6.02 Digital Communication Systems—Spring 2010

Quiz 1 - 7:30-9:30pm (Two Hours)
Wednesday, March 4th, 2010

Check your section	Section	Time	Room	Rec. Instr.
☐	1	10-11	13-5101	Lizhong Zheng
☐	2	11-12	13-5101	Lizhong Zheng
☐	3	1-2	38-166	Tania Khanna
☐	4	2-3	38-166	Tania Khanna

Directions: The exam consists of 5 problems on 14 pages. Please make sure you have all the pages. **Enter all your work and your answers directly in the spaces provided on the printed pages of this exam. Please make sure your name is on all sheets. DO IT NOW!** All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained in the space provided**, not just simply written down. This examination is closed book, but students may use one 8 1/2 $\times$ 11 sheet of paper for reference. Calculators may not be used.

The cumulative distribution function for a zero-mean unit standard deviation Gaussian (Normal) random variable is:

$$\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-\hat{x}^2}{2}} \, d\hat{x}.$$

A brief table of values for Φ is

x	$\Phi(x)$
-0.5	0.31
-1.0	0.16
-1.5	0.067
-2.0	0.023
-3.0	0.0013
-4.0	0.000032

Please leave the rest of this page blank for use by the graders:

Problem	No. of points	Score	Grader
1	25		
2	31		
3	16		
4	14		
5	14		
Total	100		

Problem 1 (25 points)

The following parts of this question refer to a channel whose input is a transmitted sequence of voltage samples denoted X, and whose output is a received sequence of voltage samples, Y. The n^{th} sample values of the X and Y sequences are denoted $x[n]$ and $y[n]$ respectively.

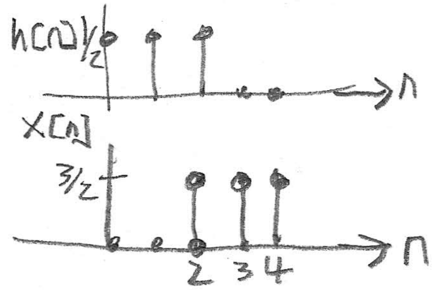
1A. (8 points) Suppose a linear time-invariant channel has a unit sample response given by

$$\begin{aligned} h[n] &= \tfrac{1}{2} && n = 0, 1, 2 \\ h[n] &= 0 && \textit{otherwise.} \end{aligned} \tag{1}$$

If the input to the channel is

$$\begin{aligned} x[n] &= \tfrac{3}{2} && n = 2, 3, 4 \\ x[n] &= 0, && \textit{otherwise} \end{aligned} \tag{2}$$

please determine the maximum value of the output of the channel, and please determine the sample index where that maximum occurs.



$y[n]_{max} = \frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2}\frac{1}{2} + \frac{3}{2}\frac{1}{2}$

$= 9/4$

$y[4] = \sum_{m=0}^{4} h[m]\, x[4-m]$

$= h[0]\, x[4] + h[1]\, x[3] + h[2]\, x[2]$

$\max_n y[n] =$ 9/4

Sample index, n, of maximum = 4

1B. (8 points) Suppose the receiver of a noisy channel receives a voltage sample equal to $0.9 + noise$ volts when the transmitter is sending a '1' bit, and a sample equal to $0.1 + noise$ volts when the transmitter is sending a '0' bit (so no intersymbol effects). If the threshold voltage is 0.5 volts, the noise is *Gaussian* with standard deviation σ, and the probability of a bit error is 0.023, please determine a numerical value for σ (note there is a cumulative distribution function table on the front page of this exam, and you may assume '1' bits and '0' bits are equally likely).

$$\frac{1}{2} P(\text{error} \mid \text{bit}=\text{'1'}) + \frac{1}{2} P(\text{error} \mid \text{bit}=\text{'0'}) =$$

$$\frac{1}{2}\Phi\left(\frac{-0.4}{\sigma}\right) + \frac{1}{2}\left(1 - \Phi\left(\frac{0.4}{\sigma}\right)\right) = \Phi\left(\frac{-0.4}{\sigma}\right) = 0.023$$

$\frac{-0.4}{\sigma}$ $= -2$ from table

Value for σ= $\sigma = 0.2$

1C. (9 points) Suppose a linear time-invariant channel has a unit sample response just like the system in part A,

$$\begin{aligned} h[n] &= \tfrac{1}{2} \qquad n = 0,1,2 \\ h[n] &= 0 \qquad \textit{otherwise} \end{aligned} \tag{3}$$

If the **output** of the channel

$$\begin{aligned} y[n] &= 1 \qquad n = 0,1 \\ y[n] &= 0, \qquad \textit{otherwise} \end{aligned} \tag{4}$$

please determine the value of the first three voltage samples of the input to the channel $x[0]$, $x[1]$, and $x[2]$.

