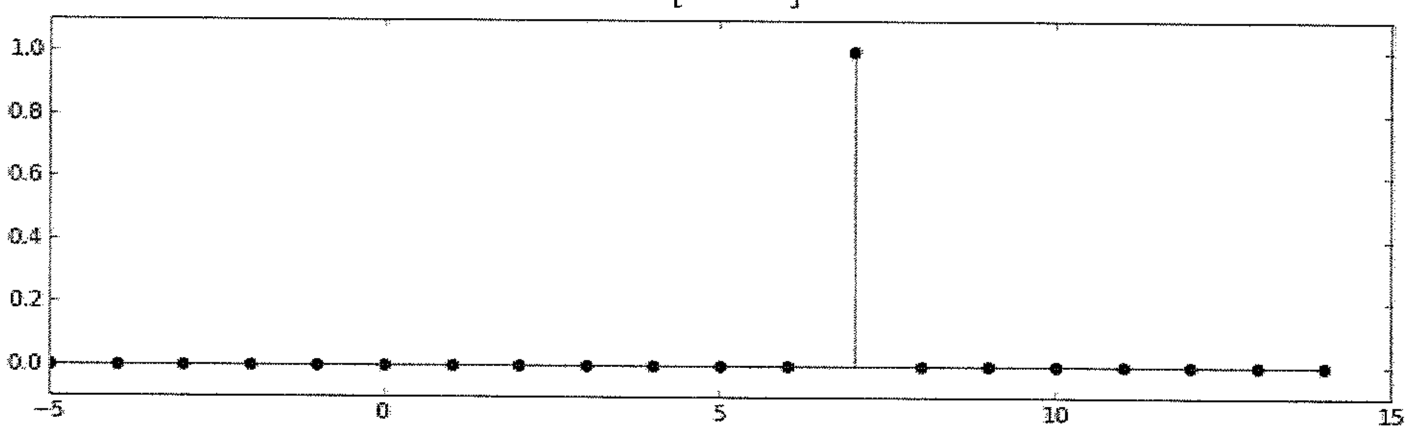
Model Wire as Causal and Linear Time-Invariant

Unit Sample Reminder

$\delta[n]$

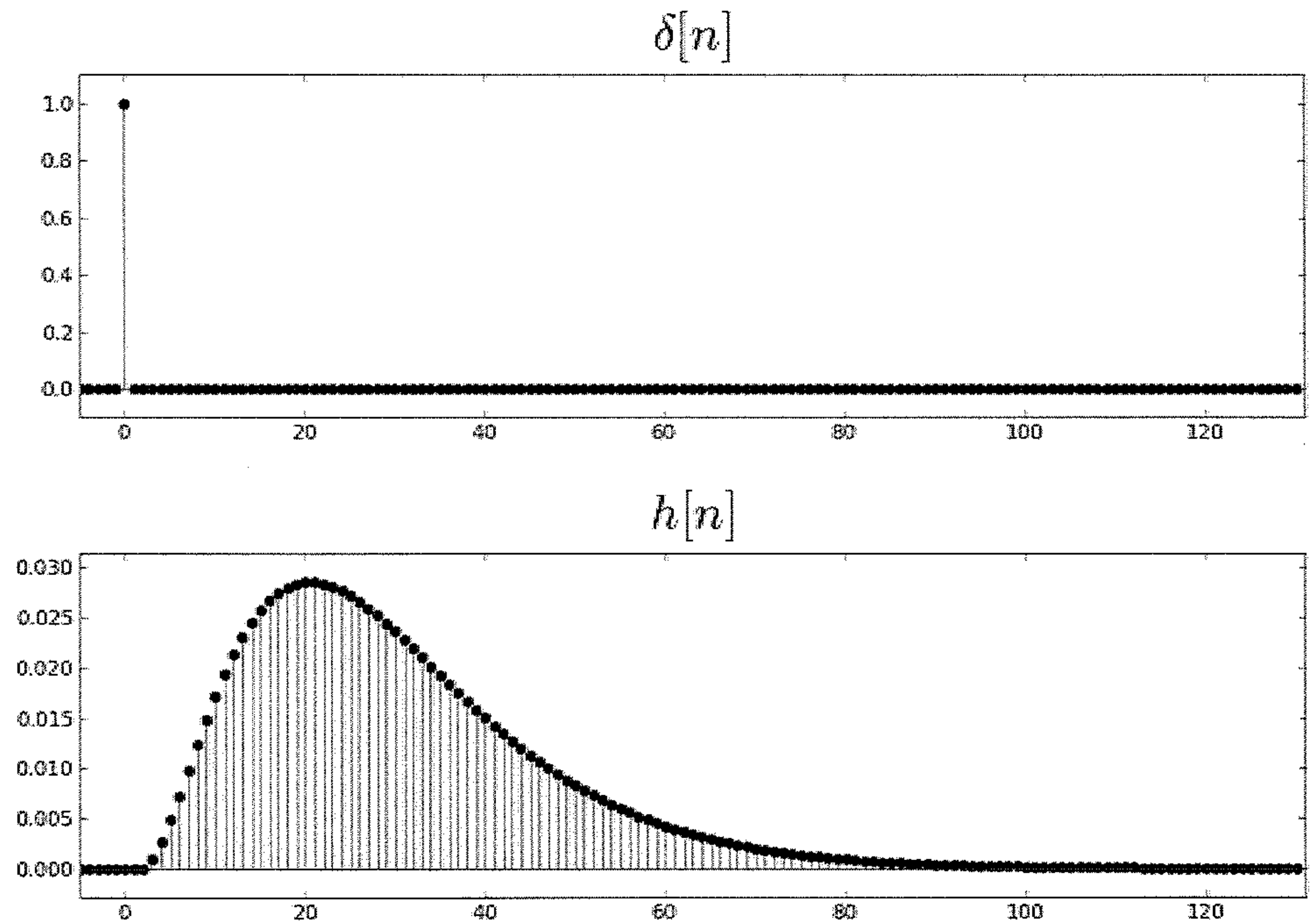


$\delta[n-7]$



Unit Sample Response (slow wire)

9

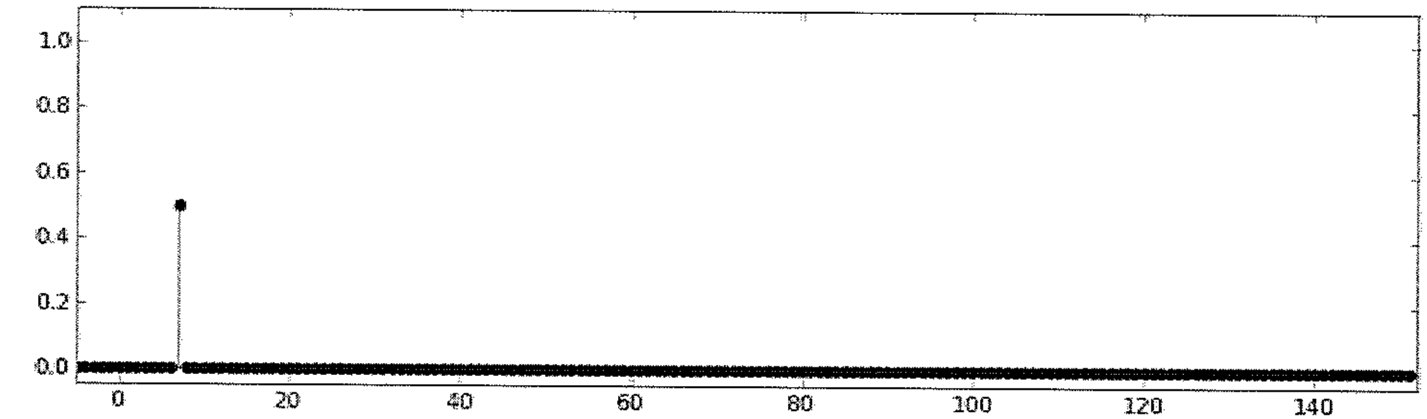


Notice h[n] is nonzero for ~ 80 samples (ISI for 40 sample/bit was two bits)

