6.02 Spring 2010
Lecture #10

- Summary of progress
- The problem of multiplexing
- Sines $\rightarrow$ LTI $\rightarrow$ Sines by experiment
- Discrete-Time Complex Exponentials
- Frequency Response
So Far - Low BER Transmission on Wires

- Problems and Analysis Techniques:
  - Intersymbol Interference (ISI)
    - LTI Systems and Unit Sample Responses
    - Eye Diagrams
  - Noise
    - Probability Density and Cumulative Distribution Functions
    - Normal (Gaussian) random variables

- Solution Approaches
  - Techniques based on LTI models
    - Deconvolution and Decision Feedback Equalization
  - Error Detection and Correction Codes
    - Parity bits, Reed-Solomon Codes, Viterbi Algorithm
New Problem - Resource Sharing

Two Approaches
- Time Division Multiplexing (TDM)
  - Each Xmit-Rcvr pair gets a time slot (how to decide?)
  - Used by wired internet
- Frequency Division Multiplexing (FDM)
  - Each Xmit-Rcvr gets part of the spectrum (explained later)
  - Like Broadcast TV and Radio, many wireless devices

Next Two Weeks on FDM
Sinusoids and LTI Systems

Three sinusoids of different frequency

\[ x_{35}[n] = \frac{2}{7}(x_1[n] + x_2[n] + x_3[n] + \frac{7}{4}) \]

Summed then scaled and shifted sinusoids
Sinusoids and LTI Systems

Summed then scaled and shifted sinusoids

IR System Output from Summed sinusoids
Sinusoids and LTI Systems

Summed then scaled and shifted sinusoids

IR System (off ceiling) Output from Summed sinusoids
IR System Response to Different Sines
Sinusoids in LTI Systems

• Sinusoids $\rightarrow$ LTI Systems $\rightarrow$ Sinusoids
  – Different Frequencies do not seem to mix
    • Xmit: encode data on different frequencies (Modulation)
    • Rcvr: decode data from different frequencies (Filtering)
  – Different Frequencies have different “Gain”
    • E.G. the IR Channel:
      – higher frequencies $\rightarrow$ lower “gain”.

• Analyzing Sinusoids in LTI systems
  – Model as Eternal signals ($-\infty < n < \infty$)
  – Difficulty Computing response to $x[n] = \cos \Omega n$
  – Simplify response computation using complex exponentials

$$\cos \Omega n = \frac{e^{j\Omega n} + e^{-j\Omega n}}{2} \quad \sin \Omega n = \frac{e^{j\Omega n} - e^{-j\Omega n}}{2j}$$
Discrete Time Cosines $-\infty < n < \infty$

$\cos(\pi/8)n$

$\cos(\pi/4)n$

$\cos(\pi/2)n$

$\cos(3\pi/4)n$

$\cos\pi n$

Why is pi highest frequency?

Lecture 1, Slide #10
Correspondence

Suppose \( f_s = 4 \cdot 10^6 \)

\[ \omega = \frac{1}{8} \quad \text{2 500 000 cycle/sec} \]

\[ \cos \frac{1}{8} \pi n \quad \Rightarrow \quad \cos 2 \pi \left( \frac{f_s}{16} \right) \]

\[ \cos \frac{\pi}{8} n \quad \Rightarrow \quad \cos 2 \pi \left( \frac{f_s}{8} \right) \quad \text{500 000 cycles} \]

\[ \cos \frac{3\pi}{4} n \quad \Rightarrow \quad \cos 2 \pi \left( \frac{3f_s}{8} \right) \quad \text{1 500 000 cycles} \]

\[ \cos \frac{\pi}{2} n \quad \Rightarrow \quad \cos 2 \pi \left( \frac{f_s}{2} \right) \quad \text{2 000 000 cycles} \]

Sampling at \( 4 \cdot 10^6 \) continuous waveform
LTI Systems for Eternal Sines and Cosines

- Convolution Changes a little

\[ y[n] = \sum_{m=0}^{\infty} h[m] x[n - m] = \sum_{m=0}^{\infty} h[m] \cos \Omega[n - m] \]

- \( x[n] \) is eternal

- \( h[m] \) is still causal

- If unit sample response finite length (K)

\[ y[n] = \sum_{m=0}^{L} h[m] \cos \Omega[n - m] \]

How do we simplify?
Frequency Response with Complex Exponentials

- From convolution

\[ y[n] = \sum_{m=0}^{L} h[m]e^{j\Omega(n-m)} \]

