Problem 1. (15 points)

(A) (1 point) Two code words A and B have a Hamming distance of 2. If an even parity bit is appended to each code word, producing A' and B', what is the Hamming distance between A' and B'? A and B differ by two bits, so the even parity bit for both will be the same.

Hamming distance between A' and B': 2

(B) (3 points) Consider a channel encoding that simply replicates each bit 8 times, i.e., "0" is encoded as the code word "00000000" and "1" is encoded as the code word "11111111".

- codewords are 8 bits long $\Rightarrow n=8$
- $k=1$
- by inspection: $\left.\begin{matrix}00000000\\11111111\end{matrix}\right\} d=8$

(n,k,d) designation for this code: (8,1,8)

code rate for this code: $k/n = 1/8$

If one uses this code to detect and correct errors of up to C bits, what's the maximum possible value for C?

$C = \lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{7}{2} \rfloor$

Maximum possible value for C: 3

(C) (3 points) Three parity bits (P_1, P_2, P_3) calculated as specified below are concatenated with five message bits $(D_1, \ldots, D_5)$ to form an 8-bit code word. $\oplus$ means XOR (addition modulo 2).

$$P_1 = D_1 \oplus D_2 \oplus D_3$$
$$P_2 = D_2 \oplus D_3 \oplus D_4$$
$$P_3 = D_3 \oplus D_4 \oplus D_5$$

After a transmission involving at most a single bit error, checking the received parity bits indicates parity errors involving P_1 and P_2 but not P_3. What correction (if any) is indicated?

D_2 is the only message bit that is part of the P_1 & P_2 equations, but not P_3.

Correction (if any): D_2

Can this code be used to perform single-bit error correction assuming that the error can occur in any one of the eight code word bits? Briefly explain your reasoning.

Single-bit error correction possible (YES or NO): NO

Can't distinguish between errors to P_1 or D_1, or errors to P_3 or D_5. In other words if only P_1 indicates an error, can't tell if D_1 had an error or P_1 had an error.

(D) (3 points) A $RS(n,k)$ Reed-Solomon code encodes a block of k message symbols into a block of n code word symbols; in this problem each symbol is an 8-bit byte. A $RS(n,k)$ code can detect and correct any combination of E symbol errors and S symbol erasures so long as $2E + S \leq n - k$.

Consider an encoding scheme where a 28-byte data block is encoded as a 32-byte message block using a $RS(32,28)$ code. Then thirty-two 32-byte message blocks undergo a 32-way interleaving, transmitting the first byte of each of the thirty-two message blocks, then the second byte, and so on until all 1024 code word bytes have been sent.

If the transmission is corrupted by one error burst of B bytes, what is the longest burst that can be corrected by the encoding scheme? Briefly explain your reasoning.

RS(32,28) code can correct up to

Maximum value of B: 64

2 errors per message block ($2 \cdot 2 + 0 \leq 32 - 28$). With 32-way interleaving a burst of 64 errors will be distributed so that exactly two errors appear in each deinterleaved block ⇒ correctable! A 65th error would become the third error in one of the deinterleaved blocks.

(E) (5 points) Congratulations! You've been hired by the Registrar to come up with a binary encoding for the gender field of the records database. They want an encoding that allows detection of errors of up to three bits in the gender field (no correction required; they'll use back-up tapes to restore fields where errors have been detected). Assume that three (!) genders (Male, Female, Other) need to be encoded. Please indicate the required Hamming distance between your code words and specify an appropriate code word for each gender. Your code doesn't have to be optimal, but it must support 3-bit error detection.

detect 3-bit errors if $D - 1 \geq 3$ $\Rightarrow D \geq 4$.

Required Hamming distance: 4

Binary code word for Male: 0000 0000

Binary code word for Female: 0000 1111

Binary code word for Other: 1111 0000

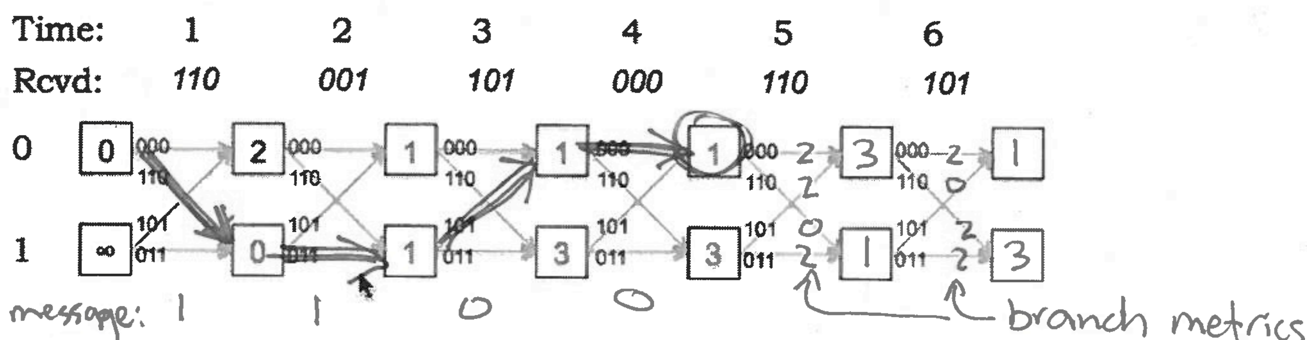
simple replication of a two bit code as many times as necessary ($= D$).

Problem 2. (18 points)

Consider a convolutional code with three generator polynomials

$$
\begin{aligned}
G_0 = 11 \Rightarrow \quad p_0[n] &= \; x[n] \oplus x[n-1] \\
G_1 = 10 \Rightarrow \quad p_1[n] &= \; x[n] \\
G_2 = 01 \Rightarrow \quad p_2[n] &= \; x[n-1]
\end{aligned}
$$

The figure below is a snapshot of the decoding trellis showing a particular state of a maximum likelihood decoder implemented using the Viterbi algorithm. The labels in the boxes show the path metrics computed for each state after receiving the incoming parity bits at time t. The labels on the arcs show the expected parity bits for each transition; the actual received bits at each time are shown above the trellis.



(A) (2 points) What is the code rate r and constraint length k for this code?

