6.02 Spring 2010
Lecture #3

- Good Channel $\rightarrow$ Eye's wide open
- LTI Channel $\Leftrightarrow$ Convolution with Unit Sample Response
- Use USR to Deconvolve: Bad Channel $\rightarrow$ Good Channel
- Deconvolution and Noise
Bits $\rightarrow$ Samples $\{X\}$ $\rightarrow$ Channel $\rightarrow$ Samples $\{Y\}$ $\rightarrow$ Bits

$X \equiv$ entire sequence
$x[n] \equiv n^{th}$ sample value

$Y \equiv$ entire sequence
$y[n] \equiv n^{th}$ sample value

Sample Rates: $\rightarrow$ 4 million Samples/Second (IR Transceiver)
$\rightarrow$ Up to gigaSamples/Second (Fastest)
Excellent Channel Transmission (using 100 Samples/bit)

$x[n]$  

$y[n]$
Excellent Channel Transmission Eye Diagram

- **Ones**
- **Eye Wide Open**
- **Threshold**
- **Zeros**
Okay Channel Transmission
(using 25 Samples/bit)

$x[n]$  
Xmit Voltage

$y[n]$  
Rcvr Voltage

Sample Number

Threshold

One Just Above Thresh
Zero Just Below Thresh

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Okay Channel Transmission Eye Diagram

Ones

Zeros

Threshold

Eye Not Wide Open

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Unusable Channel Transmission
(15 Samples/bit)

$x[n]$ Xmit Voltage

$y[n]$ Rcvr Voltage

Sample Number
Unusable Channel Transmission Eye Diagram

Threshold

No Eye.
What does an Eye Diagram Tell You?

• How many samples to use per bit
  – More samples/bit:
    • More open eye $\rightarrow$ Easier detection $\uparrow$
    • Fewer bits/second $\rightarrow$ Slower bit rate $\downarrow$

• Where to put the 1-0 threshold
  – The threshold should evenly divide the upper and lower parts of the eye.

• How to estimate errors
  – Next Week.
Faster Bit Rate with Post-Channel Processing?

- Model Wire as Causal, Linear Time-Invariant (CLTI)
- Use Unit Sample Response to Undo Channel Effects

\[ X \rightarrow \text{Channel is CLTI} \rightarrow Y \rightarrow \text{Processor Undoes Channel} \rightarrow W \approx X \]

If \[ W = X \] then only need 1 sample/bit (Perfect deconvolution)
A Glimpse at IR Channel Results

IR Channel, Light On, 50 samples/bit

Eye diagram for channel IR

Eye Open but not wide open

IR Channel plus Deconvolved, 50 samples/bit

Eye Wide Open

Deconvolved IR Channel, 5 samples/bit

Eye Wide Open 10x Faster bit rate
Causality, Time-Invariance and Linearity

- **Causality (out does not change before in)**
  - If \( x[n] = x[-\infty] \) for \( n < N \)
  - Then \( y[n] = y[-\infty] \) for \( n < N \).

- **Time-Invariance (out shifts when in shifts)**
  - If \( \text{in} = x[n] \Rightarrow \text{out} = y[n] \)
  - Then \( \text{in} = x[n - k] \Rightarrow \text{out} = y[n - k] \)

- **Linearity**
  - If \( \text{in} = x_a[n] \Rightarrow \text{out} = y_a[n] \), \( \text{in} = x_b[n] \Rightarrow \text{out} = y_b[n] \)
  - Then \( \text{in} = A x_a[n] + B x_b[n] \Rightarrow \text{out} = A y_a[n] + B y_b[n] \)
Unit Sample Reminder

\[ \delta[n] \]

\[ \delta[n-7] \]
Unit Sample Response (slow wire)

\[ \delta[n] \]

\[ h[n] \]

Notice \( h[0] = 0 \) and \( h[1] = 0 \) for 40 samples, but it was two bits.

