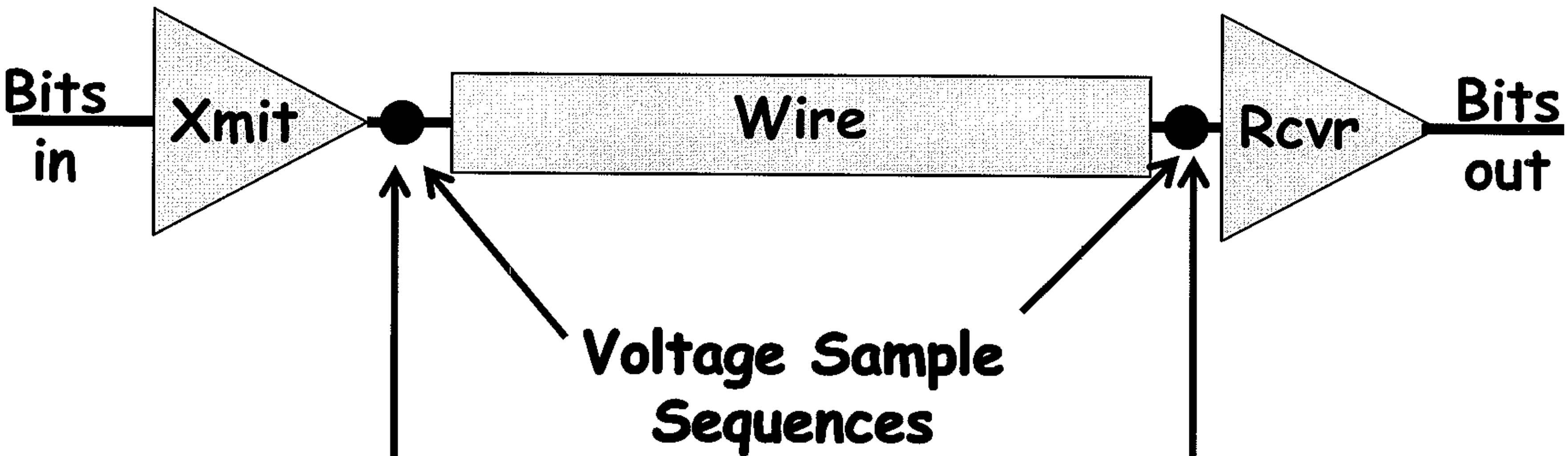
6.02 Lecture 4 – Signals and Noise

- Big Eye versus Small Eye
  - Noise problem worse for small eyes
  - Bit Error Rate
- Signals and Noise
  - Decompose into Noisefree plus Noise
  - Noise Metrics (Sample Mean and Variance)
- Probability Density Functions
  - Connection to Histograms
  - Use in estimating bit error rates