• 3 generators ⇒ 3 parity bits per message bit

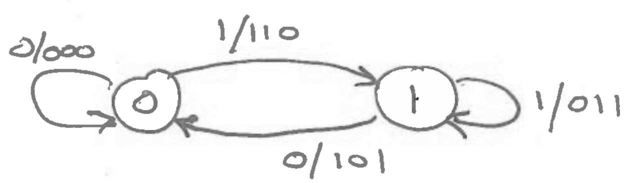
r: 1/3

trellis has 2^{k-1} states ⇒

k: 2

(B) (3 points) Draw the state transition diagram for a transmitter that uses this convolutional code. The states should be labeled with the binary string $x[n-1]...x[n-k+1]$ and the arcs labeled with $x[n]/p_0p_1p_2$ where $x[n]$ is the next message bit and p_0, p_1 and p_2 are the three parity bits computed from $x[n]...x[n-k+1]$ using G_0, G_1 and G_2.

(Draw state transition diagram)



only two states are needed!

(C) (4 points) Fill in the path metrics in the empty boxes in the trellis diagram on the previous page (corresponding to the Viterbi calculations for times 5 and 6).

see previous page **(Fill in path metrics)**

(D) (2 points) Considering the original trellis (i.e., the parts that had already been filled in before you did part C), what is the most-likely final state through time 4? How many errors were detected along the most-likely path to that state?

path metrics are 1 & 3
$\Rightarrow$ 1 is minimum
= # of error detected

Most-likely final state through time 4: 0

Number of errors detected: 1.

(E) (4 points) Again, considering only through time 4, what's the most-likely decoded message? You may find it helpful to mark the most-likely path through the trellis up until time 4. (see marked-up diagram

Most-likely decoded message through time 4: 1100

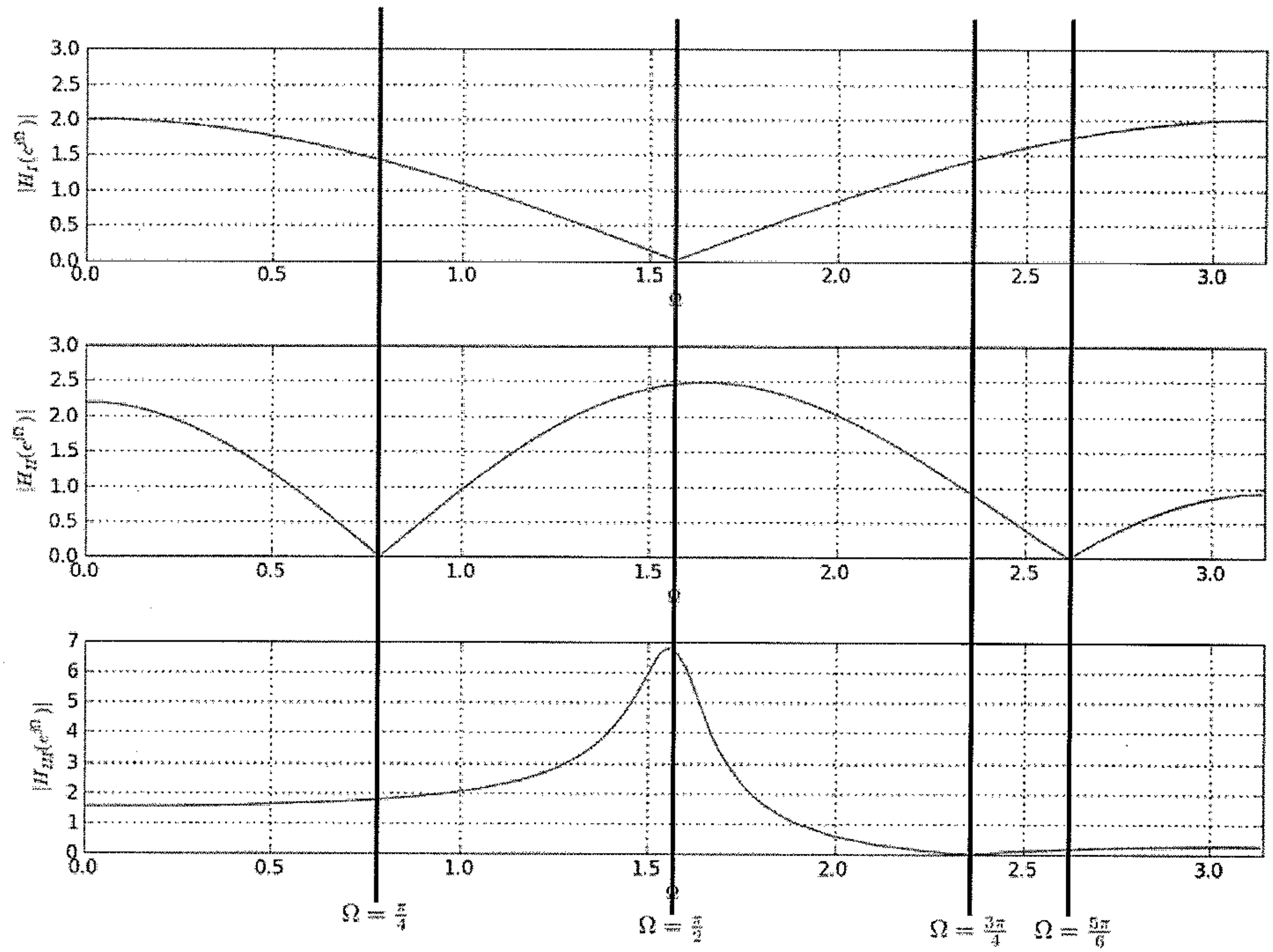
(F) (3 points) Based on your choice of the most-likely path through the trellis up until time 4, at what time(s) did the error(s) occur?

Time(s) error(s) occurred: time = 2

errors happened whenever path metric increments while following most-likely path.

Problem 3 (33 points)

In answering the four parts of this question, consider three linear time-invariant systems, denoted I, II, and III, each characterized by the magnitude of their frequency responses, $|H_I(e^{j\Omega})|$, $|H_{II}(e^{j\Omega})|$, and $|H_{III}(e^{j\Omega})|$, given in the plot below. It may be helpful to recall that $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$.



(A) (7 points) Which frequency response (I, II or III) corresponds to the system

$$(1.1)^2 y[n] + \alpha y[n-1] + y[n-2] = x[n] + \sqrt{2}x[n-1] + x[n-2]$$

and what is the numerical value of α?