Reorganizing

\[ y[n] = \left( \sum_{m=0}^{L} h[m]e^{-j\Omega m} \right) e^{j\Omega n} \]

A complex number if the sum converges

\[ y[n] = H(e^{j\Omega}) e^{j\Omega n} \]

\[ H(e^{j\Omega}), -\pi < \Omega < \pi, \text{ is the frequency response} \]
Most Interested in Magnitude

\[ |a + bi| = \sqrt{a^2 + b^2} \]

\[ \angle \theta = \frac{b}{a} \]

\[ |e^{j\theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \]

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

\[ 1 = e^0 = e^{-j\pi} = e^{j\pi} \]
Going back to Cosines

• Suppose

\[ x[n] = \cos \Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n} \]

\[ y[n] = \left( \sum_{m=0}^{L} h[m] e^{-j\Omega m} \right) \frac{e^{j\Omega n}}{2} + \left( \sum_{m=0}^{L} h[m] e^{j\Omega m} \right) \frac{e^{-j\Omega n}}{2} \]

\[ y[n] = \frac{1}{2} H \left( e^{j\Omega} \right) e^{j\Omega n} + \frac{1}{2} H \left( e^{-j\Omega} \right) e^{-j\Omega n} \]

• Just using LTI superposition property!
Channel 1 and 2 Frequency Responses

Unit-sample response for Channel 1

Magnitude of $H(e^{j\Omega})$ for Channel 1

Unit-sample response for Channel 2

Magnitude of $H(e^{j\Omega})$ for Channel 2
Non-eternal Example: Cosine starts at zero

\[ x[n] = \cos(\Omega n)u[n] \quad \Omega = \left\{ \frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10} \right\} \]

Slow Channel

Fast Channel
Summary and Next Time

- **Frequency Division Multiplexing**
  - Eternal frequencies do not mix when input to LTI systems
  - Frequency division multiplexing
  \[ x[n] = A_1 e^{j\Omega_1 n} + ... + A_K e^{j\Omega_K n} \]
  - Need Filters at receiver to separate messages.

- **Next Week**
  - How do we analyze more complicated messages
  \[ x[n] = x_1[n] e^{j\Omega_1 n} + ... + x_K[n] e^{j\Omega_K n} \]
  - Do the modulated messages still stay separated?
  - Fourier Analysis and Spectrum
  - Modulation Schemes
Only consider $-\pi \leq \omega \leq \pi$

Highest frequency

\[ e^{j \pi n} = \cos(\pi n) + j \sin(\pi n) \]
\[ = \cos(\pi n) = (-1)^n \]

Consider $\omega = \pi + \Delta$, $\Delta > 0$ (outside $-\pi \rightarrow \pi$)

\[ e^{j(\pi + \Delta) n} = \cos((\pi + \Delta) n) + j \sin((\pi + \Delta) n) \]
\[ e^{j\pi n} e^{j\Delta n} = (-1)^n \left( \cos(\Delta n) + j \sin(\Delta n) \right) \]
\[ e^{j(-\pi + \Delta) n} = e^{j(-\pi) n} e^{j\Delta n} \]
\[ = (-1)^n \left( \cos(\Delta n) + j \sin(\Delta n) \right) \]
\[ e^{j(\pi + \Delta) n} = e^{j(-\pi + \Delta) n} \]

$\omega = \pi + \Delta$ same as $\omega = \Delta - \pi$

outside $-\pi \rightarrow \pi$

Inside $-\pi \rightarrow \pi$
Connection To Sample Rate

Continuous time Signal

\[ X(t) = \cos 2\pi f t \]

\[ X(N) = \cos 2\pi f s N \]

**Sampling Period**

\[ T_s = \frac{1}{f_s} \]

**Sampling Frequency**

\[ f_s = 1/T_s \]

**Maximum Frequency**

\[ f = n T_s \]

\[ \Omega \]

\[ \frac{2\pi}{T_s} = n \Rightarrow 2\pi f s = \pi \Rightarrow f = \frac{1}{T_s} f_s \]

\[ -\pi \leq \Omega \leq \pi \] means \[ -\frac{T_s}{2} \leq f \leq \frac{T_s}{2} \]
Example of Frequency Response

\[ h[n] \]

\[ \begin{align*}
H(e^{j\omega}) &= \frac{1}{3} e^{j\omega} + \frac{1}{3} e^{-j\omega} + \frac{1}{3} \\
&= \frac{1}{3} (e^{j\omega} + e^{-j\omega}) \\
&\quad + \frac{1}{3} (\cos \omega - j\sin \omega) \\
&\quad + \frac{1}{3} \\
&= \frac{2}{3} \cos \omega + \frac{1}{3}
\end{align*} \]

\[ \text{Re } H(e^{j\omega}) \]
Example

Frequency Response in two cases

\[ h_1[n] \]

\[ \frac{1}{2} \quad \frac{1}{2} \]

\[ h_2[n] \]