$$w[n] = 2\left(y[n] - \left(\tfrac{1}{2}w[n-1] + \tfrac{1}{2}w[n-2]\right)\right)$$

$$w[0] = 2\left(1 - \left(\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 0\right)\right) = 2 = x[0]$$

$$w[1] = 2\left(1 - \left(\tfrac{1}{2}2 + \tfrac{1}{2}\cdot 0\right)\right) = 0 = x[1]$$

$$w[2] = 2\left(0 - \left(\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 2\right)\right) = -2 = x[2]$$

$x[0]$ = 2
$x[1]$ = 0
$x[2]$ = -2

Check

$$y[0] = \tfrac{1}{2}x[0] = 1 \checkmark$$

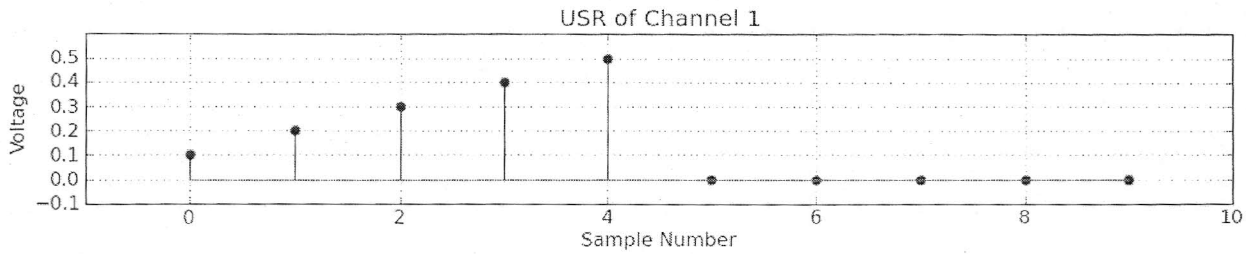
$$y[1] = \tfrac{1}{2}x[1] + \tfrac{1}{2}x[0] = 1 \checkmark$$

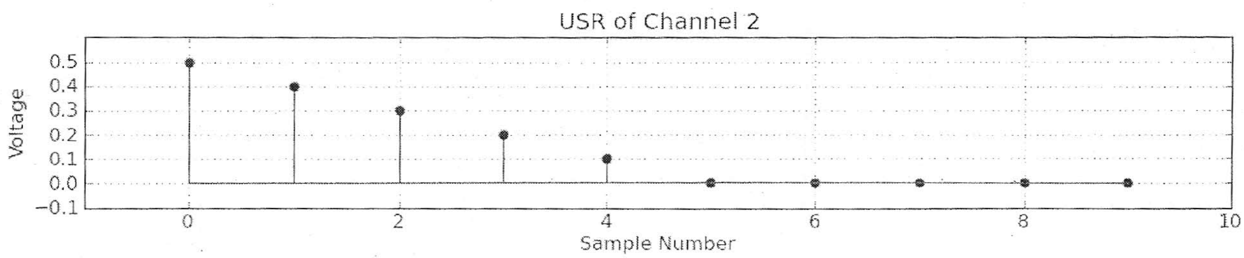
$$y[2] = \tfrac{1}{2}x[2] + \tfrac{1}{2}x[1] + \tfrac{1}{2}x[0]$$

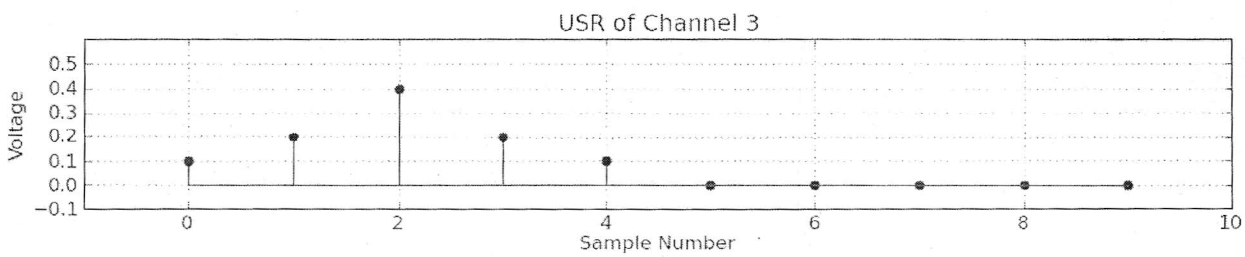
$$= -1 + 1 = 0 \checkmark$$

Problem 2 (31 points)