Notice \( h[0] \) is nonzero for 2 bit periods, then two previous bits interfere with current bit.
Shifted and Scaled Unit Sample Response

0.5\delta[n-7]

0.5h[n-7]

(Note: $X[0] = 0$ for $n > 2$ and $n < 0$.)
In General for LTI

\[ x[n] = x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2] + \ldots \]


In summation form:

\[ y[n] = \sum_{m=0}^{n} h[n-m] x[m] \]

Because \( h[0, m]=0 \) for \( m<0 \)

Because \( x[m]=0 \) for \( m<0 \)

Causality
Unit step response cannot be nonzero before the unit step occurs. \( h[n]=0 \) for \( n<0 \)

Notation:
Convolution symbol

\[ \bigtriangledown = H \ast x \]

sequences

Side Note: Polynomial multiplication

\[ \left( a[0] + a[1] z + a[2] z^2 + \ldots \right) \left( b[0] + b[1] z + \ldots \right) = \left( a[0] b[0] + a[1] b[0] z + \ldots + a[n] b[0] z^n + \ldots \right) \]
Alternative Form

\[ y[n] = \sum_{m=0}^{0} x[n-m] h[m] \]


Suppose \( h[n] = 0 \) \( n > B \)

\[ y[n] = h[0] x[n] + h[1] x[n-1] + \ldots + h[B] x[n-B] \]

(If \( n < K \), equation still holds as \( x[n-K] = 0 \) because \( x[n] = 0, n < 0 \))

\[ \text{Difference Equation} \]

*** Aside for the curious

Compare to 6.0 Right-shift operator \( \mathcal{R} \)

\[ \mathcal{R} x \Rightarrow y[n] = x[n-0] \]

\[ \mathcal{R}^2 x \Rightarrow y[n] = x[n-2] \]

\[ \Rightarrow y = (h[0] + h[1] \mathcal{R} + \ldots + h[K] \mathcal{R}^K) x \]

\[ = (\sum_{i=0}^{K} h[i] \mathcal{R}^i) x \]
We have

\[ y[n] = \sum_{m=0}^{\infty} h[n-m] \times c[m] \]

Introduce an intermediate variable

\[ l = n - m \Rightarrow m = n - l \]

Therefore

Rewriting

\[ y[n] = \sum_{l=0}^{\infty} h[l] \times c[n-l] \]

Since \( m \) and \( l \) are just used for indexing, relabel \( l \) with \( m \)

\[ y[n] = \sum_{m=0}^{\infty} x[n-m] \times h[m] \]

Alternate Form
$Y = \left( \sum_{i=0}^{k} h_{ci} x_i \right) x$

$W = \text{reconstructed } X$

How to derive $W$ from $Y$:

$W = \left( \sum_{i=0}^{k} h_{ci} y_i \right)^{-1} Y$

or

$\left( \sum_{i=0}^{k} h_{ci} x_i \right) W = Y$

Deconvolution

$x \rightarrow \text{channel} \rightarrow Y \rightarrow \text{Decom} \rightarrow W$

$h[n] w[n]+h[n] w[n-1]+\ldots+h[n-k] w[n-k] = y[n]$

Answer: Since $W$ reconstructs $X$: $W \approx X$.

It should satisfy the same difference equation as $X$.

See more general derivation in Aside.
Solving for $W$

Plug & Chug

$$W_{[0]} = \frac{1}{n_0} \left[ Y_{[0]} - (h_{[1]} W_{[-1]} + \ldots + h_{[k]} W_{[-k]}) \right]$$

All zero as $W_{[-k]} \rightarrow 0$

$$W_{[1]} = \frac{1}{n_0} \left( Y_{[1]} - h_{[1]} W_{[0]} \right)$$

$$W_{[2]} = \frac{1}{n_0} \left( Y_{[2]} - h_{[1]} W_{[1]} + h_{[2]} W_{[0]} \right)$$

$$\vdots$$
**Noise-Free Deconvolution Result** ($h[n] = 0, \ n \geq 120$)

- **Channel 1 Transmitted Samples** ($x[n]$)
- **Channel 1 Received Samples** ($y[n]$)
- **Channel 1 Deconvolved Samples** ($w[n]$)

*Notice small error; error appears near sample 120.*

Lecture 3, Slide #16

*6.02 Fall 2009*
Noise-Free Deconvolution Result ($h[n] = 0, n >= 100$)

channel1 Transmitted Samples

channel1 Received Samples

channel1 Deconvolved Samples

Notice: Worse Errors less accurate channel model.

Error appears near sample 100
How bad is noise for deconvolution?

Unit Sample Response of IR channel with lights on, lots of noise.