$\boxed{\text{System III}}$

This system has a zero at $3\pi/4$: $(1 - e^{+j3\pi/4} z)(1 - e^{-j3\pi/4} z)$

$$= 1 - 2\cos\tfrac{3\pi}{4} z + 1 \cdot z^2$$

$$= 1 + \sqrt{2} z + z^2 \Rightarrow$$

$$x[n] + \sqrt{2}\,x[n-1] + x[n-2]$$

This system has a pole at $\pi/2$

$$(1.1 - e^{j\pi/2} z)(1.1 - e^{-j\pi/2} z) = 1.1^2 - \underbrace{\cancel{2\cos\tfrac{\pi}{2}}}_{=0} + 1 \qquad \boxed{\alpha = 0}$$

Finally at $\pi/2$ $\left| \frac{1 + \sqrt{2}e^{-j\pi/2} + \overbrace{e^{-j\pi}}^{=-1}}{(1.1)^2 + e^{-j\pi}} \right| = \left| \frac{-j\sqrt{2}}{0.21} \right| \approx \frac{1.41}{0.21} \approx 6.7$

(B) (9 points) Which frequency response (I, II or III) corresponds to the system

$$y[n] = x[n] + (\sqrt{3.0} - \sqrt{2.0})x[n-1] + \beta x[n-2] + (\sqrt{3.0} - \sqrt{2.0})x[n-3] + x[n-4]$$

and what is the numerical value of β?

System II has two zeros $e^{j\pi/4}$, $e^{j5\pi/6}$

$$\Big(1 - 2\overbrace{\cancel{\cos\tfrac{\pi}{4}}}^{\sqrt{2}/2} z + z^2\Big)\Big(1 - 2\overbrace{\cancel{\cos\tfrac{5\pi}{6}}}^{-\sqrt{3}/2} + z^2\Big)$$

$$= (1 - \sqrt{2} z + z^2)(1 + \sqrt{3} z + z^2)$$

$$= 1 + (\sqrt{3} - \sqrt{2}) z + \beta z^2 + (\sqrt{3} - \sqrt{2}) z^3 + z^4$$

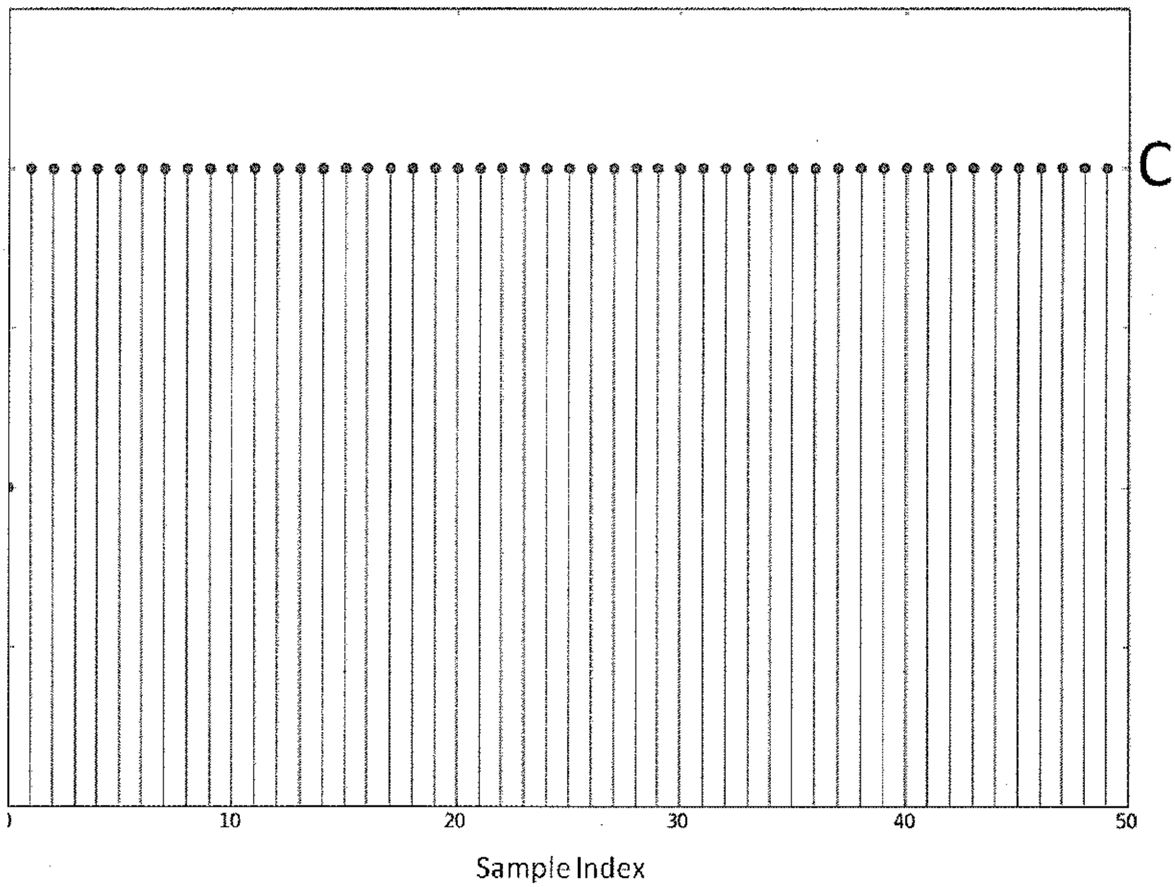
$$\Rightarrow x[n] + \ldots\ldots + \beta x[n-2] + \ldots\ldots + x[n-4]$$

$$\beta = 1 + 1 - \sqrt{2}\sqrt{3} = 2 - \sqrt{6} \approx -0.45$$

(C) (7 points) Suppose the input to each of the above three LTI systems is zero for $n < 0$ and

$$x[n] = \sin(\frac{\pi}{2.0}n) + 1.0 = \sin(\frac{\pi}{2.0}n) + \cos(0 \cdot n)$$

for $n \geq 0$. Which system, (I, II or III), produced the following plot of the output $y[n]$, and what is the value of C in the plot?