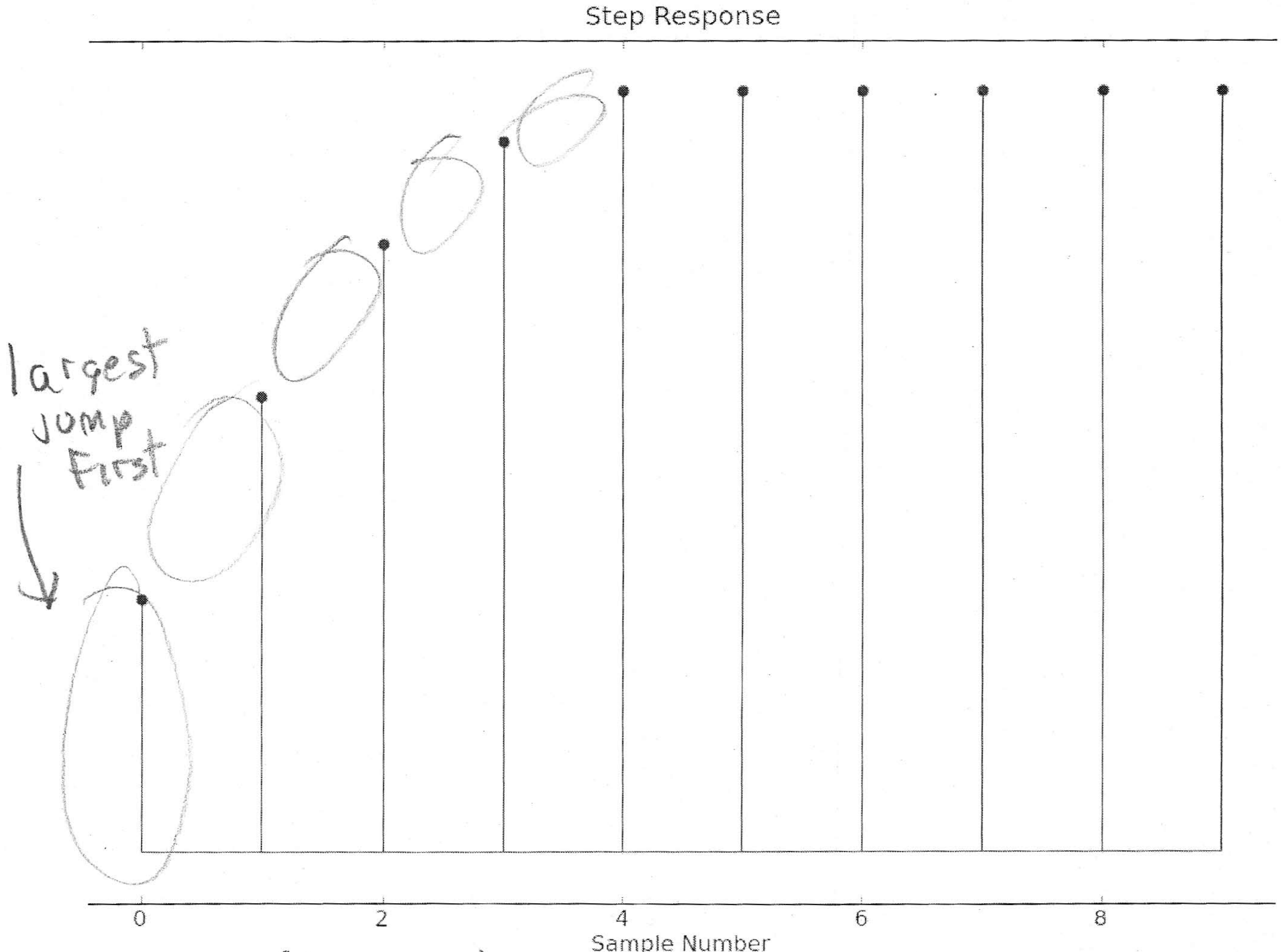
For this problem, please consider three linear and time-invariant channels, *channel one*, *channel two*, and *channel three*. The unit sample response for each of these three channels are plotted below. Please use these plots to answer all the parts of this question.







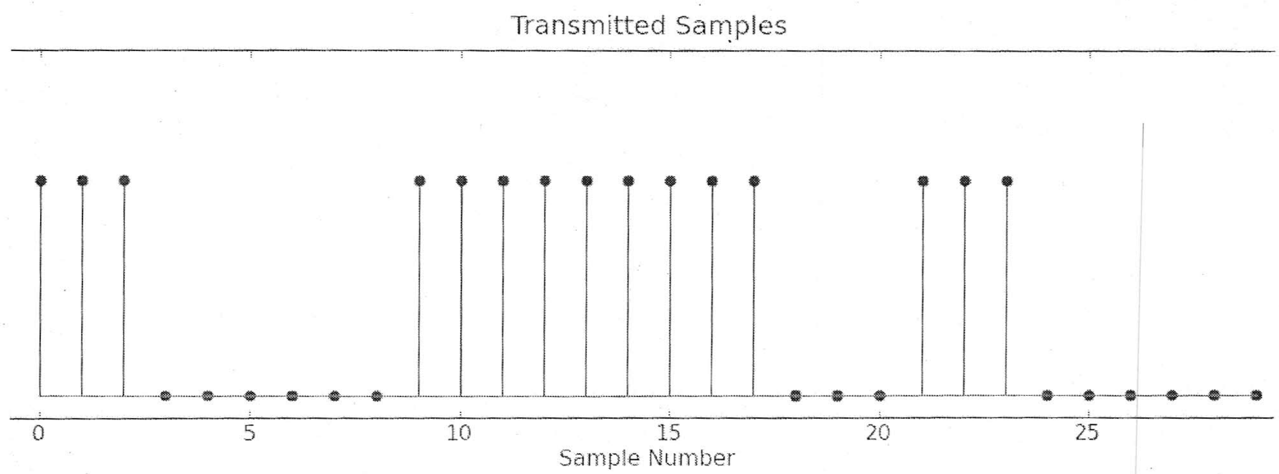
2A. (7 points) Which channel (1, 2, or 3) has the following step response, and what is the value of maximum value of the step response?

