6.02 Lecture 5 - ISI and Noise

- Inter-Symbol Interference
  - Why are eye diagrams helpful?

- ISI and noise
  - Eye Slicing
  - Evaluating BER

- Noise and Deconvolution
  - Massaging the Unit Sample response.
Block Diagram of Entire Link

X: transmitted samples
\(Y_{nf}\): received noise-free samples
N: noise samples
Y: received samples with added noise
W: processed received samples
LTI Channel + Noise + Deconvolver

\[ y_{nf}[n] = \sum_{m=0}^{m=n} h[m]x[n-m] \]
\[ y[n] = y_{nf}[n] + \text{noise}[n] \]
\[ \sum_{m=0}^{m=n} h[m]w[n-m] = y[n] \]
Noise with No ISI \( (\text{e.g. } h[n] = \delta[n - D]) \)

- Assume Gaussian PDF for noise and no intersymbol interference (ISI) yields the following picture:

\[
p(x_{\text{mit}} = 0) = p_0 \\
\mu = 0 \\
\sigma = \sigma_{\text{NOISE}} \\
p(x_{\text{mit}} = 1) = p_1 \\
\mu = \nu \\
\sigma = \sigma_{\text{NOISE}}
\]

\[p(\text{rcv}=0 | x_{\text{mit}}=1) = p_{10}
\]

- We can estimate the bit-error rate (BER) as

\[p(\text{biterror}) = p_0 * p_{01} + p_1 * p_{10}\]
Unit Sample Response and Eye Diagrams
(35 Samples per bit)

Unit Sample Response

Receiver data

Lowest '1'

Threshold

Highest '0'

Eye diagram for channel 1

Bit detection Time
Histogram for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.01
Histogram for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.05

Number of Samples equal to V

Detection Voltage
Probability of a bit error

\[ f_{\text{noise}}(x) \]

\[ \begin{align*}
    & f_{\text{1.2 + noise}}(x) \\
    & \text{Threshold} \\
    & x_{\text{mit}} = 0 \\
    & x_{\text{mit+1}} = 1
\end{align*} \]

\[ \text{Area} = P(\text{read} = 1 \mid x_{\text{mit}} = 0, p, x_{\text{mit+1}}) = P_{\text{01}} = \text{cdf}_{\text{Noise}}(+\text{threshold}) \]

\[ \text{Area} = P(\text{read} = 1 \mid x_{\text{mit}} = 0, p, x_{\text{mit+1}}) = P_{\text{01}} = \text{cdf}_{\text{Noise}}(-0.2) \]

\[ 0.2 \]

Typically = \frac{1}{4}

\[ P(\text{bit-error}) = P_{\text{00}} P_{\text{00}} + P_{\text{01}} P_{\text{01}} + P_{\text{10}} P_{\text{10}} + P_{\text{11}} P_{\text{11}} \]
Example USR, 3 Samples/bit

Unit sample response of channel

Received Samples

Eye Diagram

Receiver transition point
Bit detection sample (middle of receiver bit period)
Example Magnified, 3 Samples/bit

Received Samples

Eye Diagram

Bit Detection Time
What Voltage Values Can the bit detection sample have (Assuming No Noise).

<table>
<thead>
<tr>
<th>Previous Bit</th>
<th>Current Bit</th>
<th>Next Bit</th>
<th>Bit Detection Sample Voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0 volts</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.2 volts</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.2 volts</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.4 volts</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.6 volts</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.8 volts</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.8 volts</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.0 volts</td>
</tr>
</tbody>
</table>

Note - Sample 8 is bit detection sample (maximum value).
Aside (More Formal) (Can Skip to page 21)

A note about Definitions

Joint pdf

$f_{\text{vsample, bit}}(V, b)$

Continuous

Marginal pdf (Summed over all cases of $b$)

$f_{\text{vsample}}(V) = f_{\text{vsample, bit}}(V, b = 0) + f_{\text{vsample, bit}}(V, b = 1)$

$f_{\text{sample}}(V | \text{bit=0}) \cdot P(\text{bit=0})$

$f_{\text{sample}}(V | \text{bit=1}) \cdot P(\text{bit=1})$

Conditional pdf
Aside Continued

What if there is additive noise?

\[ Y[0] = X[0] + \text{noise}[n] \]

At bit detection sample (Assume all bit patterns equally likely)

What is \( f_X(v) \) ?

\[ f_X(v) = f_X(v | x_{mit} = 0, 0, 0) \cdot \frac{1}{8} \]

\[ + f_X(v | x_{mit} = 0, 0, 1) \cdot \frac{1}{8} \]

\[ + f_X(v | x_{mit} = 1, 0, 0) \cdot \frac{1}{8} \]

\[ + f_X(v | x_{mit} = 1, 0, 1) \cdot \frac{1}{8} \]

\[ + f_X(v | x_{mit} = 1, 1, 0) \cdot \frac{1}{8} \]

\[ + f_X(v | x_{mit} = 1, 1, 1) \cdot \frac{1}{8} \]
Aside Continued

What if additive noise is Gaussian with std. dev = σ

\[ f_x(v) = \frac{1}{\sqrt{2\pi} \sigma} \cdot \frac{1}{8} \cdot \left( e^{-\frac{v^2}{2\sigma^2}} + 2e^{-\frac{(v-0.2)^2}{2\sigma^2}} + e^{-\frac{(v-0.4)^2}{2\sigma^2}} + e^{-\frac{(v-0.6)^2}{2\sigma^2}} + 2e^{-\frac{(v-0.8)^2}{2\sigma^2}} + e^{-\frac{(v-1.0)^2}{2\sigma^2}} \right) \]

