6.02 Lecture 4 – Signals and Noise

- Signals and Noise
  - Sources of Noise
  - Noise causes bit errors

- Analyzing Noise
  - Characterizing Noise (PDF, CDF, Variance)
  - Decomposition into Noise-free plus Noise

- Computing Bit Error Rates
  - No ISI → Two cases.
  - Shape of Noise matters.
Noise Can Be Due to Fundamental Processes

Electron Moving Through Crystal with Vibrating Atoms

http://www4.nau.edu/meteorite/Meteorite/Images/Sodium_chloride_crystal.png

Randomized path leads to noisy current flow
Effect of Many Interactions Can Be Modeled as Noise

Many components connected by thin wires (that have inductance and resistance) to single power supply - 1000's of devices switching on and off creates "noisy" power supply.
6.02 IR Transceiver Noise Source - Room Light

20 Samples/bit

Receiver Waveform

Threshold

Eye Diagram
Noisy Signal Can Cause Bit Errors

Noise Free

Noise

Threshold

Potential Bit Errors
Key Noise Questions For A Channel

- For a given Transmission Scheme:
  - What's the Bit Error Rate (BER)?
    - BER: Fraction of erroneously received bits
- If the Signal is increased:
  - How much is BER reduced?
- If ISI is reduced:
  - Is BER reduced significantly?

To Answer these questions

- Need to Characterize the Noise
  - What “shape” (probability density function)?
  - How “big” (variance)?
Bit Error Rates

Bad Name - Really the probability that a given bit will be in error

Examples:

Really Good Channel: BER = 1 error in $10^{12}$ bits
Okay Channel: BER = 1 error in $10^4$ bits
Lousy Channel: BER = 1 error in $10^2$ bits

For this course: Model only additive noise

- Many other types of noise (e.g., phase noise where transmitter bit period varies)
- Additive noise is easy to analyze

Additive noise

\[ \text{Signal} + \text{Noise} = \text{Noise-free Signal} + \text{Noise} \]
Experiment to see Noise "Shape"

- Create histograms of sample values from trials of increasing lengths
- Assumption of independence and stationarity implies histogram should converge to a shape known as a probability density function (PDF)

Slide thanks to C. Sodini and M. Perrott
"Shape" = Probability Density Function PDF

- Define $X$ as a random variable whose PDF has the same shape as the histogram we just obtained
- Denote PDF of $X$ as $f_X(x)$
  - Scale $f_X(x)$ such that its overall area is 1

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1$$

This shape is referred to as a Gaussian PDF
PDF $\rightarrow$ Probability and PDF $\rightarrow$ CDF

- The probability that random variable $X$ takes on a value in the range of $x_1$ to $x_2$ is calculated from the PDF of $X$ as:

$$\text{Prob}(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_X(x) \, dx$$

- The probability that random variable $X$ takes on a value less than $x_1$ is the cumulative distribution function $CDF(x_1)$

$$CDF(x_1) = \text{Prob}(x \leq x_1) = \int_{-\infty}^{x_1} f_X(x) \, dx$$
Example Probability Calculation

- Verify that overall area is 1:

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = \int_{0}^{2} 0.5 \, dx = 1
\]

- Probability that \( x \) takes on a value between 0.5 and 1.0:

\[
\text{Prob}(0.5 \leq x \leq 1.0) = \int_{0.5}^{1.0} 0.5 \, dx = 0.25
\]
CDF of a Uniform Manifolding

\[ f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]

CDF of Uniform Manifolding

\[ F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases} \]

\[ F(x) = \int_{x_0}^{x} f(x) \, dx \]

\[ x_0 = \frac{a + b}{2} \]

\[ F(x) = \frac{x - a}{b - a} \]

\[ x_0 = \frac{a + b}{2} \]

\[ F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases} \]
Noise Modeled Using Normal (Gaussian) PDF

\[ f_X(x) = \frac{1}{(\sqrt{2\pi})\sigma} e^{-\frac{x^2}{2\sigma^2}} \]

\( \sigma = \text{standard deviation} \)

Why the Normal PDF?

Histogram for 10,000 trials of sums of 1000 uniformly (right) or triangularly (left) distributed \([-1,1]\) random variables

Key Point: Sums of Noise from many sources well approximated by the Normal (Gaussian) PDF
Unit Normal (Gaussian) PDF and CDF

Normal Probability Density Function

Area = 1

Normal Cumulative Distribution Function

\[ x_1 \to +\infty, \quad \text{CDF}(x_1) \to 1 \]

\[ \text{CDF}(x_1) = \text{Prob}(x \leq x_1) = \int_{-\infty}^{x_1} f_X(x) \, dx \]

\[ x_1 \to -\infty \quad \text{CDF}(x_1) \to 0 \]

CDF(0) = 1/2
Estimating Probability of Bit Error

Note: No significant ISI
Decompose Into Noisefree + Noise
Suppose transmitted bit = 0