Eventually $y[n] = |H(e^{j\frac{\pi}{2}})| \sin(\frac{\pi}{2}n + \phi_a)$

$+ |H(e^{j0})| \cdot (1 \text{ or } -1)$

Since $y[n]$ does not oscillate $H(e^{j\pi/2}) = 0$

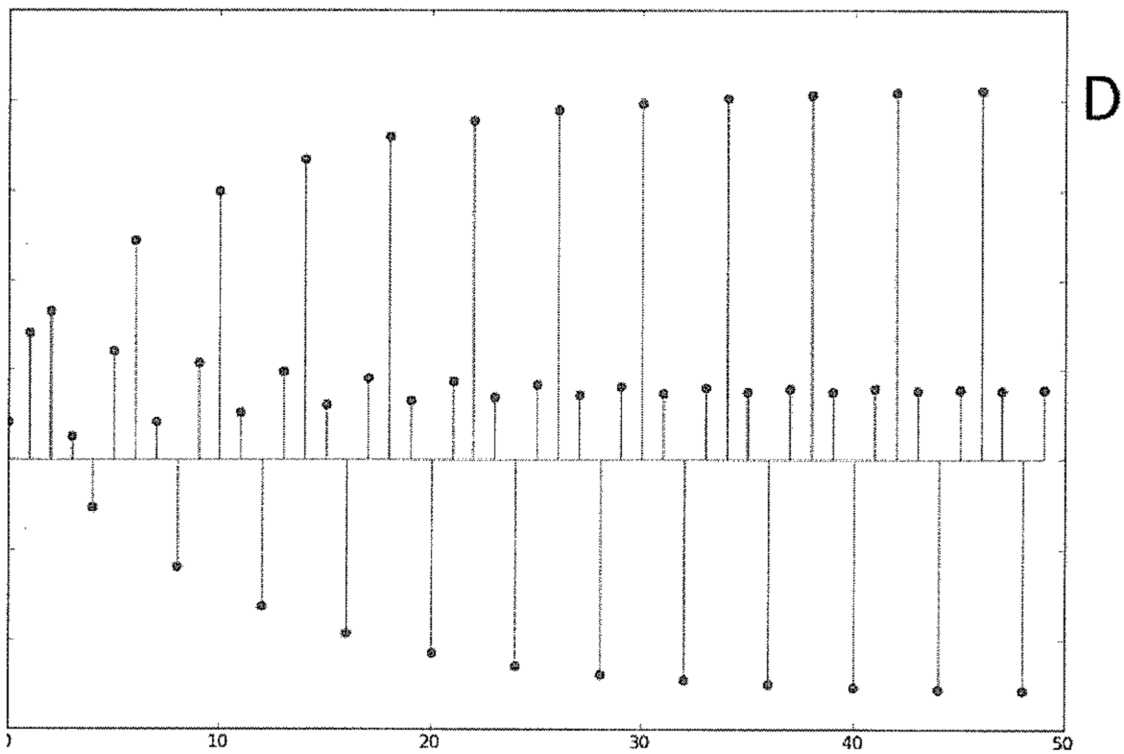
System I

and $|H(e^{j0})|$ for system 1 $= 2 \Rightarrow \boxed{C=2}$

(D) (10 points) Suppose the input to each of the above three LTI systems is zero for $n < 0$ and

$$x[n] = \sin(\frac{\pi}{2.0}n) + 1.0 = \sin(\frac{\pi}{2.0}n) + \cos(0 \cdot n)$$

for $n \geq 0$. Which system, (I, II or III), produced the following plot of the output $y[n]$, and what, approximately (within 10 percent) is the value of D in the plot?



Since $y[n] = |H(e^{j\pi/2})| \sin(\frac{\pi}{2}n + \phi) + |H(e^{j0})| \cdot (1 \text{ or } -1)$

and since the sin is only shifted upwards a fraction of its amplitude

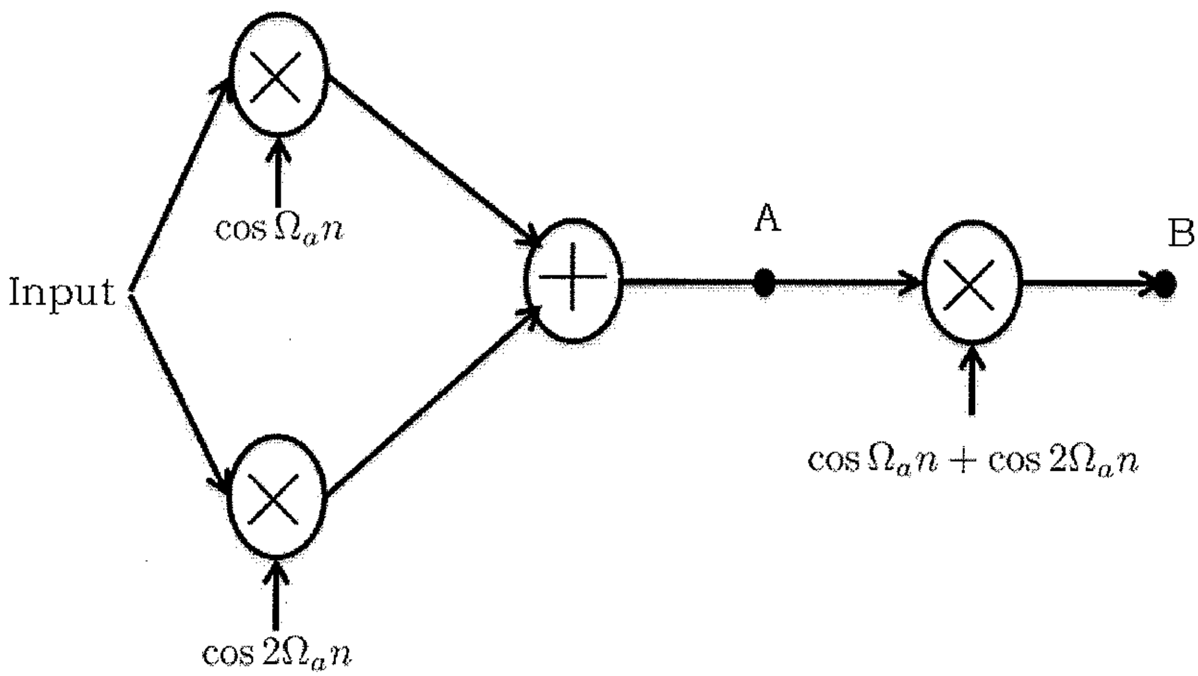
$|H(e^{j0})| < |H(e^{j\pi/2})| \Rightarrow \boxed{\text{System III}}$