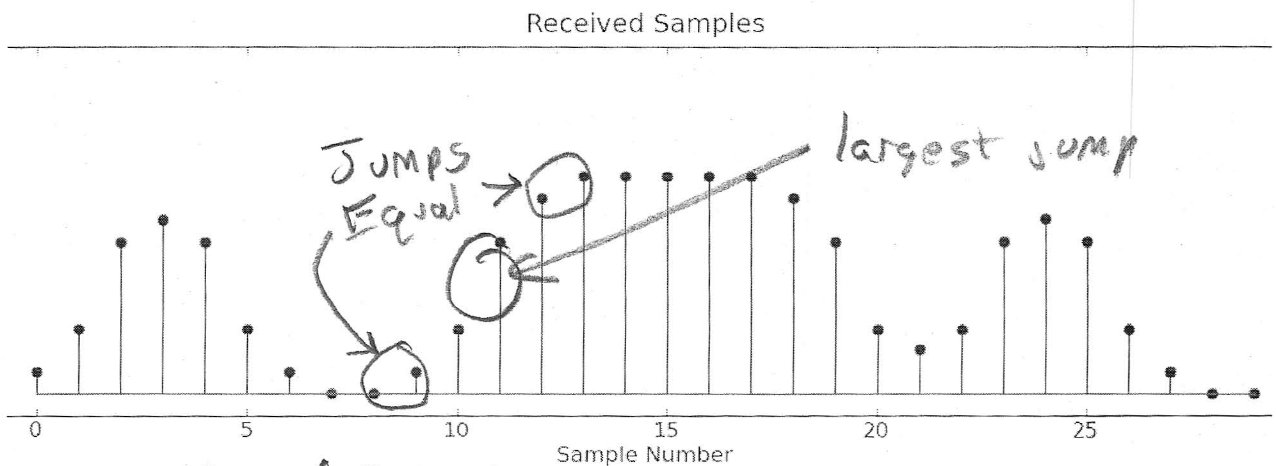


Channel = Channel 2 (largest jump first)

Maximum Value = 0.5 + 0.4 + 0.3 + 0.2 + 0.1 = 1.5

2B. (8 points) Which channel (1, 2, or 3) produced the pair of transmitted and received samples in the graph below, and what is the value of voltage sample number 24 (assuming the transmitted samples have the value of either one volt or zero volts)?