Eye diagram generated from deconvolution using above U.S.R. => Just junk!
Example $x[n]$ and $h[n]$
Evaluating $y[0] = \sum_{m=0}^{0} h[0-m] \cdot x[m]$

$x[m]$

$h[0-m]$

Flipped $h$
Evaluating $y[1] = h[0] \cdot x[0] + h[1] \cdot x[0]$

Flipped $h$
Shifted by 2
Relating Step Response to Unit Sample Response

\[ x \rightarrow \text{Channel} \rightarrow y \]

If \( x[n] = u[n] \)
\[ \begin{align*}
    x[0] &= u[0] \\
    x[-2] &= u[-2] \\
    x[-1] &= u[-1] \\
    x[-1] &= u[-1] \\
\end{align*} \]

\( y[n] = s[n] \) (step response)

\( h[n] \)

Example:

\( -3 \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \)

If \( x[n] = s[n] \) then \( x[n] = u[n] - u[n-1] \)

\( y[n] = s[n] - s[n-1] = h[n] \)

\( h[n] \)
If \( x[n] = u[n] \)

\[
y[n] = \sum_{m=0}^{n} h[m] \cdot x[n-m]
\]

\[
y[n] = \sum_{m=0}^{n} h[m] \cdot u[n-m]
\]

\[
y[n] = \sum_{m=0}^{n} h[m]
\]

**Example**

\[
h[n]
\]

\[
\begin{array}{cccccc}
-1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
y[0] = h[0] = 0
\]

\[
y[1] = h[0] + h[1] = \frac{1}{4}
\]

\[
\]

\[
y[3] = \sum_{m=0}^{2} h[m] = 3/4
\]

\[
y[4] = \sum_{m=0}^{3} h[m] = 1
\]

\[
y[n] = 1 \quad n > 4
\]
Convolution Examples

1) $x[n]$ $h[n]$ $\rightarrow y[n]$

$y[n] = h[n] \ast x[n]$

Note $y[n]$'s maximum value is 6 at $n=2$. Easy to see from flip & slide.
Example 2)

\[ x[n] \]

\[ h[n] \]

Flip & Slide

\[ h[3-n] \]

Note: Example 1 \[ \square \ast \square = \Delta \] Each Convolution smooths

Example 2 \[ \Delta \ast \square = \Delta \]
Example 3)

\[ x_1[n] = u[n] \]
\[ x_2[n] = (-1)^n u[n] \]

\[ h[n] = \frac{1}{2} \cdot g[n] \]
\[ y_1[n] = \frac{1}{2} \cdot g[n] \]
\[ y_2[n] = \frac{1}{2} \cdot g[n] \]

Example 4)

\[ h[n] = 8[n-2] \]
\[ y[n] = x[n-2] \]
Deconvolution Example

Example 1)

\( x[n] = u[n] \)
\( h[n] = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1] \)

\( \Rightarrow y[n] = \frac{1}{2} \delta[n+1] \)

\[ h[0]w[0] + h[1]w[-1] = y[0] \]
\[ w[0] = \frac{1}{2} (y[0] - h[1]w[-1]) \]
\[ w[0] = \frac{1}{2} (y[0] - h[1]w[-1]) \]
\[ w[0] = 1 \]

\( w[1] = \left( \frac{1}{2} \right) \left( y[1] - h[0]w[0] \right) \]
\[ w[1] = \left( \frac{1}{2} \right) \left( y[1] - h[0]w[0] \right) \]
\[ w[n] = 1, \ n \geq 0 \Rightarrow \overline{w} = \bar{X} \]

Example 2)

\( x[n] = (-1)^n u[n] \), same \( h[n] \)

\[ y[n] = \frac{1}{2} \]

\[ w[n] = \left( \frac{1}{h[0]} \right) \left( y[n] - h[n]w[n-1] \right) \]
\[ w[n] = \left( \frac{1}{h[0]} \right) \left( y[n] - h[n]w[n-1] \right) \]
\[ w[n] = z \left( y[n] - \frac{1}{2} w[n-1] \right) \]
Example 2 (Cont.)

\[ W[0] = 2 \left( y[0] - \frac{1}{2} W[-1] \right) \]
\[ = 2 \left( \frac{1}{2} - 0 \right) = 1 \]

\[ W[1] = 2 \left( y[1] - \frac{1}{2} W[0] \right) \]
\[ = -1 \]

\[ W[2] = 2 \left( y[2] - \frac{1}{2} W[1] \right) \]
\[ = 1 \]

\[ W[3] = (-1)^n u[n] \]
Example 3

\[ x[n], \quad h[n] = 3[n-2] \]

\[ \Rightarrow y[n] = x[n-2] \]

Deconvolution Equations


\[ \Rightarrow W[n-2] = y[n] = x[n-2] \]

\[ \frac{W[n]}{W[n+2]} = y[n+z] \]

\[ \Rightarrow W[n-z] = x[n-2] \]

\[ W[n+z] = x[n] \]

Note: To compute \( W[n] \), you need \( y[n+z] \). The deconvolution equation does NOT describe a causal system.