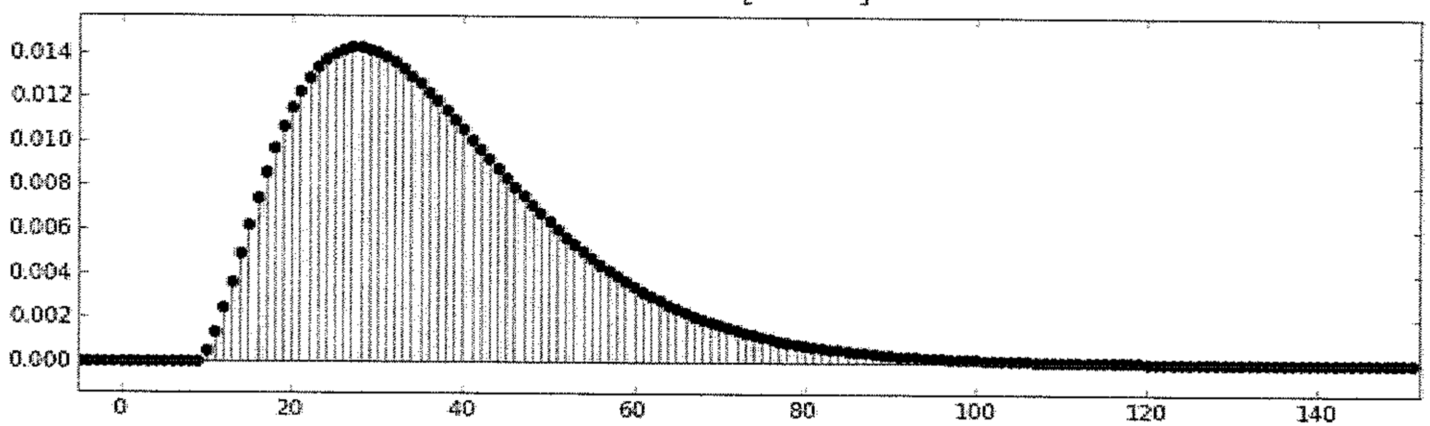
(That is, when h[n] is nonzero for 2 bit periods, then two previous bits "interfere" with current bit)

Shifted and Scaled Unit Sample Response

$0.5\delta[n-7]$

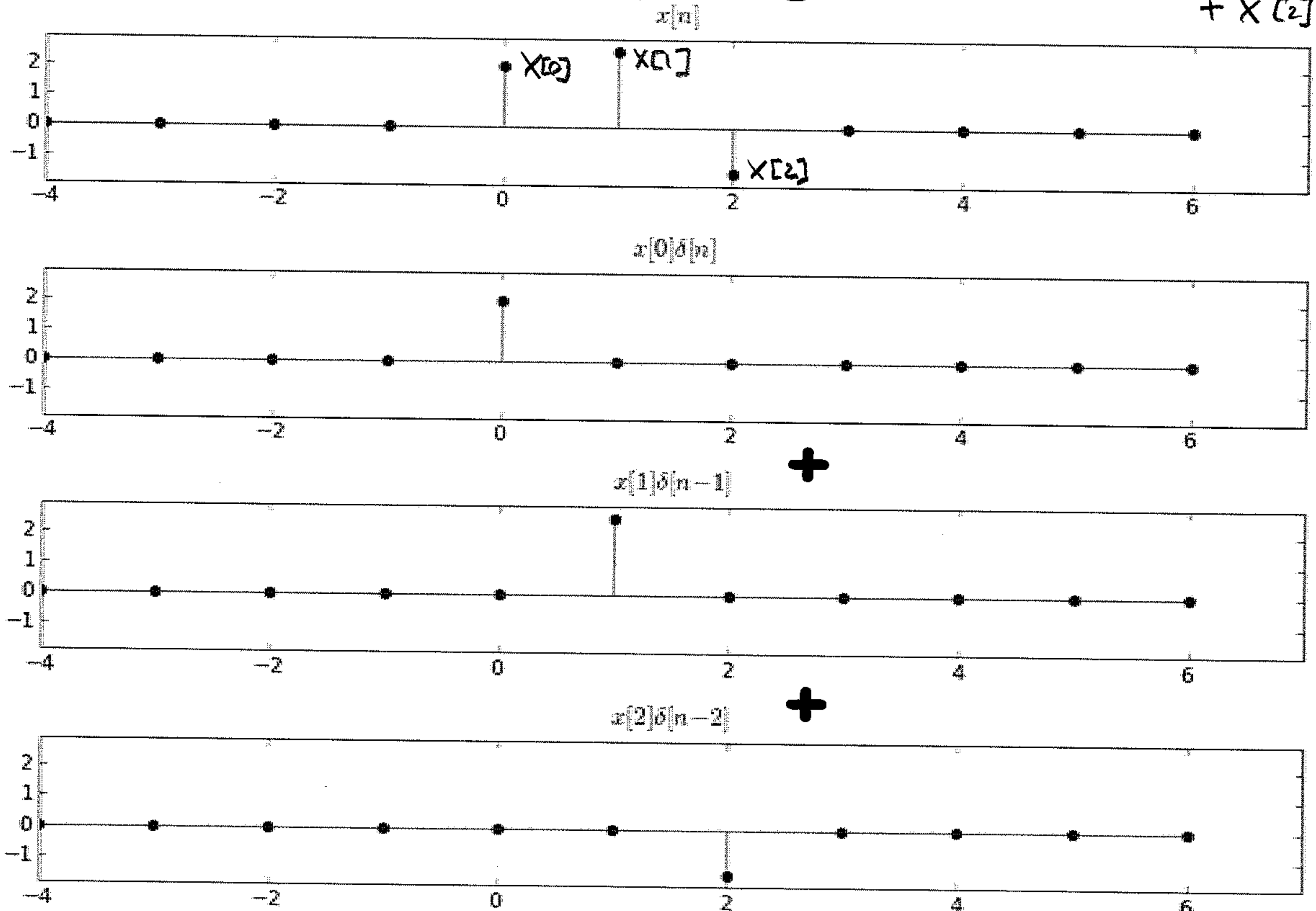


$0.5h[n-7]$



11

Decomposing $X[n] = X[0]\delta[n] + X[1]\delta[n-1] + X[2]\delta[n-2]$



(Note: $X[n]=0$ $n>2$ $n<0$)