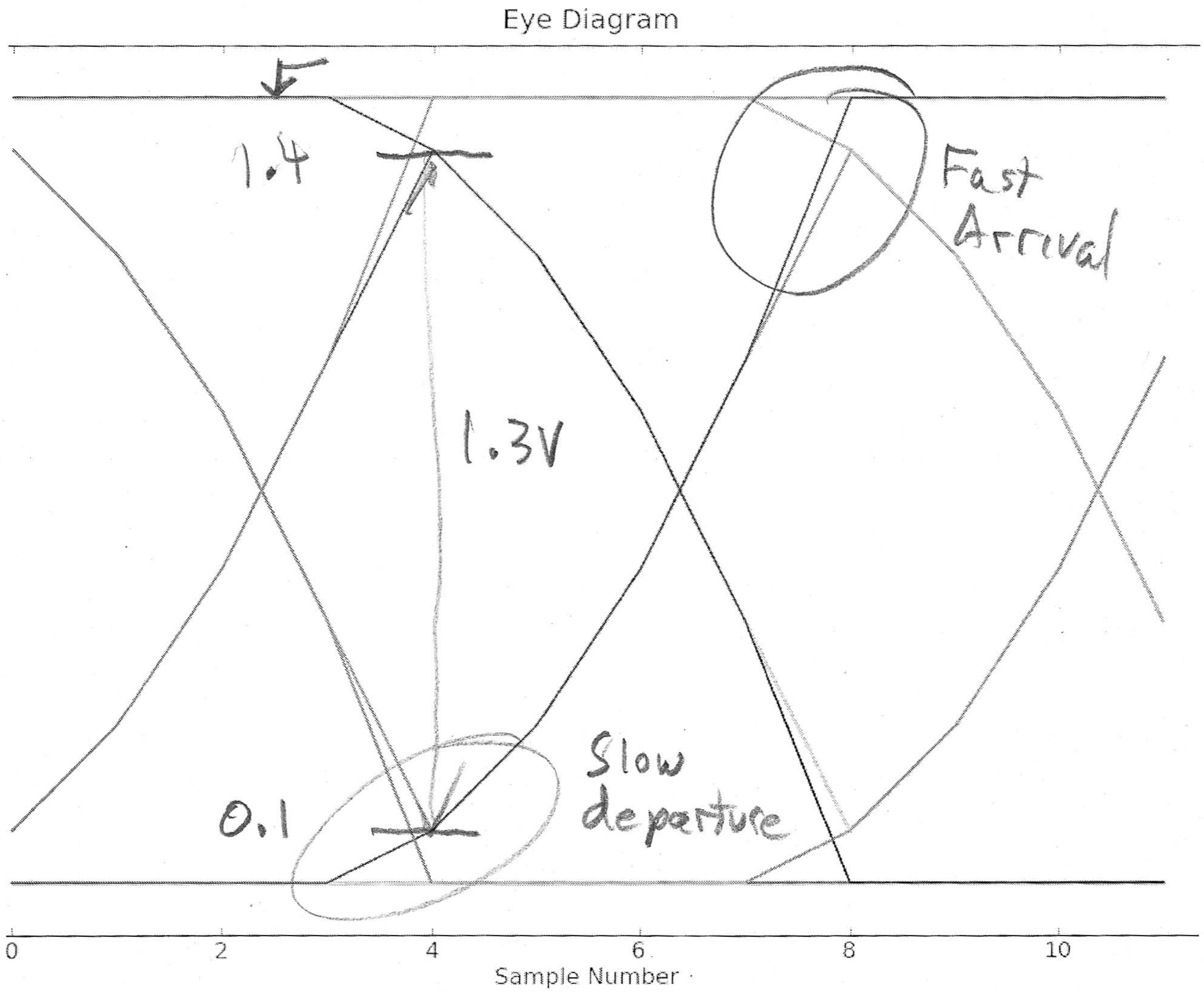


Channel = 3 (First and last jumps equal, largest jump in middle)

y[24] = 0.8

$$y[24] = \sum h[m]\, x[24-m] = h[0]\cdot 0 + h[1]\cdot 1 + h[2]\cdot 1 + h[3]\cdot 1 + h[4]\cdot 0 = 0.2 + 0.4 + 0.2$$

2C.(8 points) Which channel (1, 2, or 3) produced the eye diagram below (based on 4 samples per bit), and how wide open is the eye at its widest (lowest voltage associated with a transmitted '1' bit - highest voltage associated with a transmitted '0' bit)?



Channel = 1 (Slow departure, Fast Arrival)

Width of the eye at widest = 1.3

$\Sigma h[n]$ for 1 = 1.5

Max of eye $\approx 1.5 - 0.1$

Min of eye $= 0.0 + 0.1$

2D.(8 points) Suppose an approximate deconvolver was generated for each of channels, using the difference equation given by

$$w[n] = \frac{1}{h[0]+h[1]}\left(y_{nf}[n+1] + noise[n+1] - \sum_{m=1}^{m=3} h[m+1]w[n-m]\right) \quad (5)$$

where $y_{nf}[n] = \sum_{m=0}^{m=4} h[m]x[n-m]$ is the noise-free output of the particular channel being deconvolved. When the deconvolver was used for two of the channels, $\lim_{n\to\infty} |w[n]|$ approached infinity, but when the deconvolver was used with the remaining channel, it worked reasonably well. For which channel did the approximate deconvolver work well, (1, 2, or 3)? You must justify your answer to receive full credit.

Channel for which the approximate deconvolver worked well = Channel 2

Justification (Needed for full credit):

Channel 1 $\quad \frac{h[2]+h[3]+h[4]}{h[1]+h[0]} = \frac{1.2}{0.3} = 4 > 1$

Might blow up (might be unstable)

Channel 2 $\quad \frac{0.3+0.2+0.1}{0.4+0.5} = \frac{2}{3} < 1$

Can not blow up

Channel 3 $\quad \frac{0.4+0.2+0.1}{0.2+0.1} = \frac{0.7}{0.3} = 2\tfrac{1}{3} > 1$

Might blow up

Must be two!

Problem 3 (16 points)

Suppose a channel has both noise and intersymbol interference, and further suppose the voltage at the receiver is:

$8.0 + noise$ volts when the transmitter sends a '1' bit preceded by a '1' bit,
$6.0 + noise$ volts when the transmitter sends a '1' bit preceded by a '0' bit,
$2.0 + noise$ volts when the transmitter sends a a '0' bit preceded by a '1' bit,
$0.0 + noise$ volts when the transmitter sends a '0' bit preceded by a '0' bit,

In answering the following parts, please assume the receiver **uses** 4.0 **volts as the threshold** for deciding the bit value.