Note:
- Two different bit patterns
- Generate the same bit detection
- Sample voltage
PDF for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.01

Detection Voltage
Example with Noise (std dev = 0.03), 3 Samples/bit
PDF for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.03
Example with Noise (std dev = 0.05), 3 Samples/bit

Received Samples

Eye Diagram

Bit Detection Time
PDF for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.05
PDF for Voltage at Detection Time

Gaussian Noise with Std Dev = 0.1
Example 6: Gaussian additive noise with standard deviation $\sigma$

For bit $b = 1$ case

$$f_{\text{usample}}(v \mid b = 1)$$

$$P(\text{rcv} = 0 \mid b = 1) =$$

$$= \text{CDF}_{\text{Normal}} \left( \frac{0.5 - 1}{\sigma} \right) = P_{\text{transmitted one received zero}}$$

$$\sigma$$

$$\mu$$

$$\Phi$$

Bit Error Probability

$$\frac{\text{Prob}(\text{rcv} = 0 \mid \text{sent} = 1)}{\text{Prob}(\text{sent} = 1)} + \frac{\text{Prob}(\text{rcv} = 1 \mid \text{sent} = 0)}{\text{Prob}(\text{sent} = 0)}$$

For bit $b = 0$ case

$$1 - \text{CDF}_{\text{Normal}} \left( \frac{0.5}{\sigma} \right) = P_{\text{Prob}(\text{rcv} = 1 \mid \text{sent} = 0)}$$
How do we use CDF of $μ=0, \sigma=1$ Gaussian to estimate bit errors.

Errors for $x=0.1,0.5$ case.

Errors for $x=0.7,0.8$ case.
Shifting, Scaling & Adding

\[
\begin{align*}
P(\text{error}) &= \\
&= \frac{1}{8} \cdot \left[ (1 - \Phi \left( \frac{0.5}{\sigma} \right)) + 2(1 - \Phi \left( \frac{0.5 - 0.2}{\sigma} \right)) + \\
&\quad \quad \quad (1 - \Phi \left( \frac{0.5 - 0.4}{\sigma} \right)) + \right. \\
&\quad \quad \quad \left. \Phi \left( \frac{0.5 - 0.6}{\sigma} \right) + \right. \\
&\quad \quad \quad \left. 2 \left( \Phi \left( \frac{0.5 - 0.8}{\sigma} \right) \right) + \right. \\
&\quad \quad \quad \left. \Phi \left( \frac{0.5 - 1.0}{\sigma} \right) \right]
\end{align*}
\]

Note \( \Phi(-a) = 1 - \Phi(a) \)

\[
\Rightarrow P(\text{error}) = \frac{1}{4} \left[ (1 - \Phi \left( \frac{0.5}{\sigma} \right)) + \\
&\quad 2 (1 - \Phi \left( \frac{0.5}{\sigma} \right)) + \\
&\quad 2 (1 - \Phi \left( \frac{0.5}{\sigma} \right)) \right]
\]
Slow Wire and 20 Samples per bit
Deconvolution Great Unless There's Noise

Why does noise destroy deconvolution?
- Wait a few weeks for better analysis tools
Then why study deconvolution?

- Many applications in biology, aerospace, medical imaging, etc

Deltavision 3D fluorescence microscopy of a diving *S. pombe* cell. Before deconvolution (Left) and after deconvolution (Right).

Deconvolution of an image from Hubble Space Telescope
We can "Fix" Deconvolution

Unit-sample response of channel scale_ir
Deconvolver

Recall Stability from 6.01

Roots of some polynomial with coeffs = \( h[0], \ldots, h[K] \)

Hard to design with if \( K \gg 2 \).

What Else?

\[ W[n] = \frac{h[0]}{h[0]} W[n-1] + \ldots + \frac{h[K]}{h[0]} W[n-K] + Y[n] \]

If \( Y[n] = 0 \) for large \( n \)

Then \( W[n] \) is guaranteed to go to 0 zero if

\[ \frac{M}{K} \sum_{m=1}^{K} |\frac{h[m]}{h[0]}| = \beta < 1 \]

A Sufficient Condition

\[ \Rightarrow \forall a \in \mathbb{R}, \quad |h[a]| > \sum_{m=1}^{K} |h[m]| \]

How?
Example - Bad Case

\[ h[n] \]

\[ \sum_{n=0}^{\infty} h[n] = 1 \]

\[ \frac{1}{6} \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \rightarrow \]