Transmission Setup and Notation



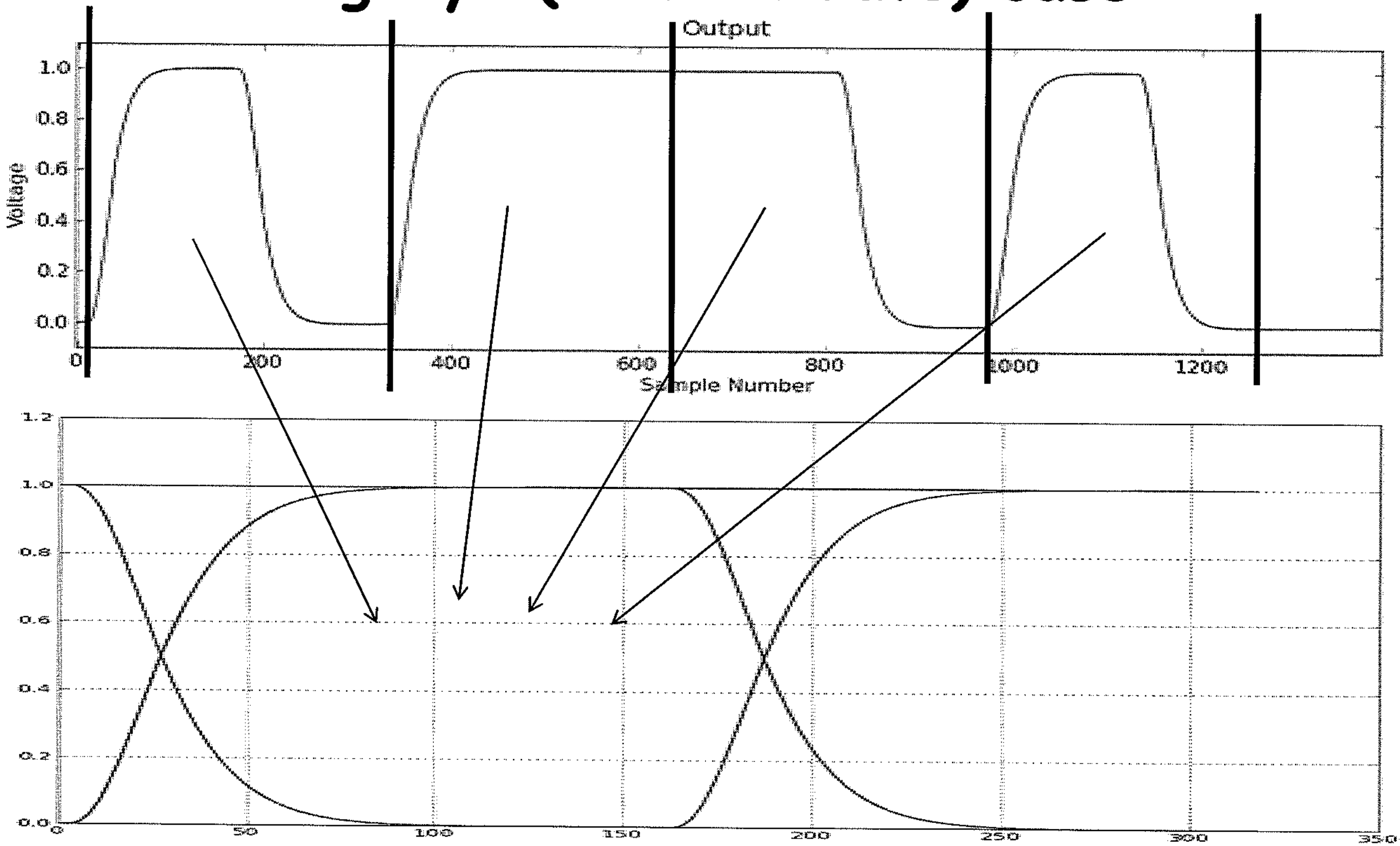
$X \equiv$ entire sequence

$x[n] \equiv n^{th}$ sample value

$Y \equiv$ entire sequence

$y[n] \equiv n^{th}$ sample value

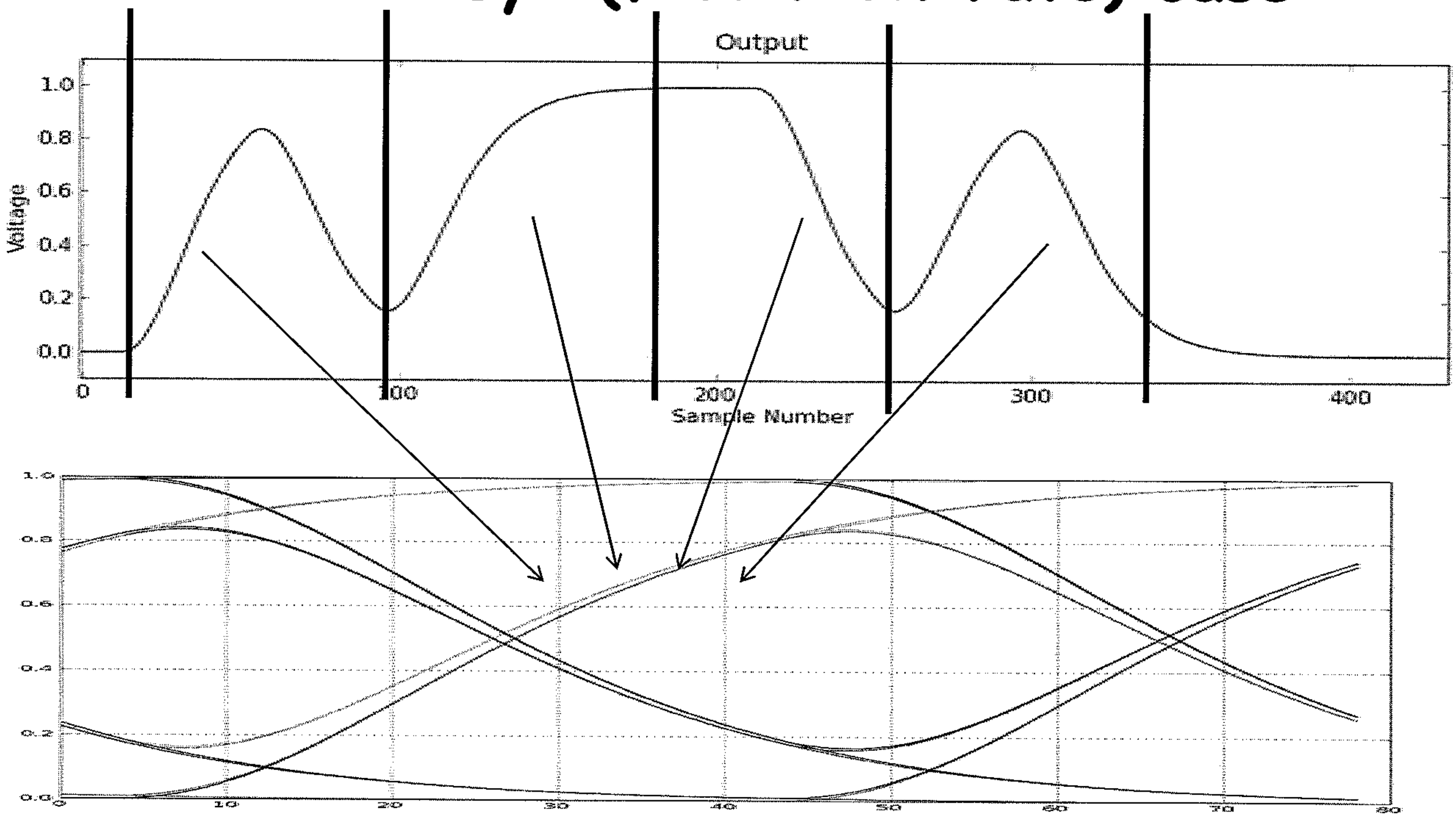
Big Eye (slow bit rate) case



Eye Diagram Generated with 160 samples per bit

Smaller Eye (faster bit rate) case

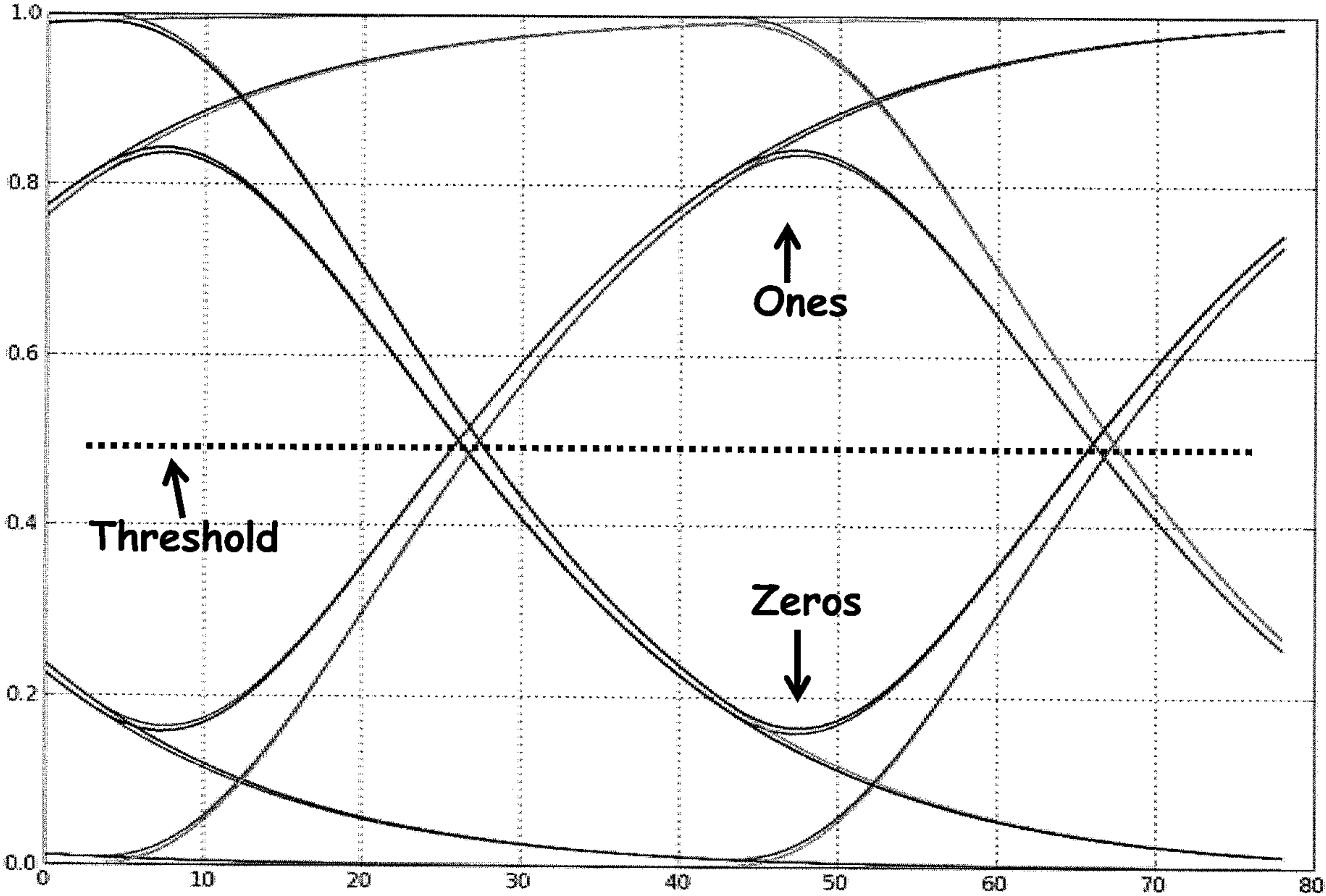
4



Eye diagram generated from 40 samples per bit and using a 200 bit long random sequence.