3A.(8 points) Suppose $noise$ is Gaussian with standard deviation $\sigma = 1$, and all bit patterns are equally likely. Using the the tables at the beginning of the exam, please determine the probability of a bit error.

$$\frac{1}{4}(1-\Phi(8-4)) + \frac{1}{4}(1-\Phi(6-2)) + \frac{1}{4}\Phi(-2) + \frac{1}{4}\Phi(-4)$$
$$= \frac{1}{2}\Phi(-4) + \frac{1}{2}\Phi(-2) = \frac{1}{2}(3.2\cdot 10^{-5}) + \frac{1}{2}(0.023)$$

Probability of a bit error = 0.011516

3B.(8 points) Again suppose $noise$ is Gaussian with standard deviation $\sigma = 1$, and suppose that for a particular set of transmitted data, which we will refer to as *checkerboard* data, there is an increased probability of unequal contiguous bits. That is, for checkerboard data, the probability of two '0' bits in a row is $\frac{1}{6}$, the probability of two '1' bits in a row is $\frac{1}{6}$, the probability of a '1' bit preceded by a '0' bit is $\frac{1}{3}$ and the probability of a '0' bit preceded by a '1' bit is $\frac{1}{3}$. Approximately what is the ratio of the probability of bit error for the checkerboard case to the probability of bit error for the uniform (all bit patterns equally likely) case. Be sure to justify your answer.

$\frac{P(biterror)checkerboard}{P(biterror)uniform} \approx$ 4/3

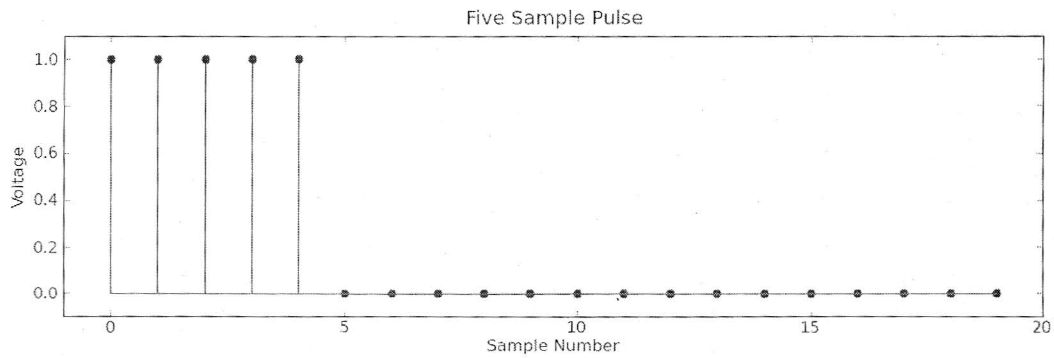
Justification:

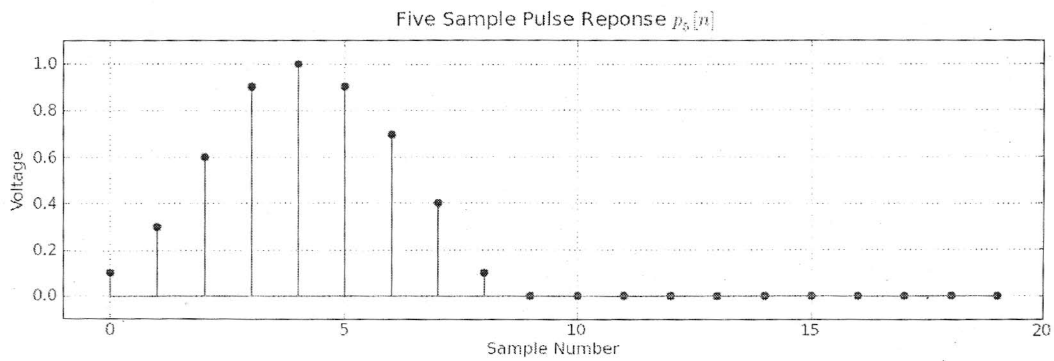
≈ 0 $(<< 0.023)$

$$\frac{\frac{1}{3}\Phi(-4) + \frac{2}{3}\cancel{\Phi(-2)}}{\frac{1}{2}\Phi(-4)^{\approx 0} + \frac{1}{2}\cancel{\Phi(-2)}}$$