Amplitude of $\sin(\pi/2\, n + \phi) \doteq 6.5$ $|H(e^{j0})| \simeq 1.5$

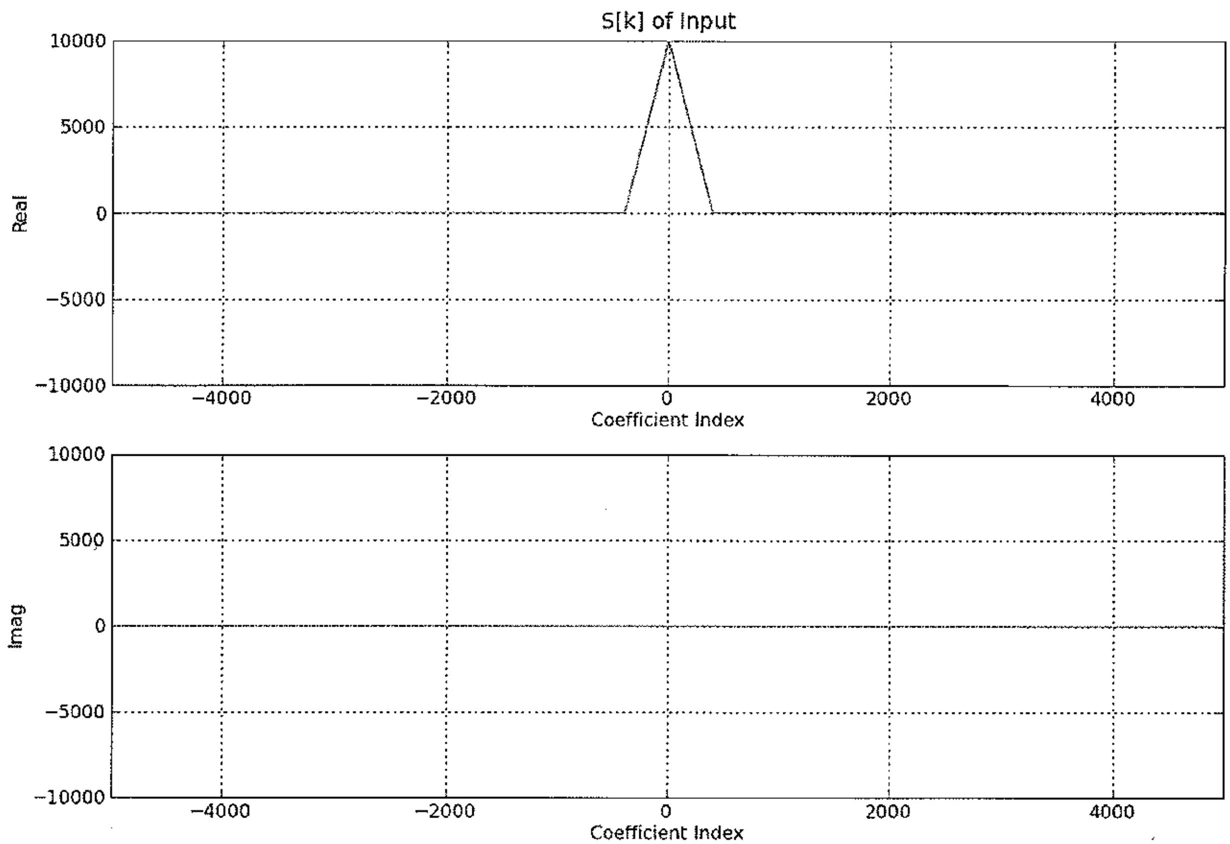
so $\boxed{D = 8}$

Problem 4 (20 points)

Consider the simple modulation-demodulation system below, where $\Omega_a = \frac{1000*2*\pi}{10001}$.