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[ H(e^{j\omega}) = \sum_{m=0}^{\infty} h[m] e^{-j\omega m} \]

\[ H_1(e^{j\omega}) = h_1[0] e^{-j\omega 0} + h_1[1] e^{-j\omega 1} \]

\[ = \frac{1}{2} \left( 1 + \cos \omega \right) - \frac{1}{2} j \sin \omega \]

\[ H_2(e^{j\omega}) = \frac{1}{3} e^{-j\omega 0} + \frac{1}{3} e^{-j\omega 1} + \frac{1}{3} e^{-j\omega 2} \]

\[ = \frac{1}{3} \left( 1 + \cos \omega + \cos 2\omega \right) \]

\[ \frac{1}{3} \left( \sin \omega + \sin 2\omega \right) \]

\[ \text{Re} \left( H_1(e^{j\omega}) \right) \]

\[ \frac{1}{3} \]

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[ \text{Im} \left( H_1(e^{j\omega}) \right) \]

\[ \text{Re} \left( H_2(e^{j\omega}) \right) \]

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[ \text{Im} \left( H_2(e^{j\omega}) \right) \]
Note if \( x[n] = (-1)^n = e^{j\pi n} \)

\[
Y_1[n] = H_1(e^{j\pi}) e^{j\pi n} = 0 \cdot e^{j\pi n} = 0
\]

Makes sense convolve \( h_1 \) with \( Y \)

\[
\begin{array}{c|c|c|c}
\text{Y}[n] & 1 & 1 & 1 \\
\hline
0 & -1 & -1 & -1
\end{array}
\]

\[
h[0-m] \Rightarrow Y[0] = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0
\]

\[
Y[n] = (-1)^n + 1
\]

\[
Y_2[n] = H_2(e^{j\pi}) e^{j\pi n} = \frac{1}{3} e^{j\pi n} = \frac{1}{3} (-1)^n
\]

Makes sense

\[
\begin{array}{c|c|c|c|c}
\text{Y}[n] & 1 & 1 & 1 & 1 \\
\hline
0 & -1 & -1 & -1 & -1
\end{array}
\]

\[
h[0-m] \Rightarrow Y[0] = \frac{1}{3}
\]

\[
h[1-m] \Rightarrow Y[1] = -\frac{1}{3}
\]

\[
h[2-m] \Rightarrow Y[2] = \frac{1}{3}
\]
Suppose \( X[n] = \cos \pi n = \frac{1}{2}(e^{j\pi n} + e^{-j\pi n}) \)

\[ y[n] = \sum h[m] \cos \omega n - m \]

\[ y[n] = H(e^{j\pi}) \frac{1}{2} e^{j\pi n} + H(e^{-j\pi}) \frac{1}{2} e^{-j\pi n} \]

Consider \( \omega = \frac{\pi}{2} \) and system 2

\[ H(e^{j\frac{\pi}{2}}) = 0 - \frac{1}{2} j \]

\[ H(e^{-j\frac{\pi}{2}}) = 0 + \frac{1}{2} j \]

\[ y[n] = -\frac{1}{2} j \frac{1}{2} e^{j\pi n} - \frac{1}{2} j \frac{1}{2} e^{-j\pi n} \]

\[ = -\frac{1}{2} j \left( \cos \frac{\pi n}{2} + j \sin \frac{\pi n}{2} \right) \]

\[ = -\frac{1}{2} j \left( 2j \sin \frac{\pi n}{2} \right) \]

\[ y[n] = -\frac{1}{2} \sin \frac{\pi n}{2} \]

Same Amplitude, Different Phase

Suppose \( h[n] = \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \)

(Analyzed Before)

\[ y[n] = H(e^{j\pi}) \frac{1}{2} e^{j\pi n} + H(e^{-j\pi}) \frac{1}{2} e^{-j\pi n} \]

\[ = \frac{1}{2} \left( e^{j\pi n} + e^{-j\pi n} \right) \]

\[ = \frac{1}{2} \cos \pi n \]