Received Voltage = 0 volts + noise

\[ P(\text{V+noise} > \text{thresh}) = \int_{\text{threshold}}^{\infty} f_{\text{noise}}(x) \, dx \]

has a PDF of 0 volts + noise

\[ \int_{\text{threshold}}^{\infty} f_{\text{noise}}(x) \, dx = 1 - \int_{-\infty}^{\text{threshold}} f_{\text{noise}}(x) \, dx \]

\[ = 1 - \text{CDF}_{\text{noise}}(\text{threshold}) \]

\[ \equiv P_{01} \]

Probability transmitted zero, received a 1.
Suppose Transmitted bit = 1 volt + noise

Received voltage = 1 volt + noise

\[ P_{1|0} = P(V_1|\text{noise} < \text{threshold}) = \int_{-\text{threshold}}^{\text{threshold}} \text{CDF} \text{ noise}(x) \, dx \]

Probability of Error: Sum Two Cases

\[ = \frac{\text{Prob(Transmitted a zero)} \cdot P_{0,1}}{\text{Prob(Transmitted a one)} \cdot P_{0,1}} \]

Typically

\[ = \frac{1}{2} \]
Mean and Variance

- The mean of random variable $X$, $\mu_X$, corresponds to its average value
  - Computed as
    $$\mu_X = \int_{-\infty}^{\infty} x f_X(x) \, dx$$

- The variance of random variable $X$, $\sigma_X^2$, gives an indication of its variability
  - Computed as
    $$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) \, dx$$

- The standard deviation of a random variable $X$, is denoted $\sigma_X$
Example Mean and Variance Calculation

- **Mean:**

\[
\mu_X = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{2} x \frac{1}{2} \, dx = \frac{1}{4} x^2 \bigg|_0^2 = 1
\]

- **Variance:**

\[
\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) \, dx = \int_{0}^{2} (x - 1)^2 \frac{1}{2} \, dx
\]

\[
= \frac{1}{6} (x - 1)^3 \bigg|_0^2 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\]
Visualizing Mean and Variance from PDF

Changes in mean of $x$

$\mu_x = A$

Changes in variance of $x$

$\mu_x$

• Changes in mean shift the center of mass of PDF
• Changes in variance narrow or broaden the PDF
  - Variance tells us how “big” the noise is!
Why Does "Shape" Matter?

Question: How much better is 2 than 1?

Case 1 (Noise Free)

Transmitted '1'

0.9V

0.4V

0

Threshold

Sample

Transmitted '0'

Case 2 (Noise Free)

Transmitted '1'

1.0V

0.5

0.5V

Threshold

Transmitted '0'
Uniform Noise

PDF of Noise

\[ f_{\text{noise}}(x) \]

\[ \text{Area} = 0.6 \cdot \frac{1}{2} = 0.3 \]

Case 1 Bit Error

\[ \frac{1}{2} \cdot P(\text{noise} > 0.4) + \frac{1}{2} \cdot P(\text{noise} < -0.4) \]

\[ P_0 = \frac{1}{2} \cdot 0.3 + \frac{1}{2} \cdot 0.3 = 0.3 \]

Marginaly Better:

Case 2 Bit Error

\[ \frac{1}{2} \cdot P(\text{noise} > 0.5) + \frac{1}{2} \cdot P(\text{noise} < -0.5) \]

\[ = \frac{1}{2} \cdot 0.25 + \frac{1}{2} \cdot 0.25 = 0.25 \]
Normal (Gaussian) Noise

PDF of Noise

\[ f_{\text{noise}}(x) \] with \( \sigma = 0.1 \)

\[ = 1 - \text{cdf}_{N_{0.1}}(0.4) \]
\[ = \frac{1}{2} \cdot 3.17 \cdot 10^{-5} + \frac{1}{2} \cdot 3.17 \cdot 10^{-5} = 3.17 \cdot 10^{-5} \]

Case 1 bit Error

\[ P_0 \cdot P(\text{noise} > 0.4) + P_1 \cdot P(\text{noise} < -0.4) \]
\[ = \frac{1}{2} \cdot 1 - \text{cdf}_{N_{0.1}}(0.4) + \frac{1}{2} \cdot \text{cdf}_{N_{0.1}}(-0.4) \]

100x lower bit error rate!

Case 2 bit Error

\[ P_0 \cdot P(\text{noise} > 0.5) + P_1 \cdot P(\text{noise} < -0.5) \]
\[ = \frac{1}{2} \cdot 2.97 \cdot 10^{-7} + \frac{1}{2} \cdot 2.87 \cdot 10^{-7} = 2.87 \cdot 10^{-7} \]
Summary

- Assume Gaussian PDF for noise and no intersymbol interference (ISI next time) yields the following picture:

\[
\begin{align*}
\text{p}(\text{xmit}=0) &= p_0 \\
\mu &= 0 \\
\sigma &= \sigma_{\text{NOISE}}
\end{align*}
\]

\[
\begin{align*}
\text{p}(\text{rcv}=0|\text{xmit}=1) &= p_{10} \\
\text{p}(\text{rcv}=1|\text{xmit}=0) &= p_{01}
\end{align*}
\]