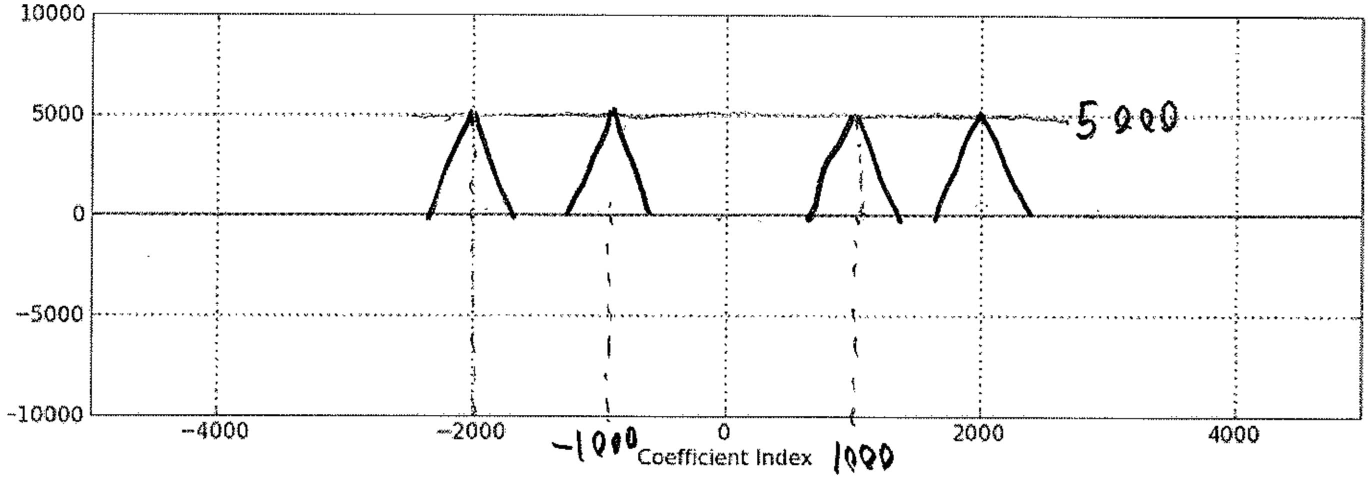


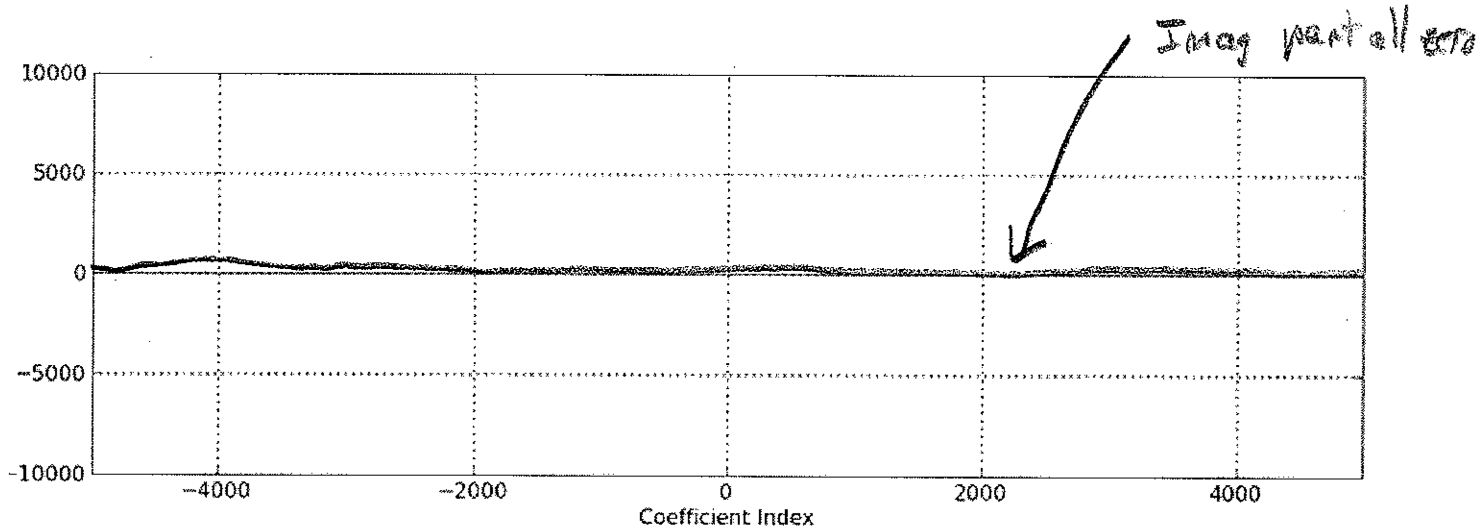
The Fourier Series coefficients for the input to the modulation-demodulation system is plotted below for the case $N = 10001$. Note that the Fourier coefficients are nonzero only for $-400 \leq k \leq 400$.



On the two sets of axes below, please plot the Fourier series coefficients for the signal at location **A** and **B** in the above diagram. Be sure to label key features such as values and coefficient indices for peaks.

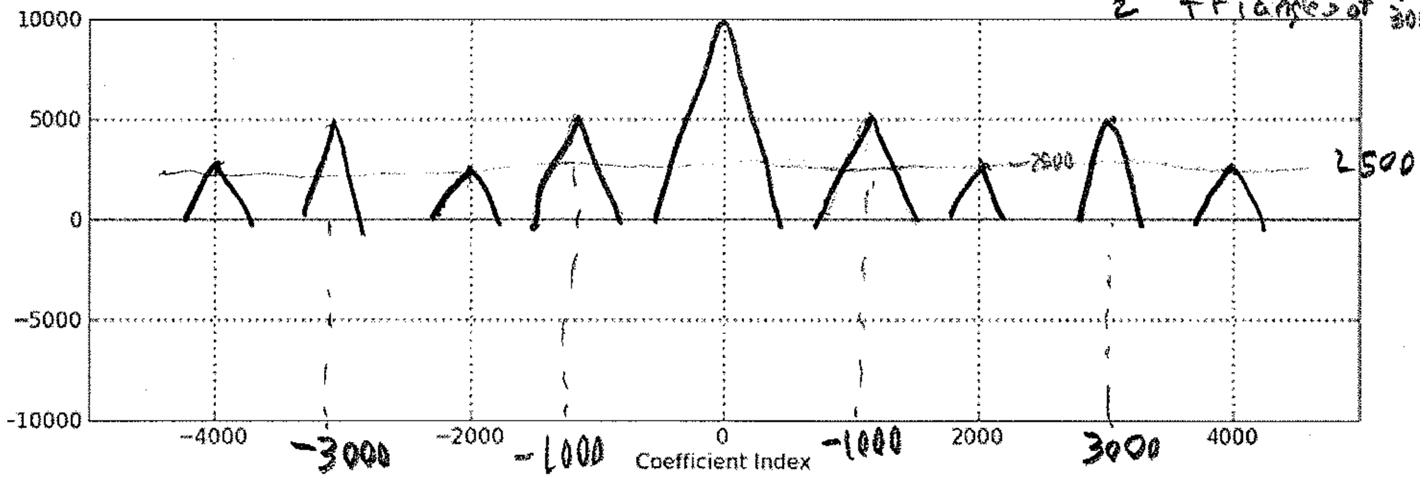
Plot of Fourier Coefficients of signal at Point A



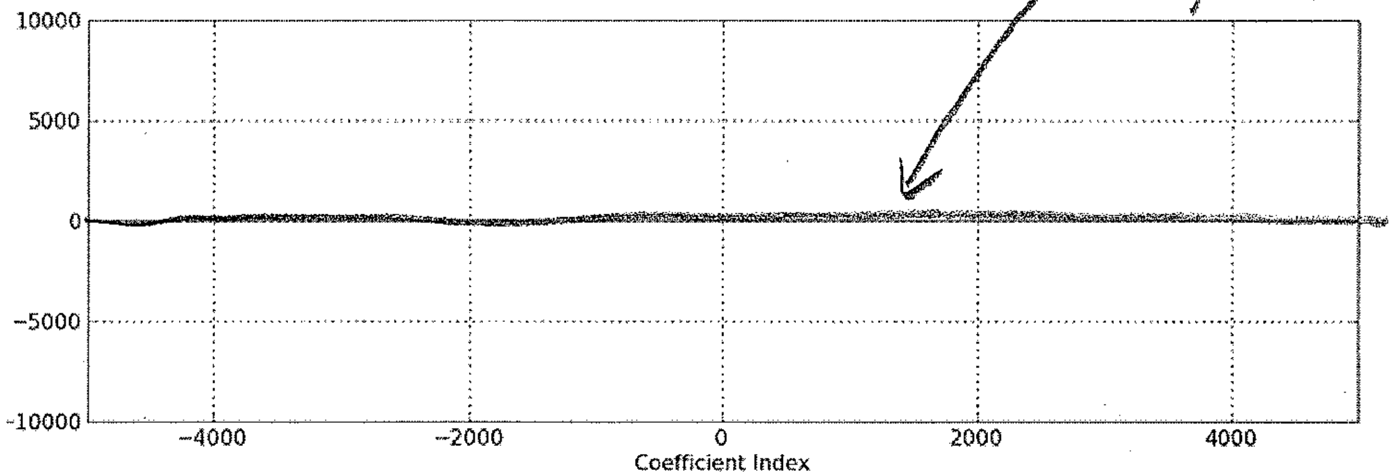


4 triangles at zero
2 triangles at 1000, -1000
1 triangle at 2000, -2000
1/2 triangles at 3000, -3000