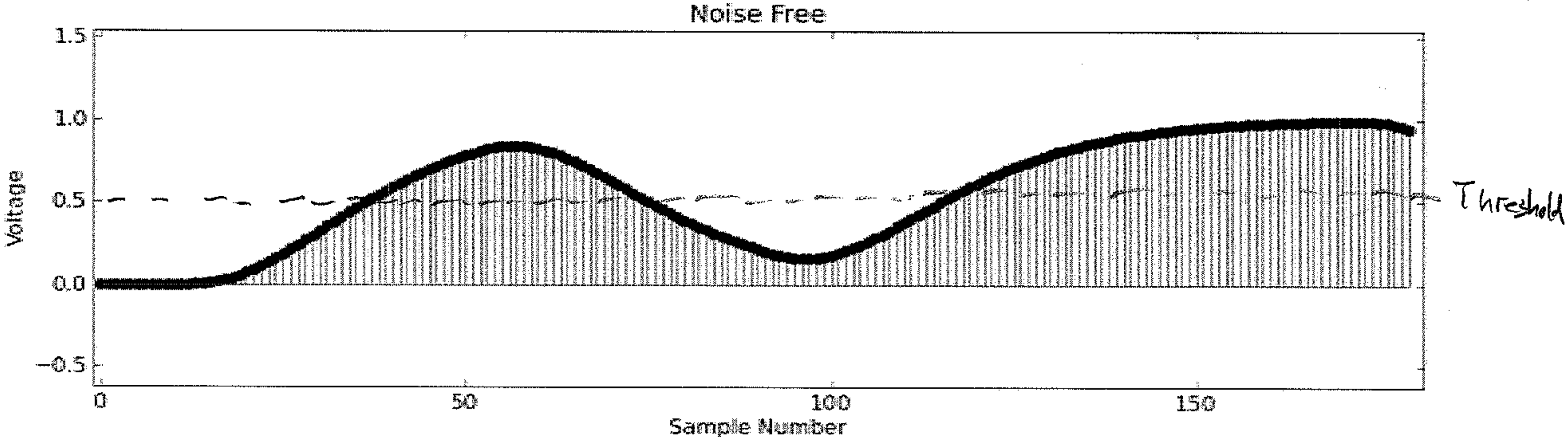
5

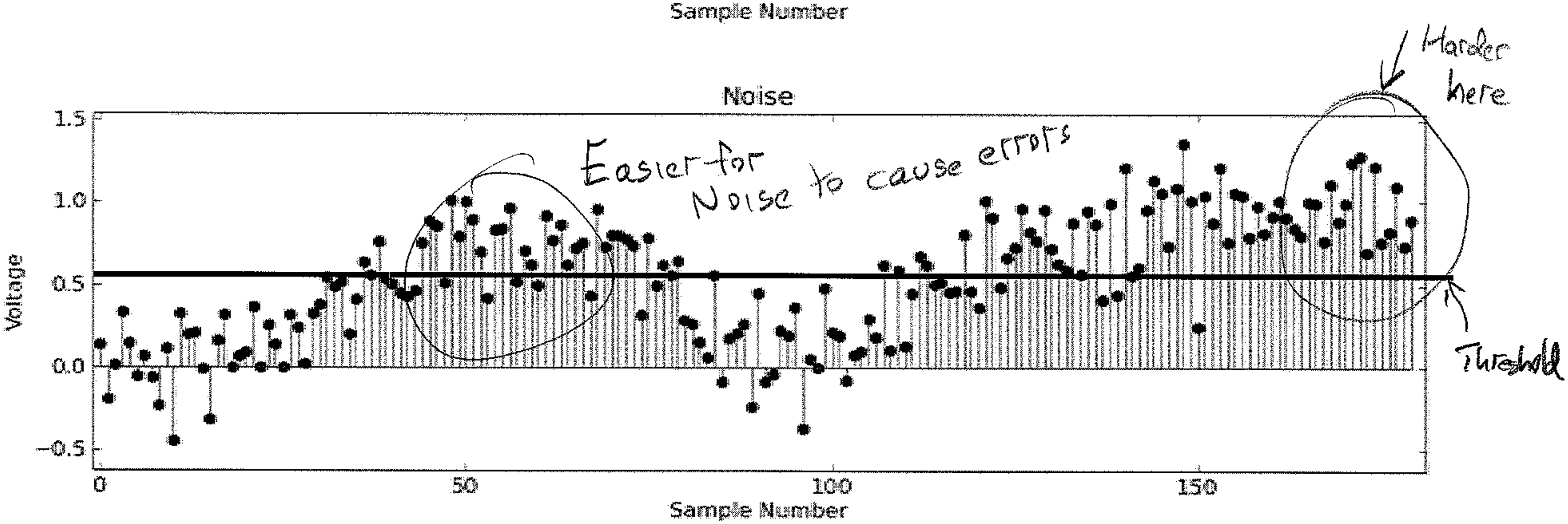
Why is a small eye bad?