- We can estimate the bit-error rate (BER) as

\[
p(\text{biterror}) = p_0 \times p_{01} + p_1 \times p_{10}
\]
Example 5

Suppose a non-eye diagram is:

-0.6   0.6

Sample time

and the noise is given by:

\[ f_{\text{noise}}(\text{noise}) \]

\[
\frac{5}{6} \]

Note: \[
\int_{-0.6}^{0.6} f_{\text{noise}}(x) \, dx = 1.2 \cdot \frac{5}{6} = 1
\]

\[ \mu_{\text{noise}} = 0 \]

\[ \sigma^2_{\text{noise}} = \frac{5}{6} \left( \frac{1}{3} x^3 \bigg|_{-0.6}^{0.6} \right) = \frac{3}{25} \]
Probability of Error

Nominal Zero

\[
\begin{align*}
\int_{-0.6}^{0.6} f_{\text{noise}}(\text{noise}) & \quad \text{area} = 0.1 \cdot \frac{5}{6} = \frac{1}{12} = \text{Prob error given bit is a zero} \\
& \quad = P_{01}
\end{align*}
\]

Nominal One

\[
\begin{align*}
\text{area} = 0.1 \cdot \frac{5}{6} = \frac{1}{12} = \text{prob err given bit is a one} = P_{10}
\end{align*}
\]

Probability of Error =

\[
P_{10} \cdot \text{Prob}(\text{bit}=1) + P_{01} \cdot \text{Prob}(\text{bit}=0)
\]

Equally likely case: \( P(\text{bit=error}) = \)

\[
\frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{12}
\]
Suppose 1's are twice as likely as zeros

\[ P(\text{bit} = \text{err}) = P_{10} \cdot \frac{2}{3} + P_{01} \cdot \frac{1}{3} \]

\[ \uparrow \text{prob(bit}=1) \]

If \( \text{thresh} = 0.5 \text{V} \) and

\[ P_{\text{noise}} \]

\[ \frac{5}{6} \]

\[ -0.6 \quad 0.6 \]

then

\[ P(\text{bit} = \text{err}) = \frac{1}{12} \cdot \frac{2}{3} + \frac{1}{12} \cdot \frac{1}{3} = \frac{1}{12} \]

What if threshold is moved?

Assuming \( V_{\text{thresh}} \geq 0.4 \)

\[ P_{10} = (V_{\text{thresh}} - 0.4) \cdot \frac{5}{6} \]

Assuming \( V_{\text{thresh}} \leq 0.6 \)

\[ P_{01} = (0.6 - V_{\text{thresh}}) \cdot \frac{5}{6} \]

\[ 0.4 \quad V_{\text{thresh}} \quad 1 \]

\[ 0 \quad V_{\text{thresh}} \quad 0.6 \]
Probability of Error with moved threshold

\[ P(\text{bit} = \text{error}) = \frac{3}{4} \left( \frac{1}{2} - \frac{5}{8} \right) V_{\text{thresh}} + \frac{6}{10} \frac{5}{18} - \frac{4}{10} \]

Leaving \( V_{\text{thresh}} = 0.4 \) (No bit transmitted)

\[ P(\text{bit} = \text{error}) = \frac{3}{10} < \frac{1}{2} \]

\[ \text{if } V_{\text{thresh}} = 0.5 \]

\[ \text{prob (bit = error)} = \frac{7}{18} - \frac{1}{18} = \frac{1}{9} \]
Suppose one tried to balance the errors when transmitted 1's and 0's.

That is \( p_{01} \cdot p_0 = p_{10} \cdot p_1 \)

Then

\[
\frac{3}{5} (V_{\text{thresh}} - 0.4) \frac{5}{6} = \frac{1}{3} (0.6 - V_{\text{thresh}}) \frac{5}{6}
\]

\[
\Rightarrow \left( \frac{10}{18} + \frac{5}{18} \right) V_{\text{thresh}} = \frac{10}{18} \frac{4}{10} + \frac{5}{18} \frac{6}{10}
\]

\[
\frac{5}{6} V_{\text{thresh}} = \frac{4}{18} + \frac{3}{18}
\]

\[
V_{\text{thresh}} = \frac{7}{15}
\]
Probability of error with $V_{\text{thresh}} = 7.5$

\[ p_{10} = \left( \frac{14}{30} - \frac{16}{30} \right) \cdot \frac{5}{6} = \frac{1}{8} \]

\[ p_{01} = \left( \frac{18}{30} - \frac{16}{30} \right) \cdot \frac{5}{6} = \frac{3}{8} \]

\[ p = \frac{p_{10} \cdot \frac{2}{3}}{27} + \frac{p_{01} \cdot \frac{1}{3}}{27} = \frac{2}{27} + \frac{2}{27} = \frac{2}{9} \]

\[ \frac{1}{8} < \frac{2}{9} < \frac{1}{3} \]

**Note:**

- $P(\text{bit = error})$
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- $P(\text{bit = error})$
- $p_{\text{error}} = 0.5$ when $V_{\text{thresh}} = 0.5$
- $V_{\text{thresh}} = \frac{15}{30}$