Plot of Fourier Coefficients of signal at Point B

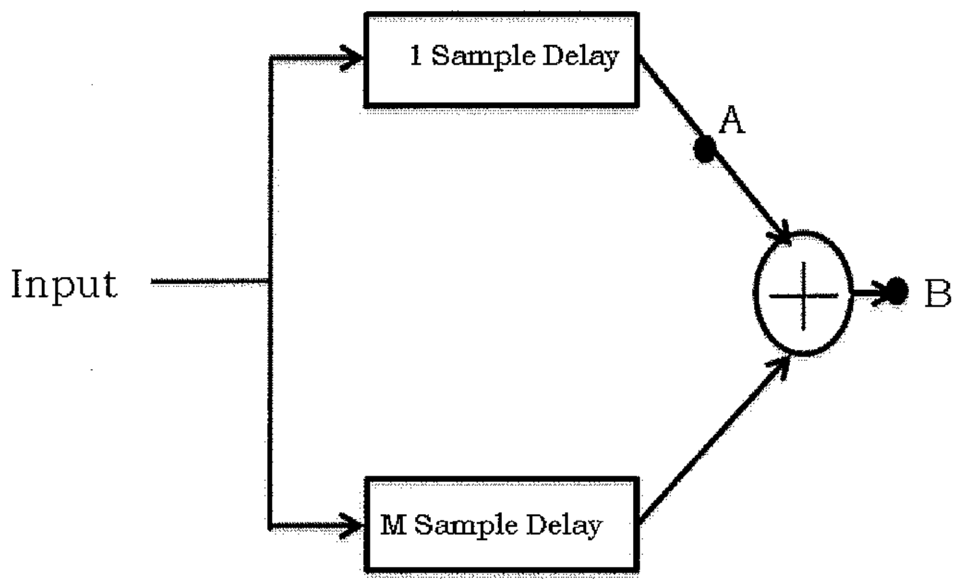


Imag = 0

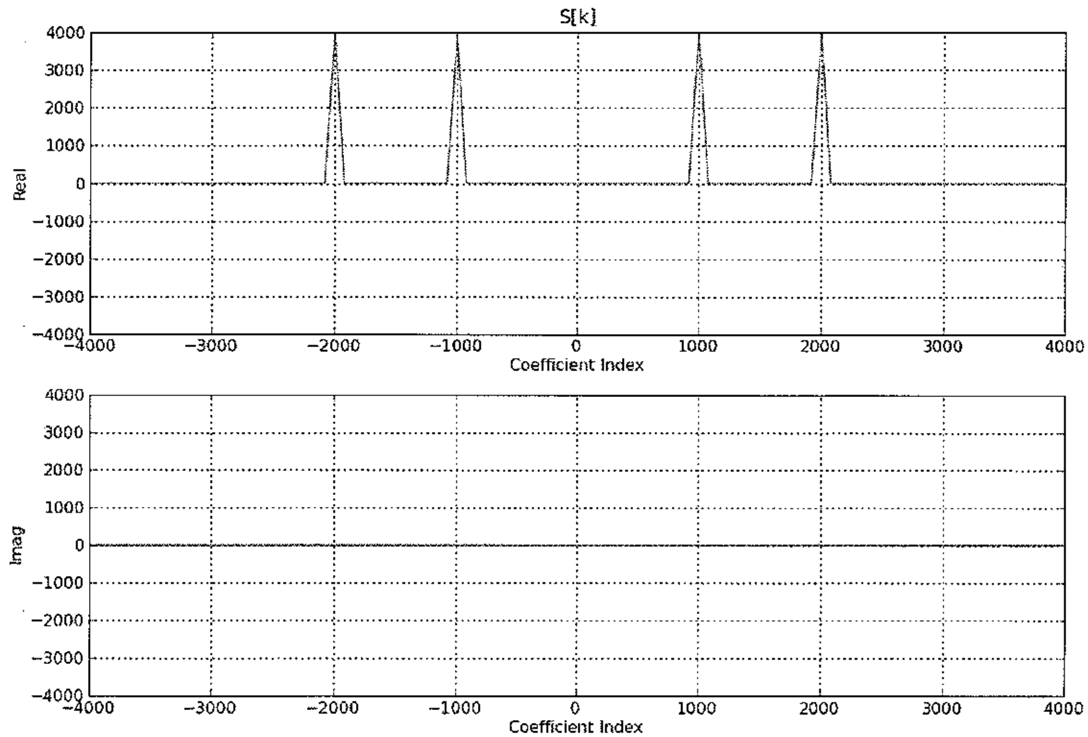


Problem 5 (14 points)