Decomolverc:

\[ w[n] = \frac{1}{6} w[n-1] + \ldots + \frac{1}{6} w[n-5] \]

\[ + \frac{1}{6} y[n] \]

\[ w[0] = \sum_{m=1}^{5} w[n-m] + 6y[n] \]

Suppose \( y[n] = 0 \) for \( n > 10 \)

\[ \text{Example} \quad w[10] = w[9] = \ldots = w[1] = 1 \]


\[ w[12] = 9 \quad (= w[11] + w[10] + \ldots + w[7]) \]

\[ w[13] = 17 \]

\[ \lim_{n \to \infty} w[n] = \infty \]

Compare: \[ \sum_{m=1}^{5} \left| \frac{w[m]}{w[0]} \right| = \frac{5}{6} \frac{1}{6} = \frac{5}{36} = 5 > 1 \]
Example - Marginal Case

\[ \sum_{n=0}^{\infty} h[n] = 1 \]

\[ h[1/2], h[1/6], \ldots, h[\pi] \]

Decomposer

\[ W[c_n] = \frac{1}{\sqrt{2}} W[c_{n-2}] + \frac{1}{\sqrt{2}} W[c_{n-3}] + \frac{1}{\sqrt{2}} Y[c_n] \]

\[ W[c_n] = \sum_{M=1}^{\infty} \frac{1}{3} W[c_{n-M}] + 2 Y[c_n] \]

Suppose Again

\[ Y[c_n] = 0 \quad n > 10 \]


\[ W[11] = 1 \left( \frac{1}{3} W[10] + \frac{1}{3} W[9] + \frac{1}{3} W[8] \right) \]

\[ W[12] = 1 \]

\[ W[13] = 1 \]
Example

\[ h[n] \]

\[ y[n] = \frac{8}{3} \sum_{k=0}^{n} h[k] = 1 \]

-1 -1 -1 -1 -1

Decomposer

\[ w[n] = \frac{1}{2} w[n-1] + \frac{1}{3} w[n-2] \]

or

\[ w[n] = \frac{1}{4} w[n-1] + \frac{1}{4} w[n-2] + \frac{1}{2} y[n] \]

Suppose Again

\[ y[n] = 0 \quad n > 0 \]

\[ w[10] = w[9] = \ldots = w[0] = 1 \]

\[ w[1] = \frac{1}{2} \left( \frac{1}{2} w[0] + \frac{1}{2} w[1] \right) \]

\[ w[2] = \frac{3}{8} \]

\[ w[3] = \frac{5}{16} \]

\[ \ldots \]
A brief Proof

If \( y[n] = 0 \quad n > N \)

and \( \sum_{M=1}^{B} \left| \frac{h[M]}{h[0]} \right| < 1 \quad \text{(Note: } h[M] = 0 \text{)} \quad M > B \)

Then \( \lim_{n \to \infty} w[n] = 0 \)

Proof

\[
w[n] = \sum_{M=1}^{B} \frac{h[M]}{h[0]} w[n-M] + \frac{1}{h[0]} y[n]
\]

\( = 0 \quad n > N \)

\[\Rightarrow |w[n]| = \left| \sum_{M=1}^{B} \frac{h[M]}{h[0]} w[n-M] \right| \quad n > N\]

\[\Rightarrow |w[n]| \leq \sum_{M=1}^{B} \left| \frac{h[M]}{h[0]} \right| |w[n-M]| \quad n > N\]

\[|w[n]| \leq \left( \sum_{M=1}^{B} \left| \frac{h[M]}{h[0]} \right| \right) \max_{1 \leq M \leq B} |w[n-M]| \quad n > N\]

\[\Rightarrow |w[n]| \leq \theta \quad \text{by assumption} \quad n > N\]

\[\Rightarrow |w[n]| < \theta \max_{1 \leq M \leq B} |w[n-M]| \quad n > N\]

\[\Rightarrow |w[n+2B]| < \theta \max_{1 \leq M \leq B} |w[n-M]| \quad n > N\]
How to Use Decay Result

Original \[ h[n] = \frac{1}{10} (0.9)^n \quad n \geq 0 \]

\[ \sum_{n=0}^{\infty} h[n] = \frac{1}{1 - 0.9} \frac{1}{10} = 1 \]

\[ \sum_{n=1}^{\infty} \left| \frac{h[n]}{h[0]} \right| = \frac{0.9}{0.1} = 9 > 1 \]

Modified \[ \hat{h}[n] = h[n] \quad n > 6 \]

\[ \hat{h}[6] = \sum_{n=1}^{6} h[n] \]

\[ \hat{h}[n] = 0 \quad n < 6 \]

\[ \sum_{n=0}^{\infty} \hat{h}[n] = 1 \]
Step Response

\[ S[n] = \sum_{m=0}^{n} h[m] = \frac{1}{10} \sum_{m=0}^{n} (0.9)^m \]
Step Response

\[ \hat{S}[n] = \sum_{m=0}^{\infty} \hat{h}[m] = s[n] \quad n \geq 6 \]

\[ = 0 \quad n < 6 \]