In General for CLTI

$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \dots$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$y[n] = x[0]\,h[n] + x[1]\,h[n-1] + x[2]\,\delta[n-2] + \dots$$

In summation form:

$$y[n] = \sum_{\underbrace{m=0}_{\substack{\text{Because} \\ x[m]=0 \\ m<0}}}^{n \;\leftarrow\; \text{Because } h[n-m]=0\ (n-m)<0} h[n-m]\,x[m]$$

Causality
Unit step response can not be nonzero before the unit step occurs.
$h[n]=0$
$n<0$

Notation:

$$Y = H * X$$

sequences

Side Note Polynomial multiplication

$$\left(a[0] + a[1]z + a[2]z^2 + \dots + a[n]z^n\right)\left(b[0] + b[1]z + \dots + b[m]z^m\right)$$

Looks like convolution?

$$= \left(a[0]b[0] + (a[0]b[1] + a[1]b[0])z + \dots + a[n]b[m]z^{n+m}\right)$$

Alternative Form

$$y[n] = \sum_{m=0}^{n} x[n-m]\, h[m]$$

$$y[n] = x[n]h[0] + x[n-1]h[1] + x[n-2]h[2] + \dots$$

Suppose $h[n] = 0, \; n > K$

$$y[n] = h[0]x[n] + h[1]x[n-1] + \dots + h[K]x[n-K]$$

$$\underbrace{\text{(If } n < K \text{, equation still holds as } x[n-K] = 0 \text{ because } x[n] = 0, n < 0)}_{\text{Difference Equation}}$$

Recall Right-shift operator $\mathcal{R}$

$$V = \mathcal{R}X \Rightarrow v[n] = x[n-1]$$
$$V = \mathcal{R}^2 X \Rightarrow v[n] = x[n-2]$$

$$\Rightarrow Y = (h[0] + h[1]\mathcal{R} + \dots + h[K]\mathcal{R}^K)X$$
$$= \left(\sum_{i=0}^{K} h[i]\mathcal{R}^i\right) X$$

Deconvolution

$$Y = \left(\sum_{i=0}^{k} h[i]\,\mathcal{R}^i\right) X$$

W = reconstructed X

How to derive W from Y:

$$W = \frac{1}{\left(\sum_{i=0}^{k} h[i]\,\mathcal{R}^i\right)} Y$$

or

$$\left(\sum_{i=0}^{k} h[i]\,\mathcal{R}^i\right) W = Y$$

or

$$h[0]\,w[n] + h[1]\,w[n-1] + \ldots + h[k]\,w[n-k] = y[n]$$

Since W reconstructs X:

It should satisfy the same difference equation as X

Solving for W

Plug & Chug

$$w[0] = \frac{1}{h[0]}\left[\frac{y[0]}{h[0]} - \underbrace{\left(h[1]\,w[-1] + \ldots + h[k]\,w[-k]\right)}_{\text{all zero}}\right]$$

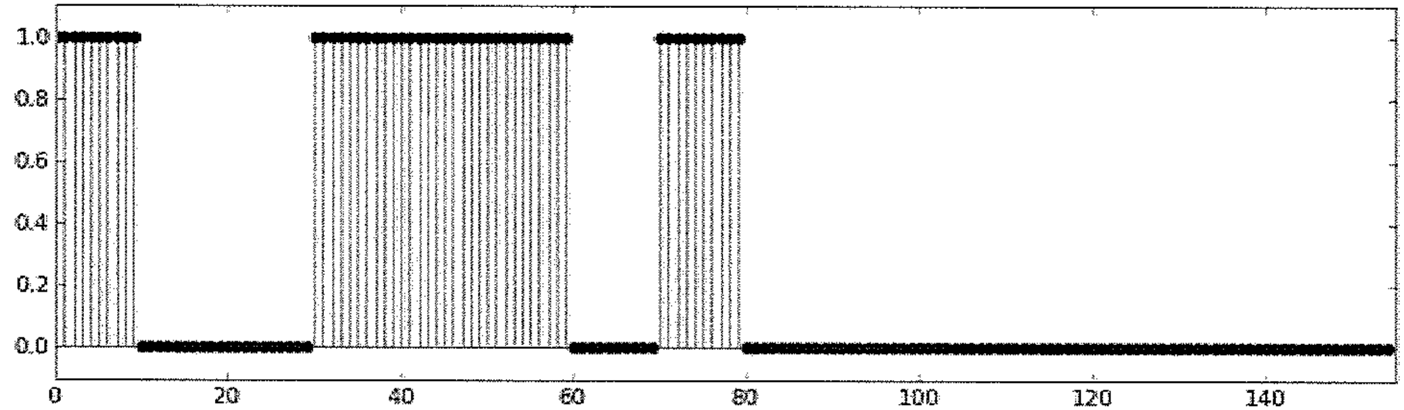
$$w[1] = \frac{1}{h[0]}\left(y[1] - h[1]\,w[0]\right)$$

$$w[2] = \frac{1}{h[0]}\left(y[2] - \left(h[1]\,w[1] + h[2]\,w[0]\right)\right)$$