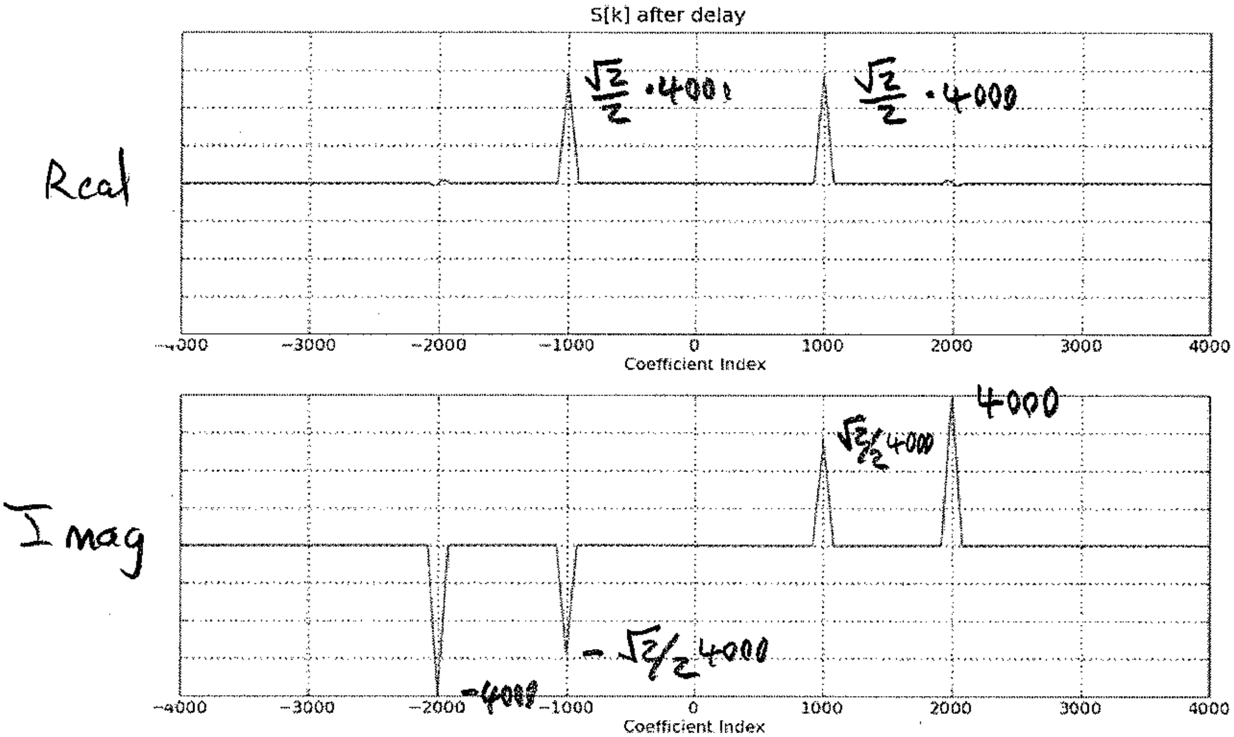
Consider the multiple delay system diagrammed below.



The input to the multiple delay system is a modulated signal that is periodic with period $N = 8001$. The Fourier Series coefficients for this modulated signal are plotted below.



(A) (10 points) Below are plots of the real and imaginary parts of the Fourier coefficients for point **A** in the multiple delay system. Determine the numerical values for the six peaks in the plots.



1 sample delay $\Rightarrow 4000\underbrace{\left(e^{j\,1000\,\frac{2\pi}{8001}(n-1)} + e^{-j\frac{2000\pi}{8001}(n-1)}\right)}_{\text{for the peak}}$

$\pi/4$ shift $\Rightarrow \cos\pi/4 + j\sin\pi/4 = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$

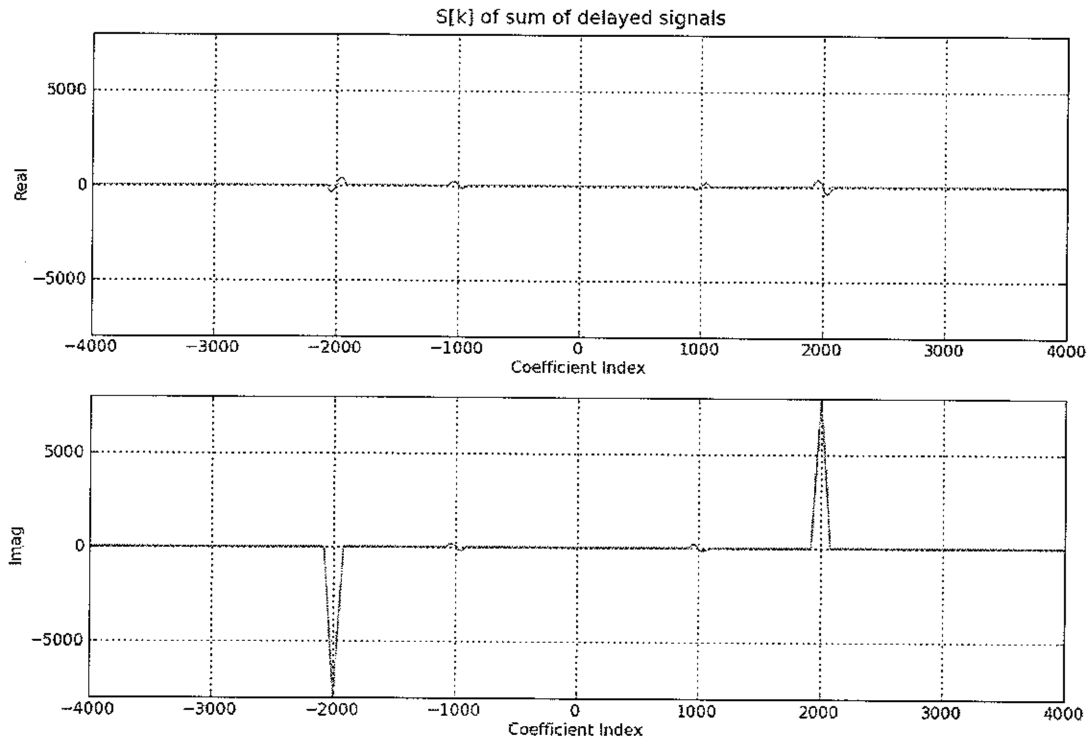
$+\,4000\underbrace{\left(e^{\frac{4000\pi}{8001}(n-1)} + e^{-j\frac{4000\pi}{8001}(n-1)}\right)}_{\text{for the peak}}$

$\pi/2$ shift

$\cos\pi/2 + j\sin\pi/2$

$= j$

(B) (4 points) Use the following plot of the Fourier series coefficients for the sum of the delayed signals (point **B** in the multiple delay diagram), to determine the smallest integer value for M, the number of samples in the second delay.



Must add a 1 sampled delayed signal to an M sample delayed signal and zero out the $\Omega = 1000 \frac{2\pi}{8001}$ term. M

$$e^{j(M-1)1000\frac{2\pi}{8001}} \cong -1 \cong e^{j\pi} \Rightarrow M-1 = 4$$

$$\boxed{M = 5}$$

End of Quiz 2!