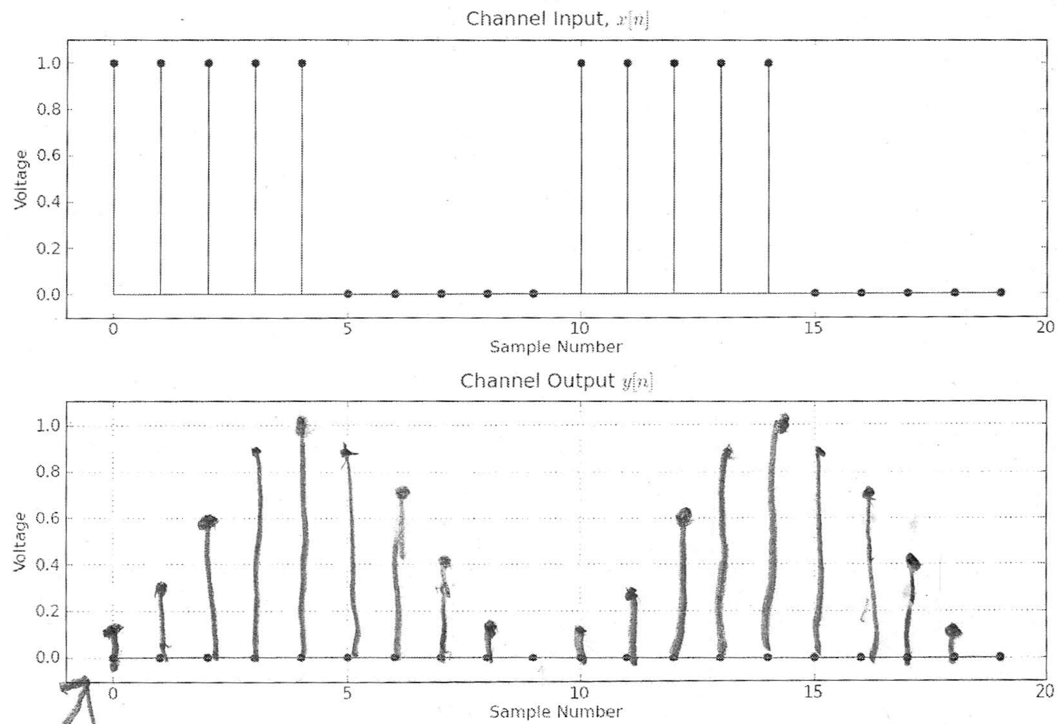
Problem 4 (14 points)

In this problem you will be answering questions about a linear time-invariant channel characterized by its response to a five-sample *pulse*, denoted $p_5[n]$.





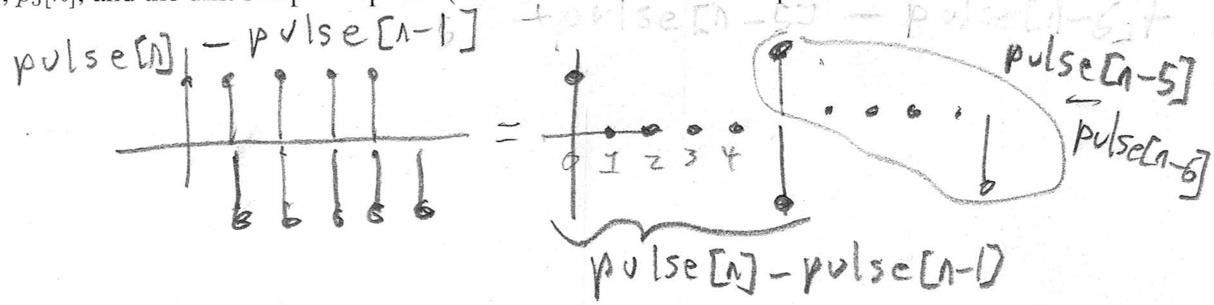
4A.(5 points) Suppose the input to the channel is as plotted below. Plot the output of the channel on the axes provided beneath the input.



$x[n]$ = pulse$[n]$ + pulse$[n-10]$

$y[n] = p_5[n] + p_5[n-10]$

4B.(9 points) The unit sample response, $h[n]$, can be related to the step response, $s[n]$ by the formula $h[n] = s[n] - s[n-1]$. Please derive a similar formula for the relation between the five-sample pulse response, $p_5[n]$, and the unit sample response (an infinite series is an acceptable form for the answer).



$$h[n] = p_5[n] - p_5[n-1] + p_5[n-5] - p_5[n-6] + p_5[n-10] - p_5[n-11] + \ldots$$

$$= \sum_{k=0}^{\infty} \left(p_5[n-5k] - p_5[n-1-5k] \right)$$

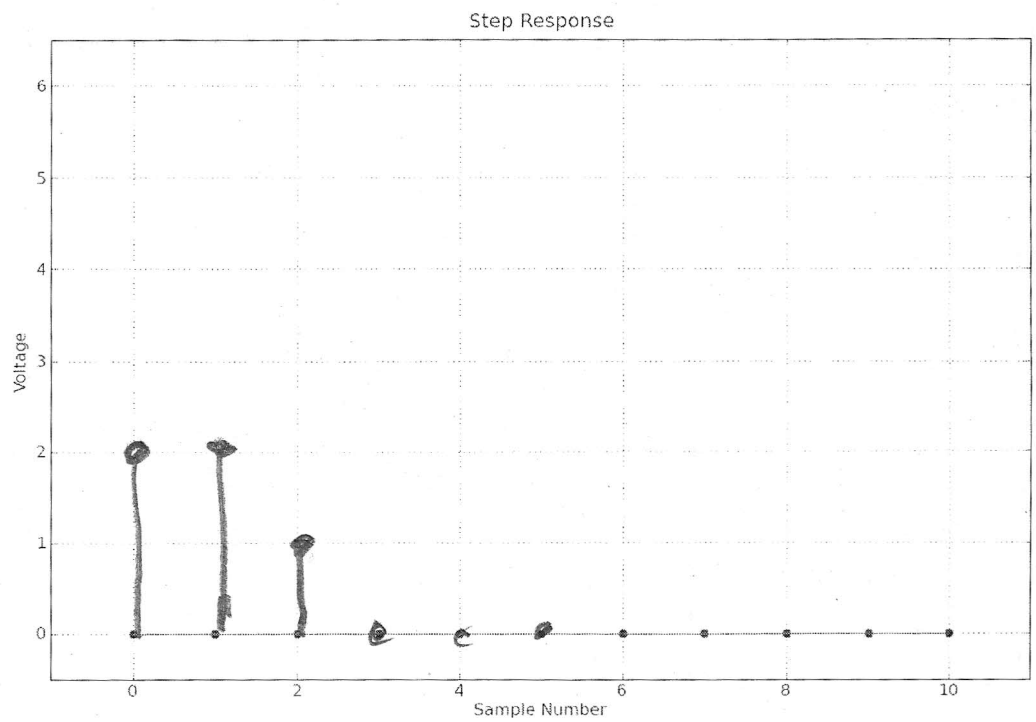
Problem 5 (14 points)