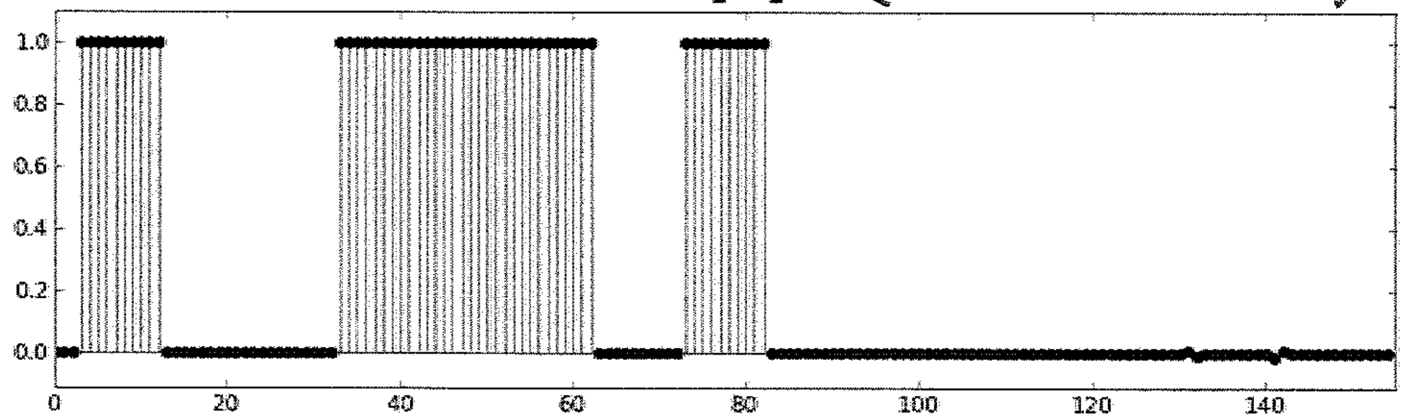
$$\vdots$$

Deconvolution Result

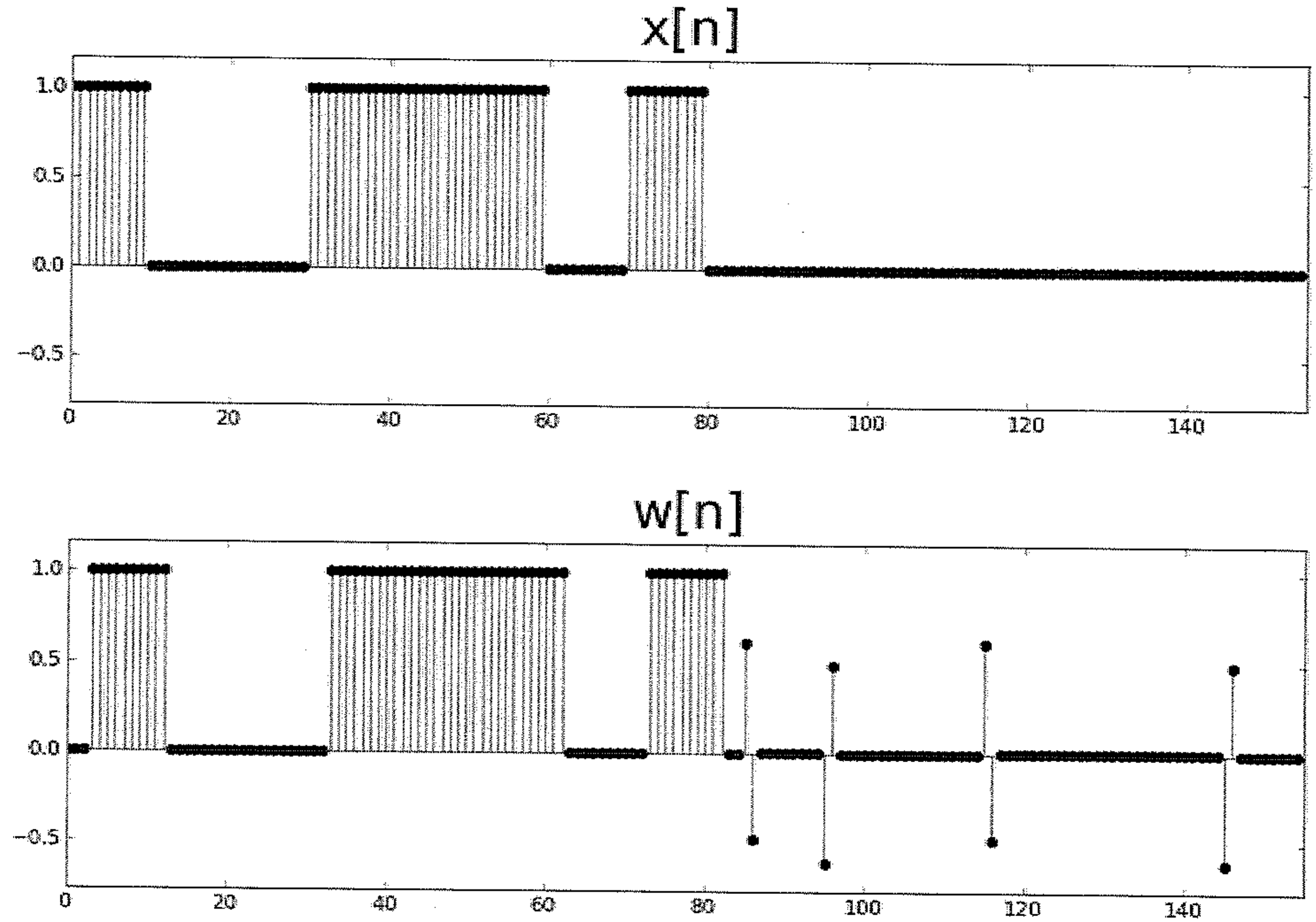
x[n]



w[n] (Derived from y[n])



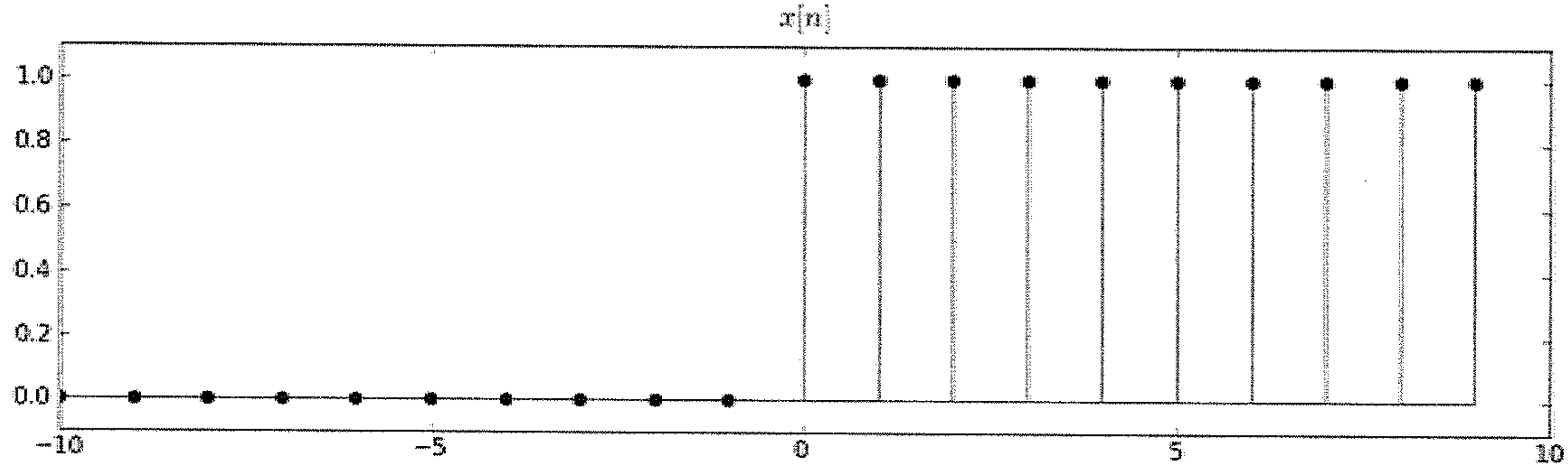
Deconvolution Result (Truncated h)