6

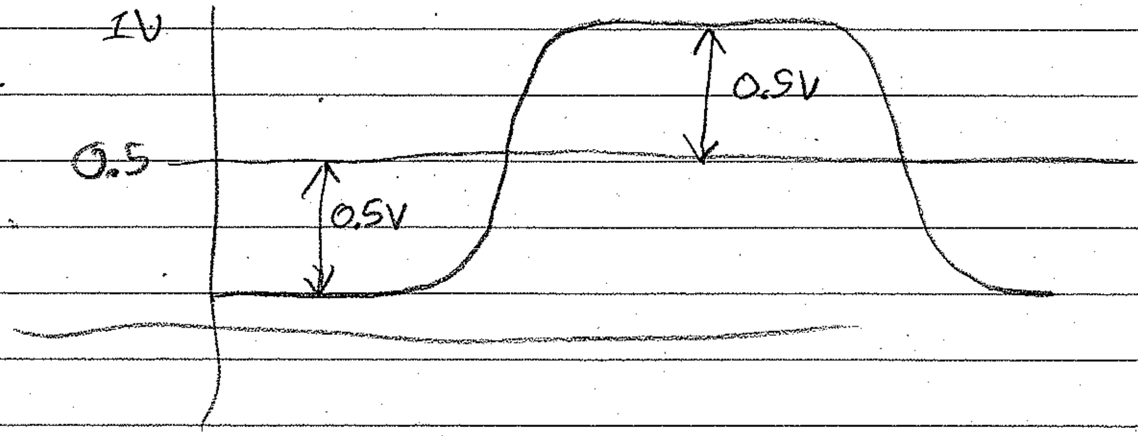
Noisy Signal Can Cause Bit Errors





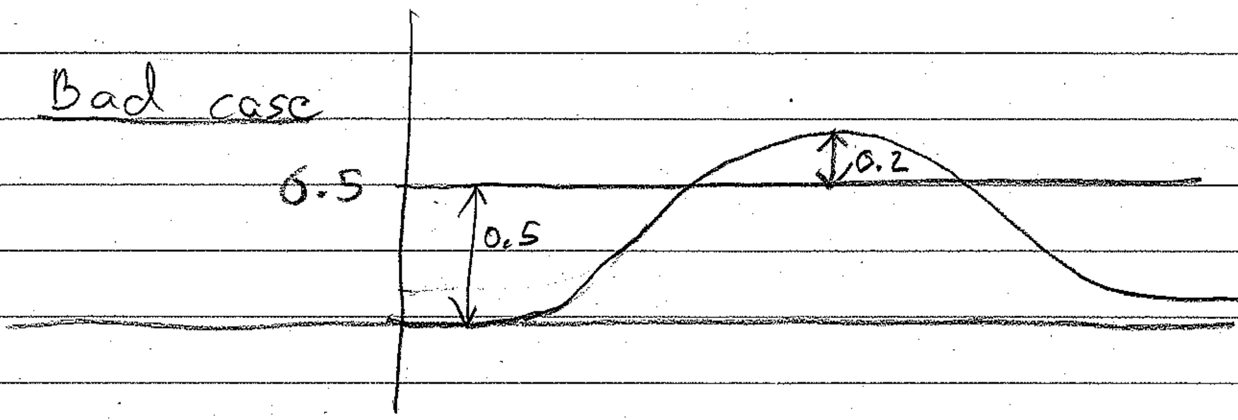
Analyzing Big Errors

Good Case



To cause an error noise > 0.5V

Bad case



To cause an error noise > 0.2V

Bit Error Rates

Bad Name - Really the probability that a given bit will be in error

Examples:

Really Good Channel: BER = 1 error in 10^{12} bits
O Kay Channel: BER = 1 error in 10^{4} bits
Lousy Channel: BER = 1 error in 10^{2} bits

For this course: Model only additive Noise

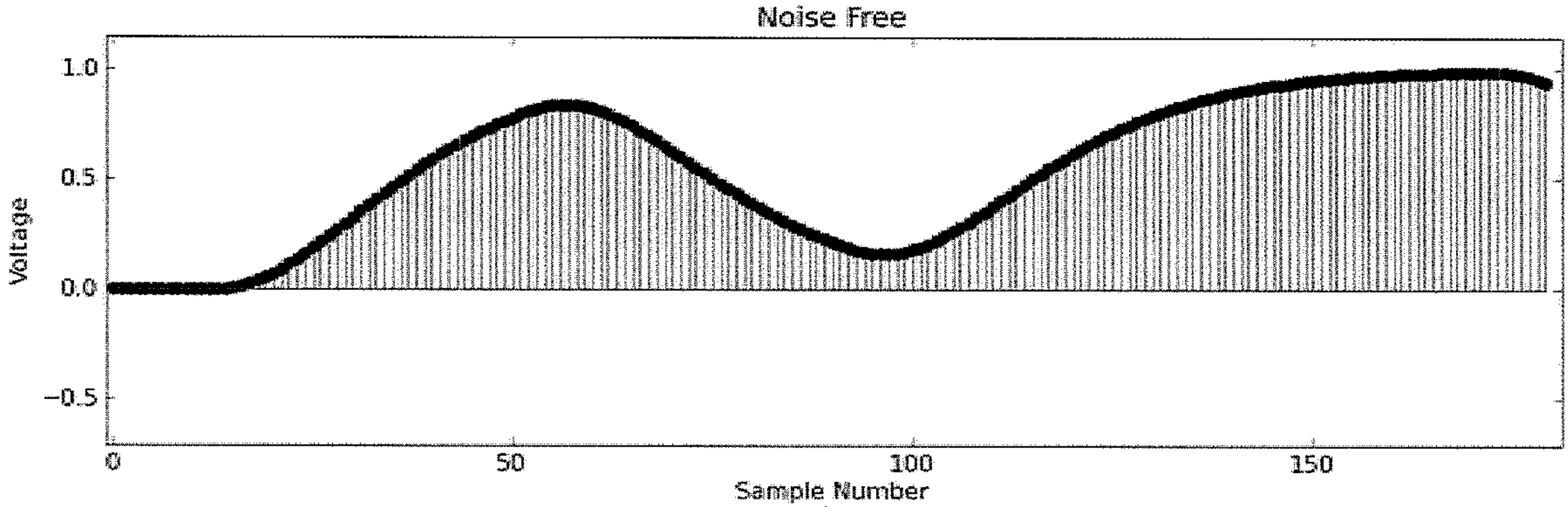
- Many other types of noise (e.g. phase noise where transmitter bit period varies)
- Additive Noise is easy to analyze

Additive noise

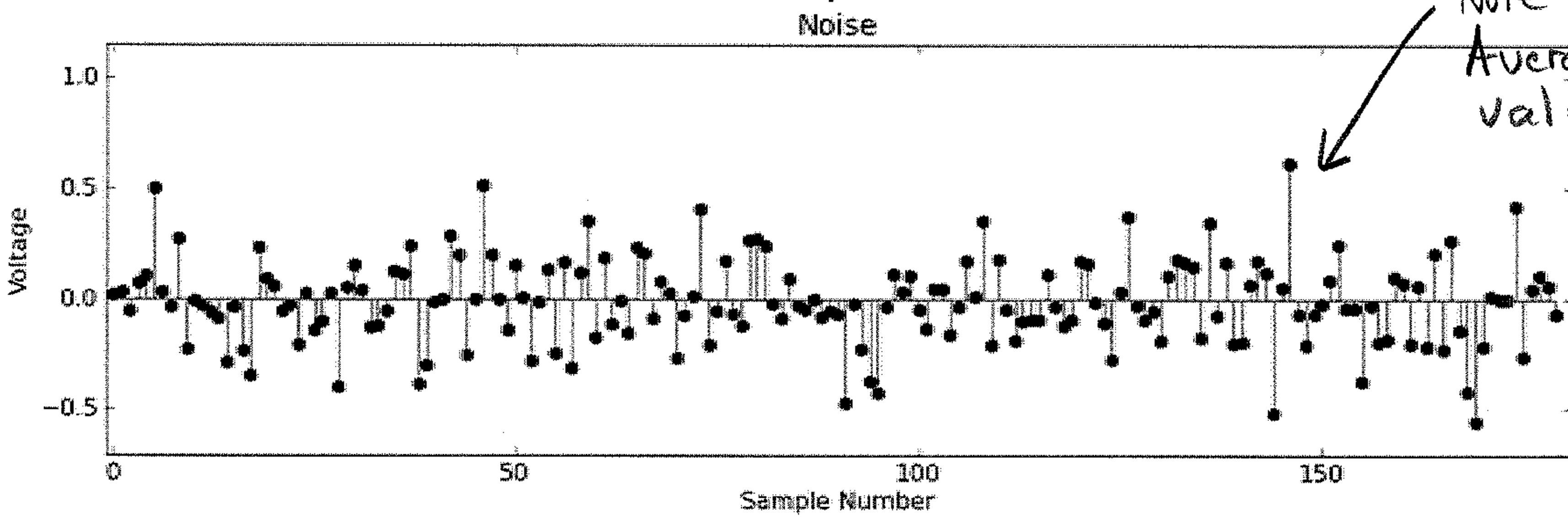
signal[n] = noise_free_signal[n] + noise[n]

Decompose Into Noisefree + Noise

9



+



Note Noise Average value ≈ 0

Noise Metrics

Sample Mean:

$$SM \equiv \frac{1}{N} \sum_{i=0}^{N-1} noise[n]$$

number of samples → N

Typically $SM \simeq 0$

Sample Variance:

$$SV \equiv \frac{1}{N} \sum_{i=0}^{N-1} \left(noise[n] - SM\right)^2$$

Sample Standard Deviation:

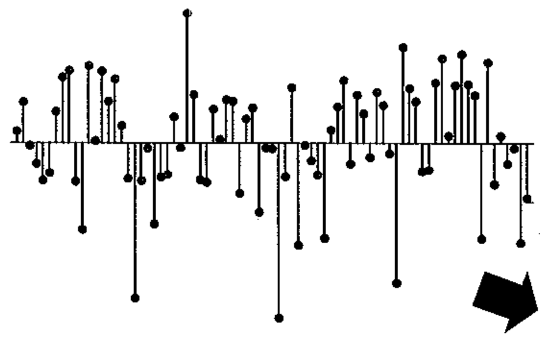
$$SSV = \sqrt{SV}$$

$$\simeq \text{Amplitude of Noise}$$

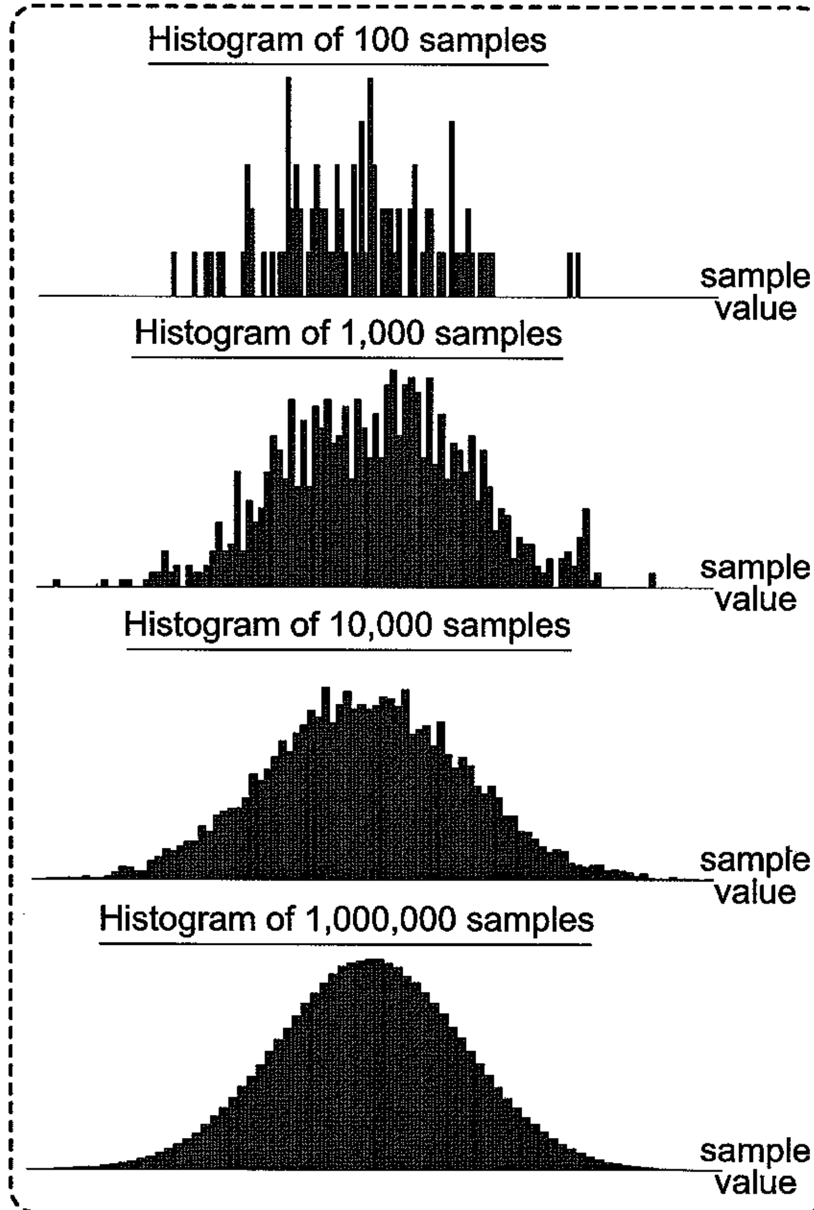
Key Point: Want distance between noise-free-signal and threshold >> Amplitude of Noise.

11

Experiment to see Statistical Distribution



- Create histograms of sample values from trials of increasing lengths
- Assumption of independence and stationarity implies histogram should converge to a shape known as a probability density function (PDF)

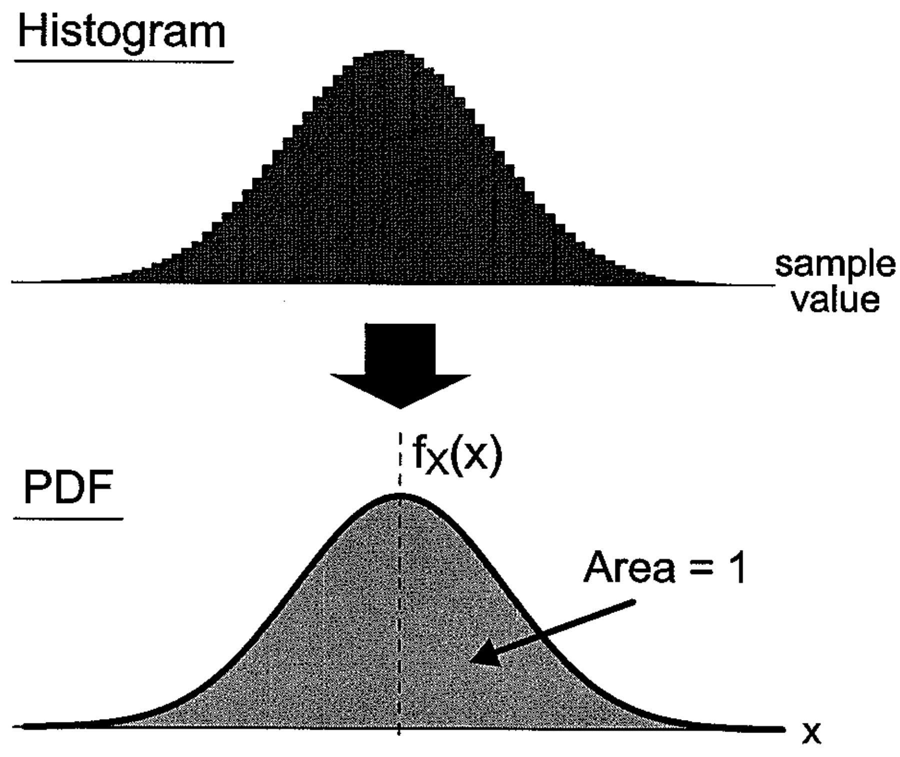


Slide thanks to C. Sodini and M. Perrott

The Probability Density Function PDF

- Define x as a random variable whose PDF has the same shape as the histogram we just obtained
- Denote PDF of x as $f_X(x)$
  - Scale $f_X(x)$ such that its overall area is 1

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1$$



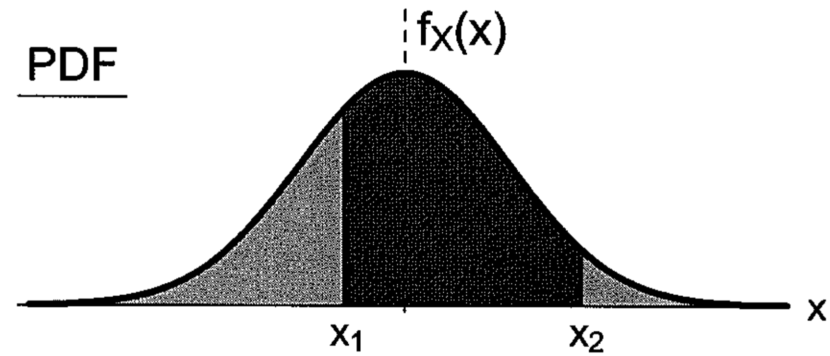
This shape is referred to as a Gaussian PDF