Suppose the perfect deconvolving difference equation for a linear time-invariant channel is

$$w[n] = \frac{1}{2}y[n] - \frac{1}{2}\left(2w[n-1] + w[n-2]\right).$$

That is, if $x[n]$ is the input to the channel and $y[n]$ is the channel output, in the noise-free case, $w[n]$ will be exactly $x[n]$.

5A.(5 points) On the axes below, plot the unit sample response of the linear time-invariant channel.



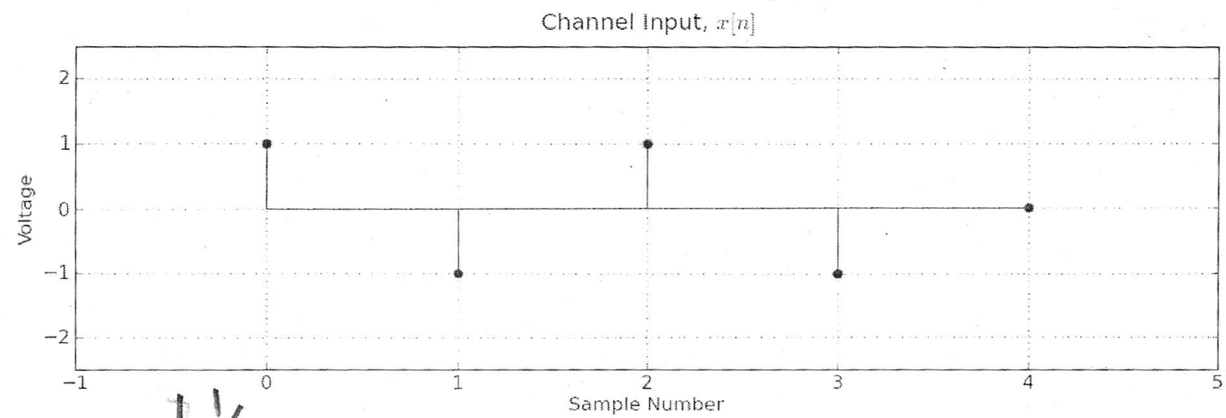
$2w[n] + 2w[n-1] + 1w[n-2] = y[n]$

h[0] h[1] h[2]

5B.(9 points) Suppose a noise spike is added to the output of the channel before deconvolving, as in

$$w[n] = \frac{1}{2}(y[n] + \delta[n]) - \frac{1}{2}\left(2w[n-1] + w[n-2]\right),$$

where $\delta[n]$ is the unit sample ($\delta[n] = 1$ when $n = 0$ and zero otherwise). Suppose the input to the channel, $x[n]$, is as plotted below. Please determine the first three values of the deconvolver output, $w[0]$, $w[1]$, and $w[2]$. (think before you compute, or you will spend 20 minutes doing algebra. Recall that if not for the noise spike, you would have a perfect deconvolver).



$w[0] =$ $1\frac{1}{2}$

$w[1] =$ $-1\frac{1}{2}$

$w[2] =$ $+1\frac{1}{4}$

End of Quiz 1!

$$W[n] = \underbrace{W_{np}[n]}_{\text{perfect deconv}} + W_{spike}[n]$$

$$= X[n] + W_{spike}[n] = \begin{array}{ll} 1 + 1/2 & n=0 \\ -1 - 1/2 & n=1 \\ 1 + 1/4 & n=2 \end{array}$$

$$W_{spike}[0] = \frac{1}{2}\delta[0] = 1/2$$

$$W_{spike}[1] = \frac{1}{2}\cdot 0 + -\frac{1}{2}(2\cdot \tfrac{1}{2}) = -1/2$$

$$W_{spike}[2] = \frac{1}{2}0 + -\frac{1}{2}\left(2\cdot\left(\frac{-1}{2}\right) + \frac{1}{2}\right) = -1/4$$