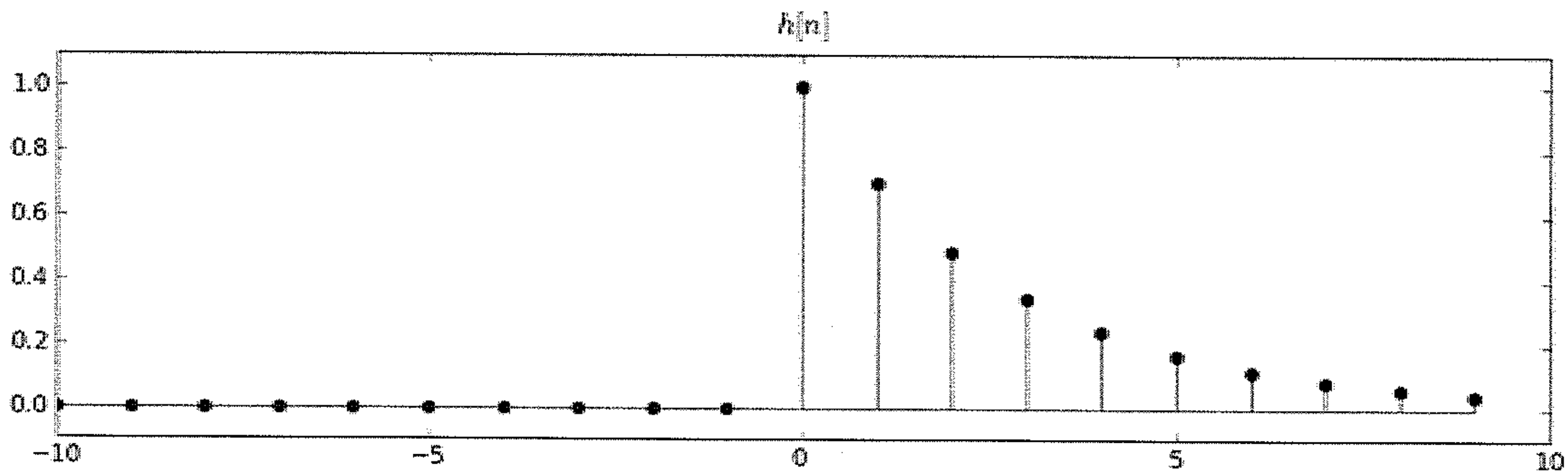


why are errors at the end?

18

Example x[n] and h[n]

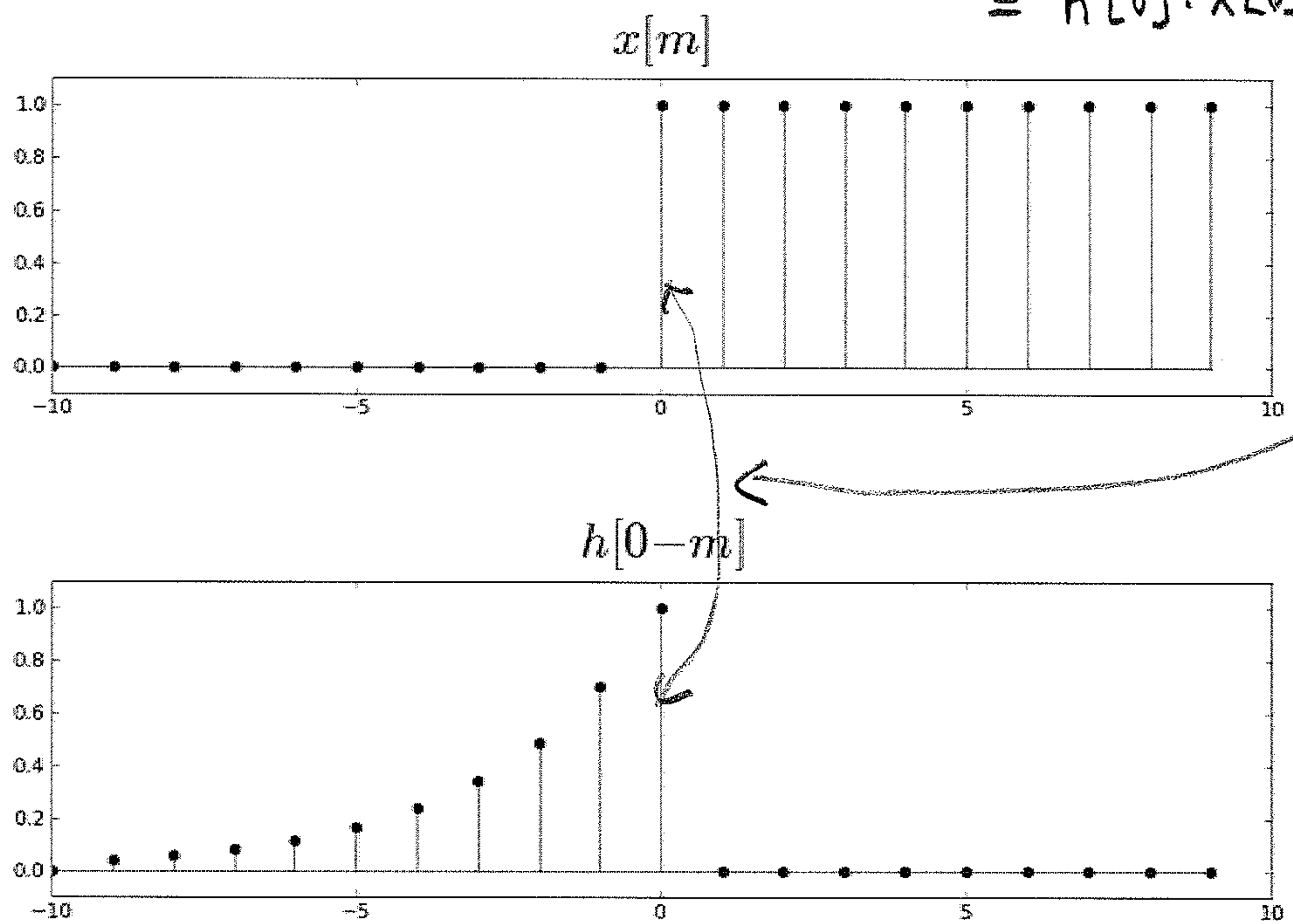




19

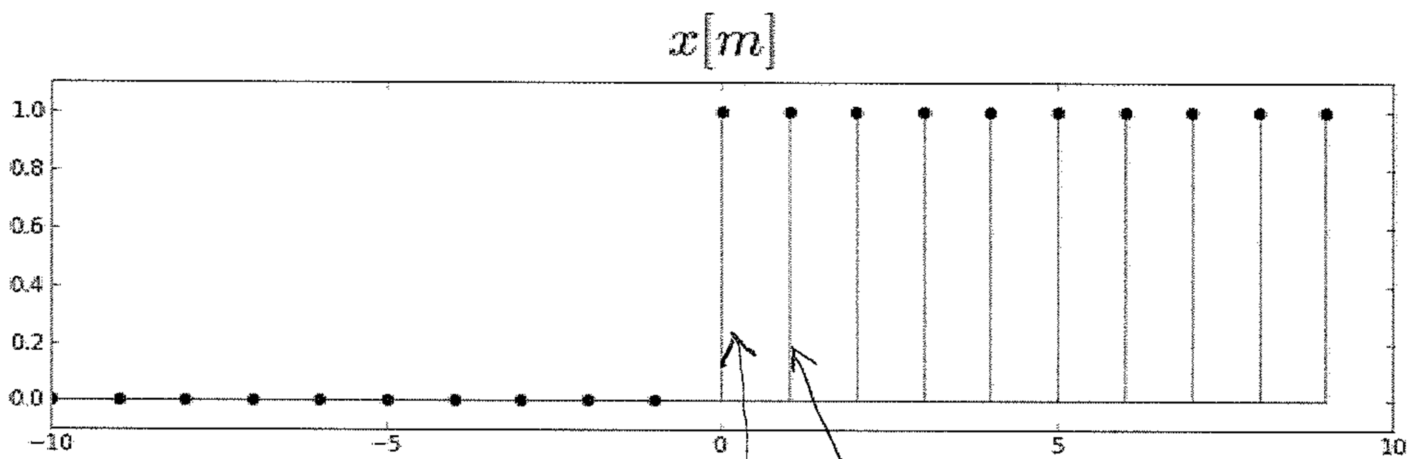
Evaluating y[0] $= \sum_{m=0}^{0} h[0-m]\, x[m]$