Slide thanks to C. Sodini and M. Perrott

Formalizing Probability

- The *probability* that random variable x takes on a value in the range of x_1 to x_2 is calculated from the PDF of x as:

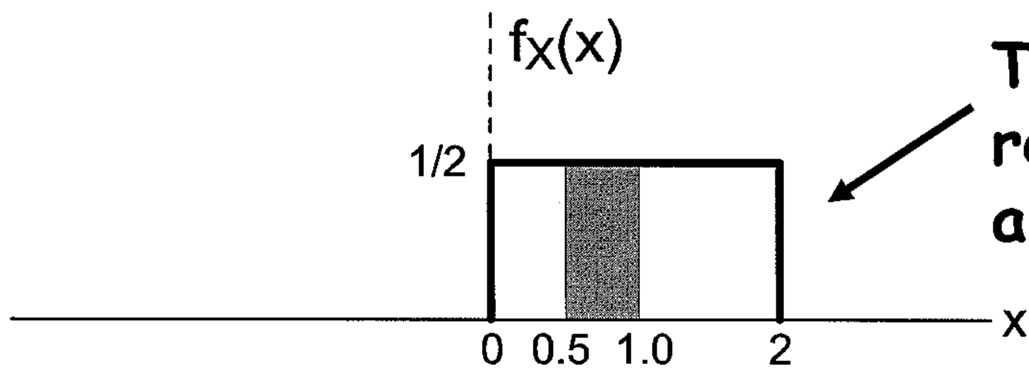
$$\mathrm{Prob}(x_1 \le x \le x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$



- Note that probability values are always in the range of 0 to 1
  - Higher probability values imply greater likelihood that the event will occur

Slide thanks to C. Sodini and M. Perrott

Example Probability Calculation



This shape is referred to as a uniform PDF

- **Verify that overall area is 1:**

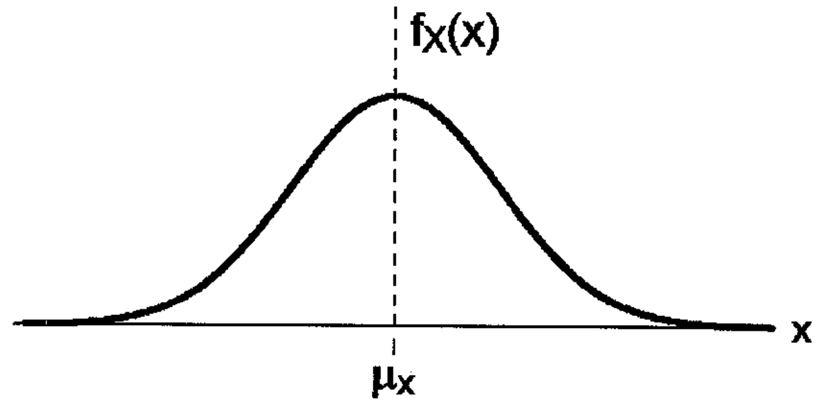
$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^2 0.5\,dx = \boxed{1}$$

- **Probability that x takes on a value between 0.5 and 1.0:**

$$\mathrm{Prob}(0.5 \le x \le 1.0) = \int_{0.5}^{1.0} 0.5\,dx = \boxed{0.25}$$

Slide thanks to C. Sodini and M. Perrott

Mean and Variance



- The mean of random variable *x*, μ_x, corresponds to its average value
  - Computed as

$$\mu_x = \int_{-\infty}^{\infty} x f_X(x) dx$$

- The variance of random variable *x*, σ_x^2, gives an indication of its variability
  - Computed as

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) dx$$

- The standard deviation of a random variable x, is denoted σ_x

Slide thanks to C. Sodini and M. Perrott

Under "Reasonable" Assumptions

In the limit as $N \to \infty$:

$$\underbrace{\lim_{N\to\infty} \sum_{n=0}^{N-1} noise[n]}_{\text{Sample Mean}} = \mu_{noise}$$

$$\underbrace{\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \left(noise[n] - \underbrace{\left(\sum_{n=0}^{N-1} noise[n]\right)}_{SM} \right)^2}_{\text{sample variance}}$$

$$= \sigma^2_{noise}$$

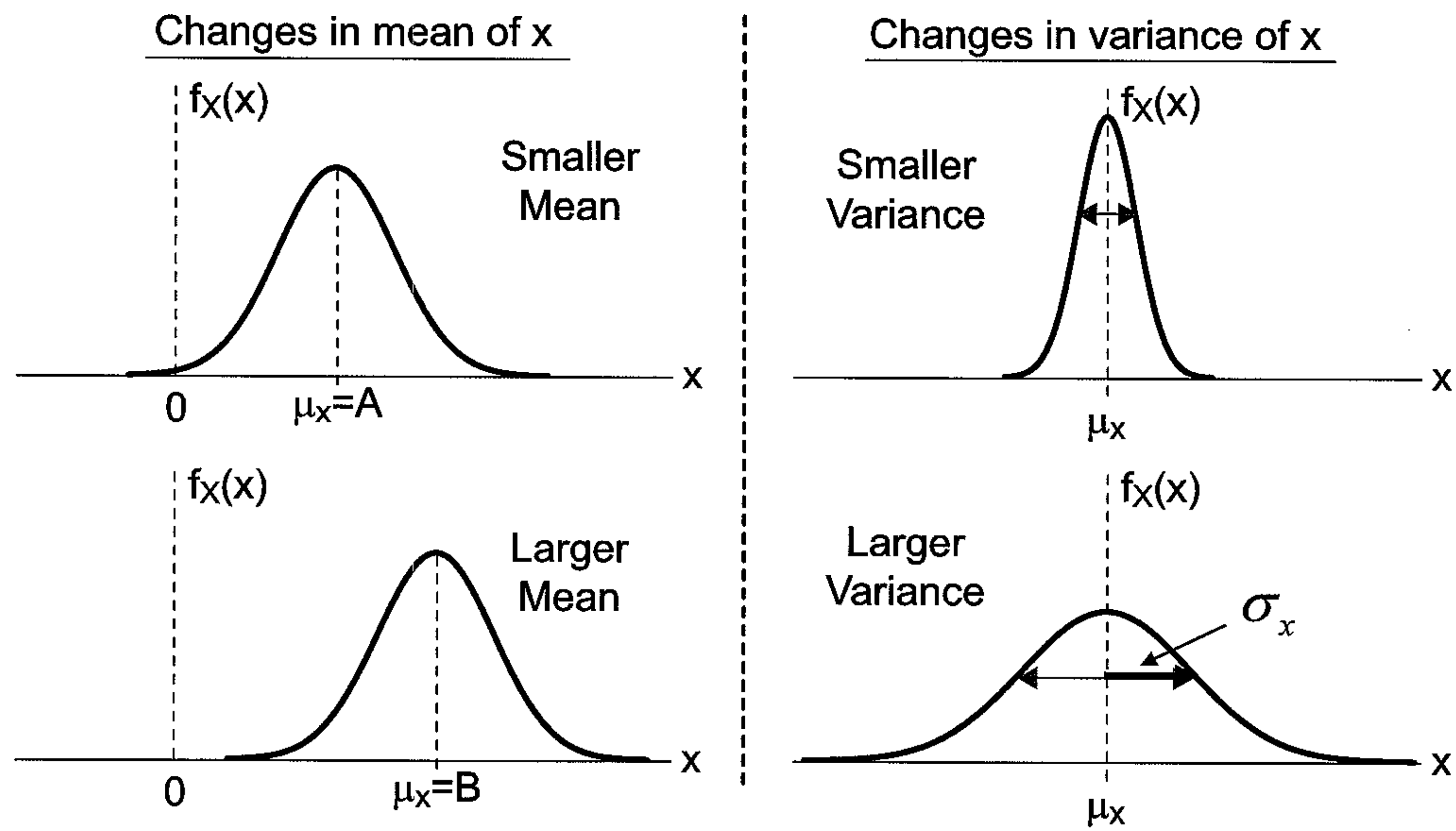
That is: If the noise is described by a probability density function