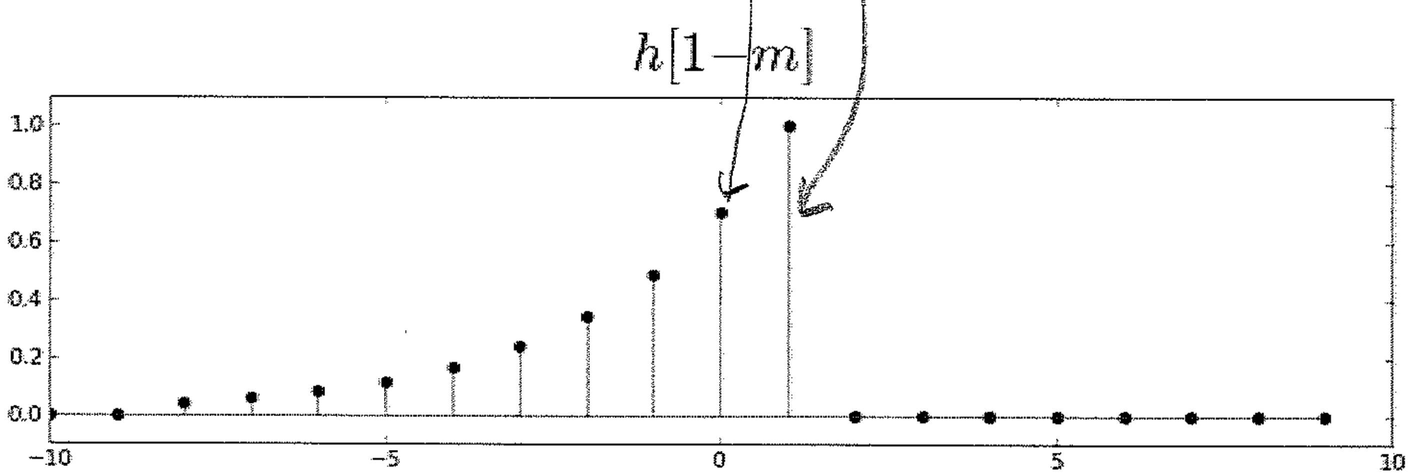
$= h[0] \cdot x[0]$



Flipped H

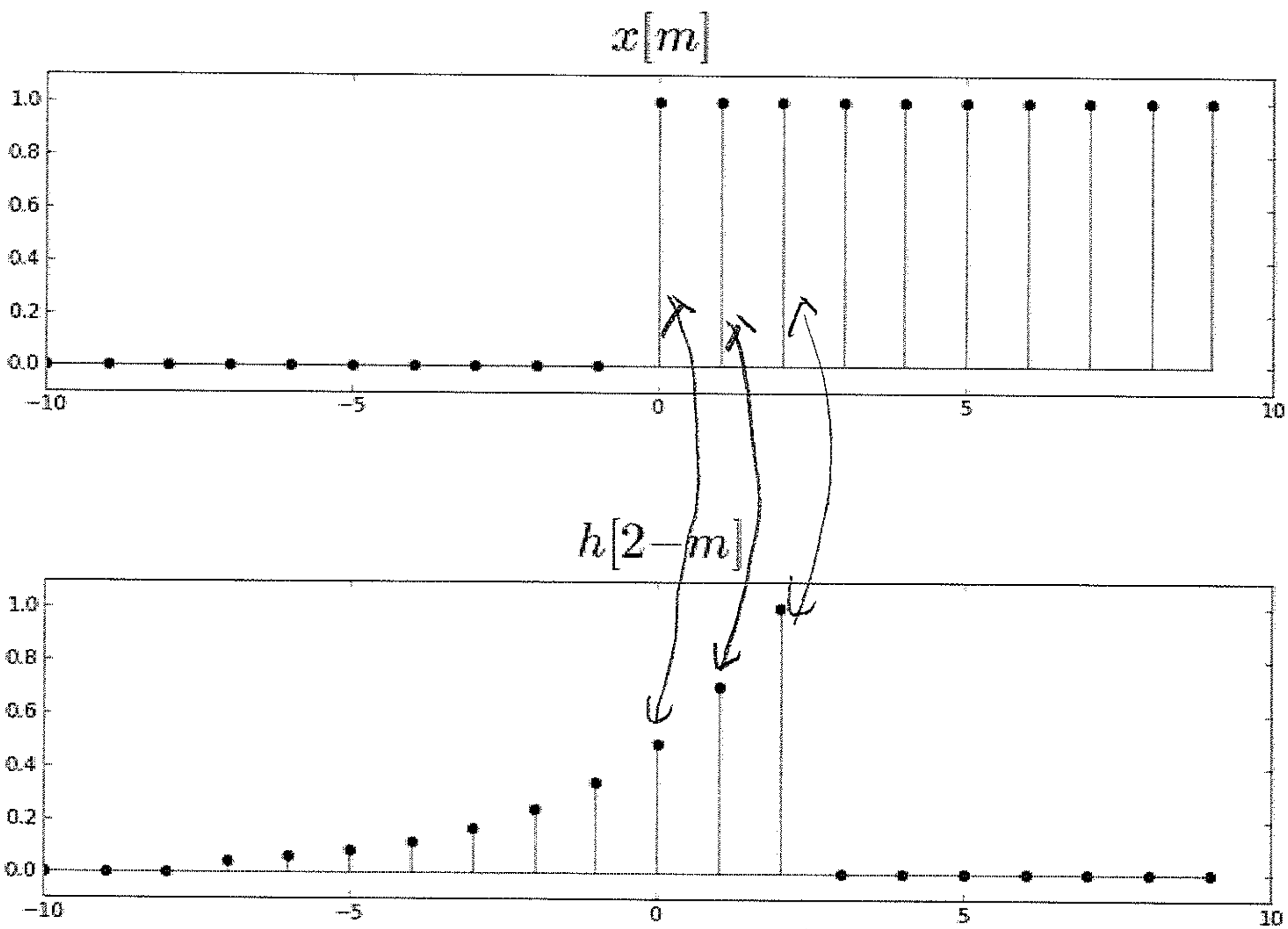
Evaluating y[1] $= h[0]x[1] + h[1]x[0]$





Flipped H, slid by 1.

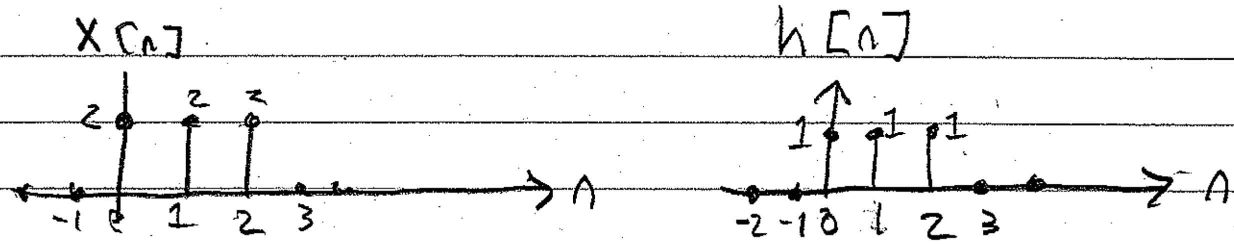
Evaluating y[2] = $x[0]h[2] + x[1]h[1] + x[2]h[0]$



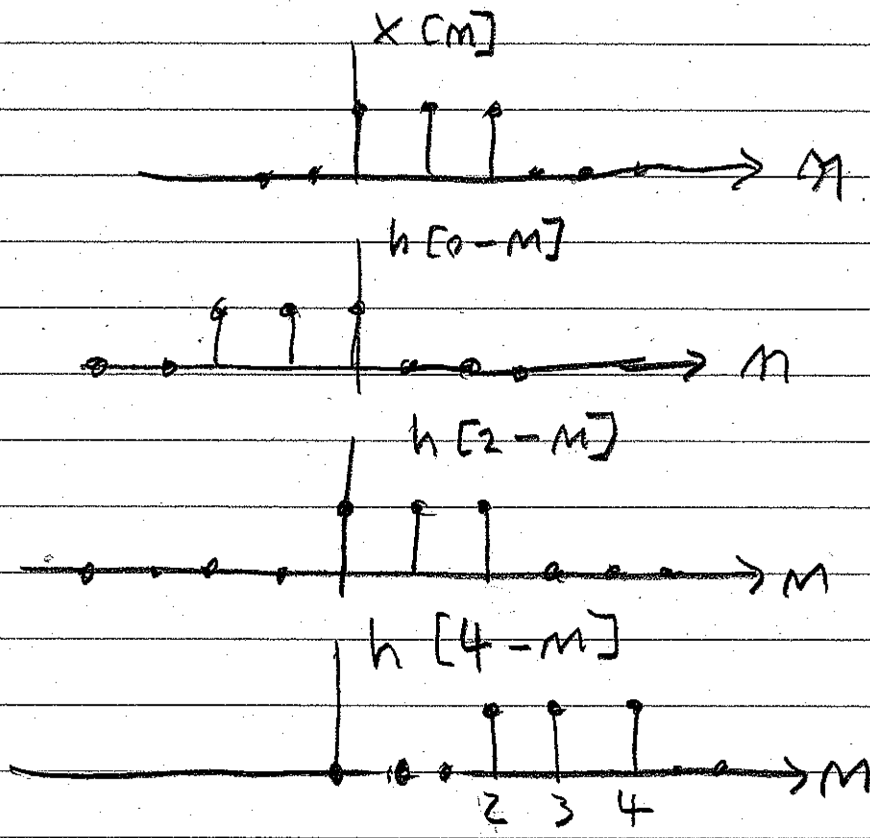
Flipped H
Slid by 2

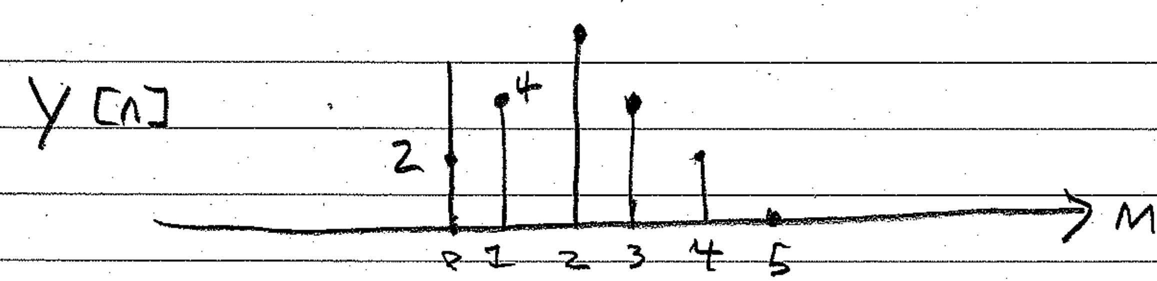
Convolution Examples

1)



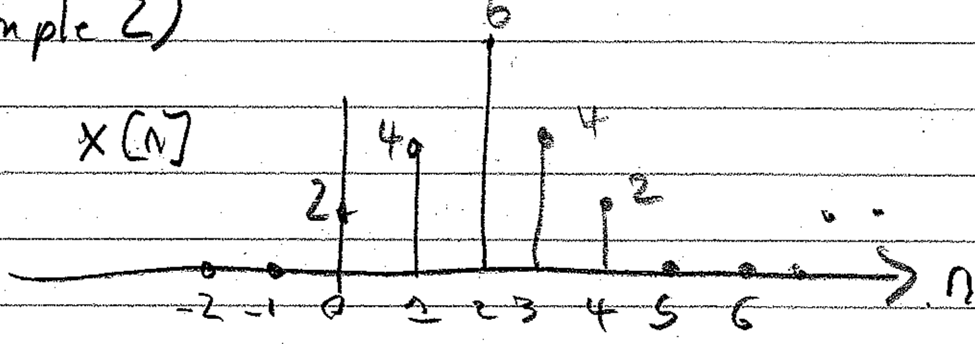
$Y = H * X$

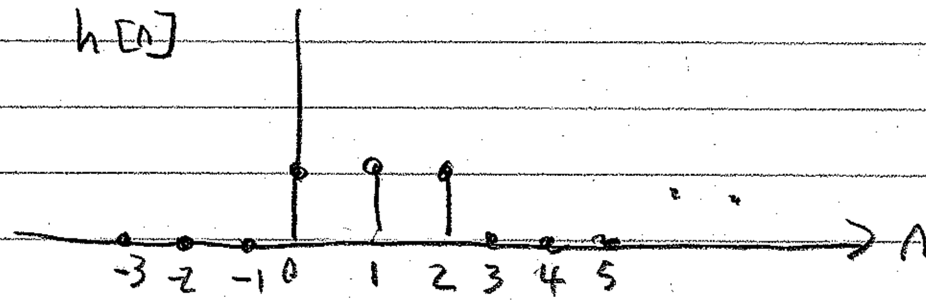




<u>Note</u> y[n]'s maximum value is 6 at n=2.
Easy to see from flip & Slide

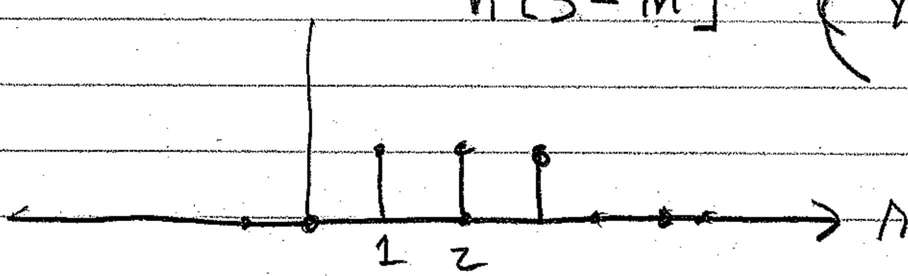
Example 2)