$f_{NOISE}(noise)$

then the sample mean and sample variance of the noise should converge to the mean and variance of f

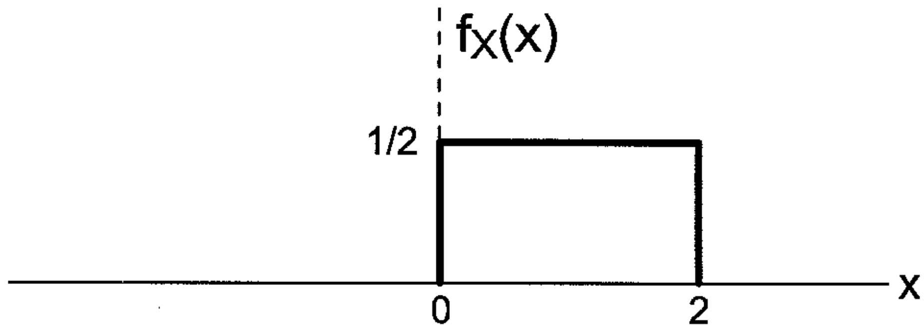
Visualizing Mean and Variance from PDF

17



- Changes in mean shift the *center of mass* of PDF
- Changes in variance narrow or broaden the PDF
  - Note that area of PDF must always remain equal to one

Example Mean and Variance Calculation



- **Mean:**

$$\mu_x = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^2 x\frac{1}{2}dx = \frac{1}{4}x^2\Big|_0^2 = \boxed{1}$$

- **Variance:**

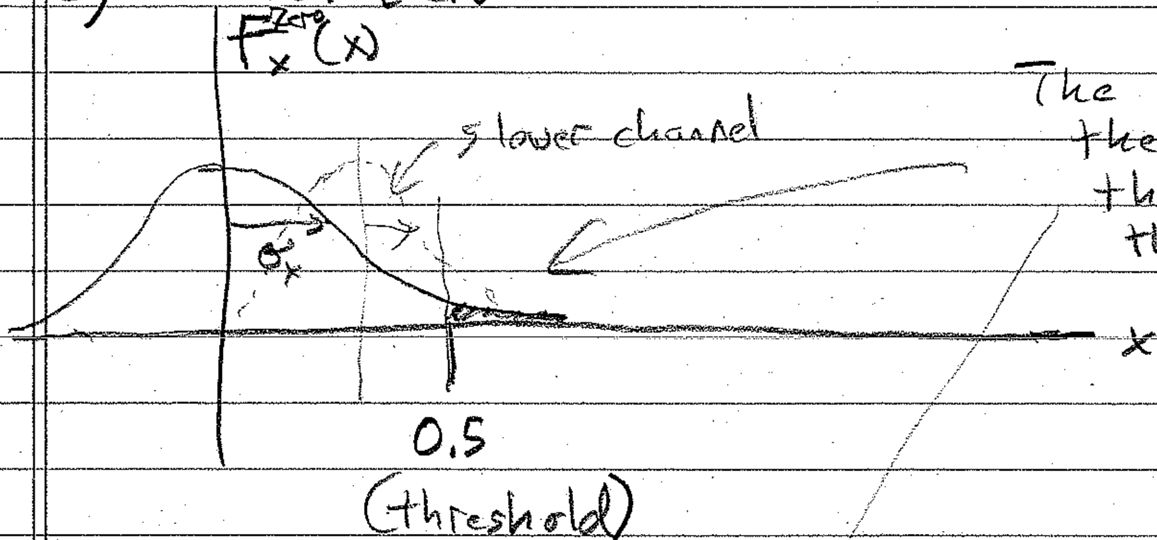
$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)dx = \int_0^2 (x-1)^2\frac{1}{2}dx$$

$$= \frac{1}{6}(x-1)^3\Big|_0^2 = \frac{1}{6} + \frac{1}{6} = \boxed{\frac{1}{3}}$$

Slide thanks to C. Sodini and M. Perrott

Consider Two Cases

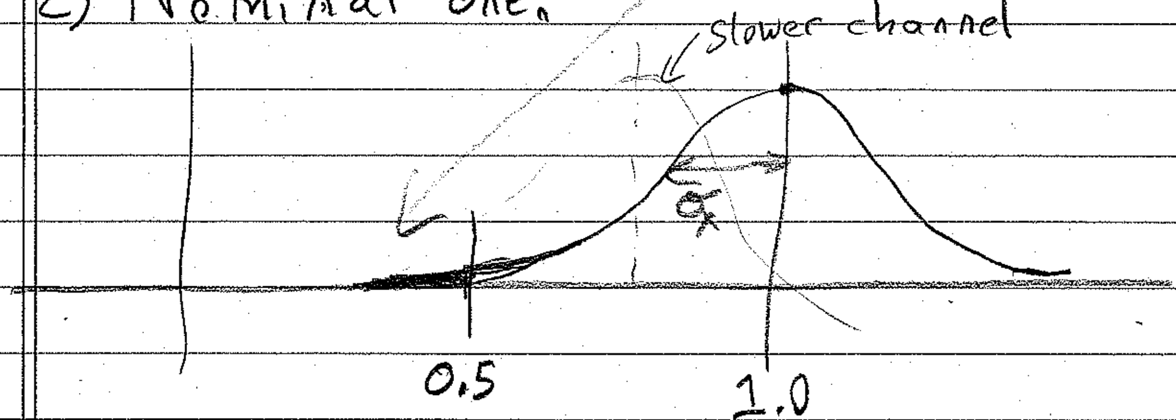
1) Nominal Zero:



The larger the standard deviation the larger the noise.

$$\int_{0.5}^{\infty} f_X^{zero}(x)\,dx = \text{Probability zero read as one} = E_{01}$$

2) Nominal one:



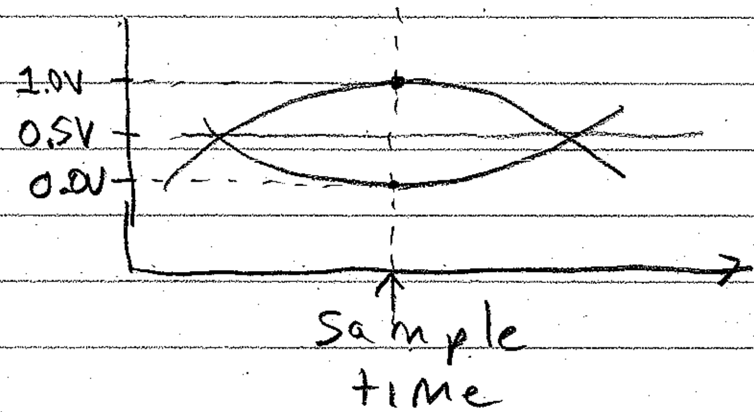
$$\int_{-\infty}^{0.5} f_X^{one}(x)\,dx = \text{probability one read as zero} = E_{10}$$

If ones & zeros equally likely

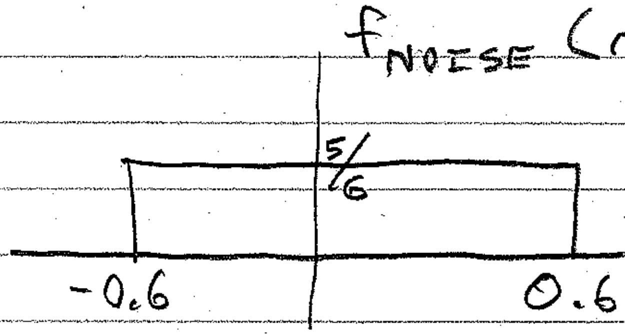
$$\tfrac{1}{2} E_{01} + \tfrac{1}{2} E_{10} = \text{Probability of error}$$

Example

Suppose a noise-free eye diagram is:



and the noise is given by:



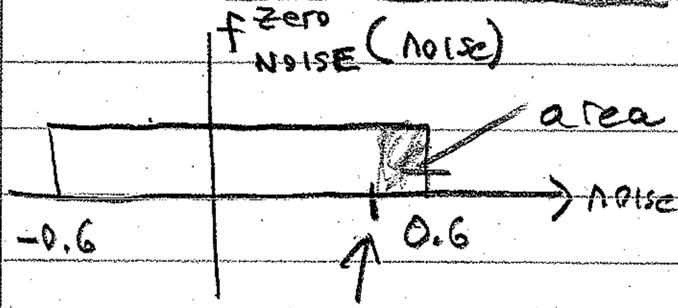
Note: $$\int_{-0.6}^{0.6} f_{NOISE}(x)\,dx = 1.2 \cdot \frac{5}{6} = 1$$

$$\mu_{noise} = 0$$

$$\sigma^2_{noise} = \frac{5}{6}\left(\frac{1}{3}x^3 \Big|_{-0.6}^{0.6}\right) = \frac{3}{25}$$

Probability of error

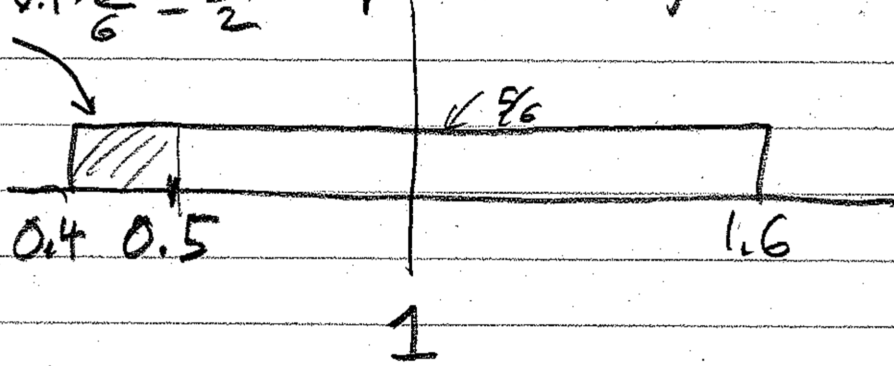
Nominal Zero



area $= 0.1 \cdot \frac{5}{6} = \frac{1}{12} =$ Prob error given bit is a zero $= E_{01}$

Nominal One

area $= 0.1 \cdot \frac{5}{6} = \frac{1}{2} =$ prob err given bit is a one $= E_{10}$



Probability of Error =

$P\ E_{01} \cdot \text{Prob}(\text{bit}=1) + E_{01} \cdot \text{Prob}(\text{bit}=0)$

Equally likely case: $P(\text{bit} = \text{error}) =$

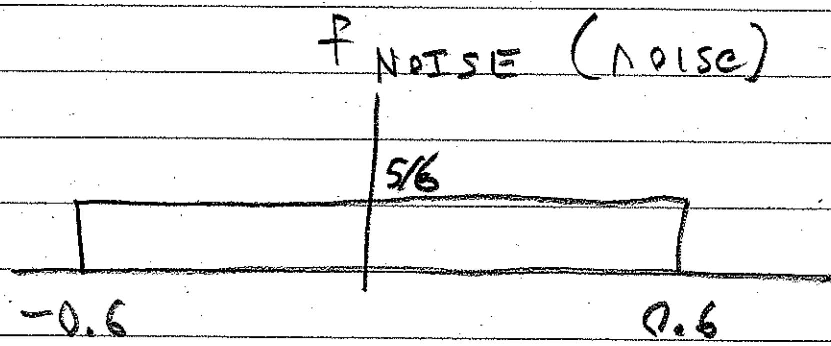
$$\frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{1}{12}$$

$$= \frac{1}{12}$$

Suppose 1^s twice as likely a Zeros

$$P(\text{bit}=\text{err}) = E_{10}\cdot \underset{\substack{\uparrow\\ \text{prob(bit=1)}}}{\frac{2}{3}} + E_{01}\cdot\frac{1}{3}$$

If thresh = 0.5 V and



then

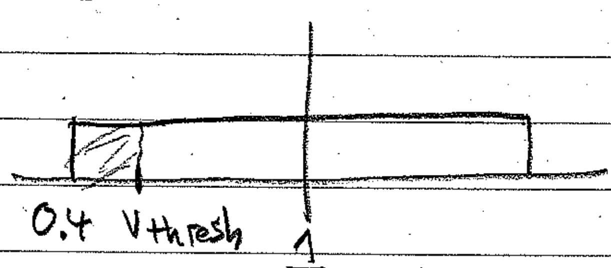
$$P(\text{bit}=\text{err}) = \overbrace{\frac{1}{12}}^{E_{10}}\cdot\frac{2}{3} + \overbrace{\frac{1}{12}}^{E_{01}}\cdot\frac{1}{3} = \frac{1}{12}$$

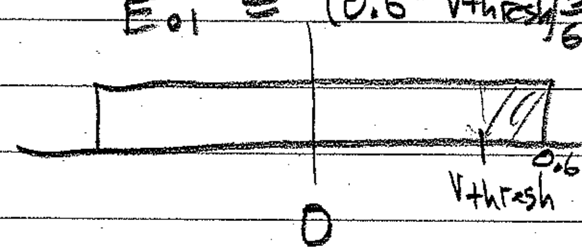
What if threshold is moved?

best threshold if $\frac{2}{3}E_{10} = \frac{1}{3}E_{01}$

$$E_{10} = (V_{thresh} - 0.4)\cdot\frac{5}{6}$$

$$E_{01} = (0.6 - V_{thresh})\frac{5}{6}$$





$V_{thresh} = 7/15$ minimizes error.