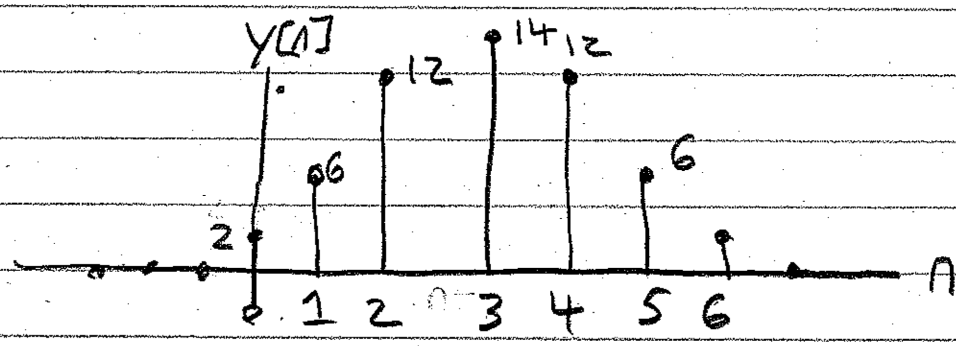




Flip & Slide

h[3 - m] (y[n] max at n=3)



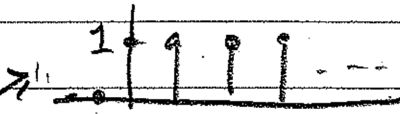


Note Example 1 □ * □ = Δ
Example 2 Δ * □ = ⌒

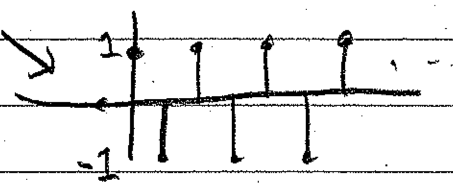
} Each Convolution smooths

Example 3)

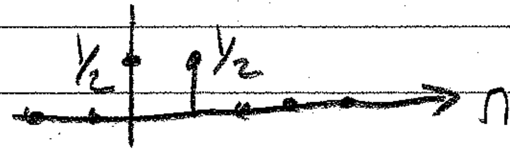
$x_1[n] = u[n]$



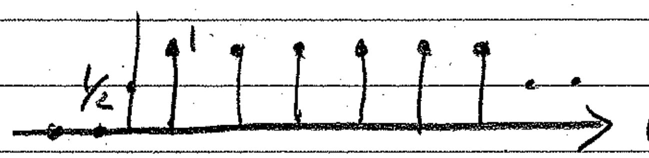
$x_2[n] = (-1)^n u[n]$



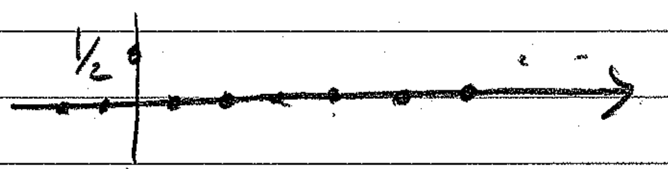
$h[n] =$



$y_1[n] =$



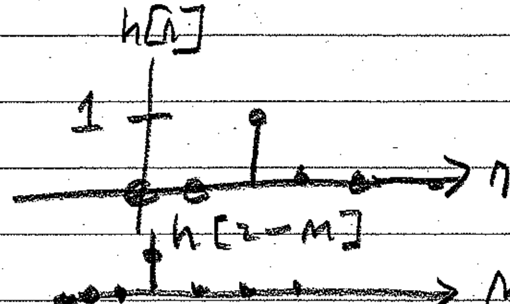
$y_2[n] =$



Easy to see by Flip and Slide

Example 4)

$h[n] = \delta[n-2]$

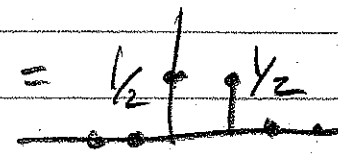


$y[n] = x[n-2]$

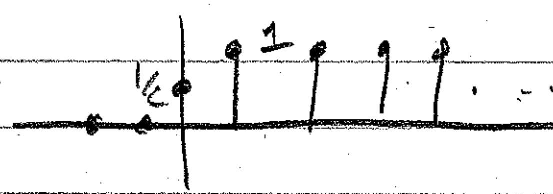
Deconvolution Example

Example 1)

$x[n] = u[n]$ $\quad h[n] =$



$\Rightarrow y[n] =$



$$h[0]\,w[n] + h[1]\,w[n-1] = y[n]$$

$$w[0] = \frac{1}{\frac{1}{2}}\left(y[0] - \cancel{h[1]\,w[-1]}^{=0}\right)$$

$$= 1$$

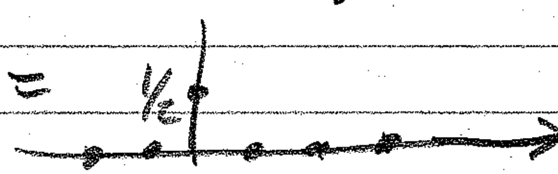
$$w[1] = \frac{1}{\left(\frac{1}{2}\right)}\left(y[1] - h[1]\,w[0]\right)$$

$$= 1$$

$$w[n] = 1 \quad n > 1$$

Example 2) $\quad x[n] = (-1)^n u[n]$, same $h[n]$

$y[n] =$



$$w[n] = \left(\frac{1}{h[0]}\right)\left(y[n] - h[1]\,w[n-1]\right)$$

$$= 2\left(y[n] - \tfrac{1}{2}\,w[n-1]\right)$$

Example 2 Cont)

$$w[0] = 2\left(y[0] - \tfrac{1}{2}\cancelto{0}{w[-1]}\right)$$

$$= 2\left(\tfrac{1}{2} - 0\right) = 1$$

$$w[1] = 2\left(\cancelto{0}{y[1]} - \tfrac{1}{2}w[0]\right)$$

$$= -1$$

$$w[2] = 2\left(\cancel{y[2]} - \tfrac{1}{2}w[1]\right)$$

$$= 1$$

$$w[n] = (-1)^n u[n]$$

Example 3

$$x[n], \quad h[n] = \delta[n-2]$$

$$\Rightarrow \quad y[n] = x[n-2]$$

Deconvolution Equation:

$$\overset{=0}{\cancel{h[0]}}\, w[n] + \overset{=0}{\cancel{h[1]}}\, w[n-1] + \overset{=1}{\cancel{h[2]}}\, w[n-2] = y[n]$$

$$\Rightarrow \quad w[n-2] = y[n] = x[n-2]$$

$$\Rightarrow \quad w[n-2] = x[n-2]$$

$$w[n] = x[n]$$

__Note__ To compute $w[n]$, you need $y[n+2]$! The deconvolution equation does __NOT__ describe a causal system!