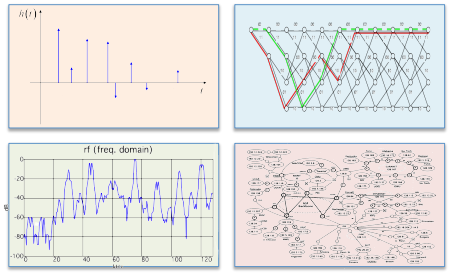


INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

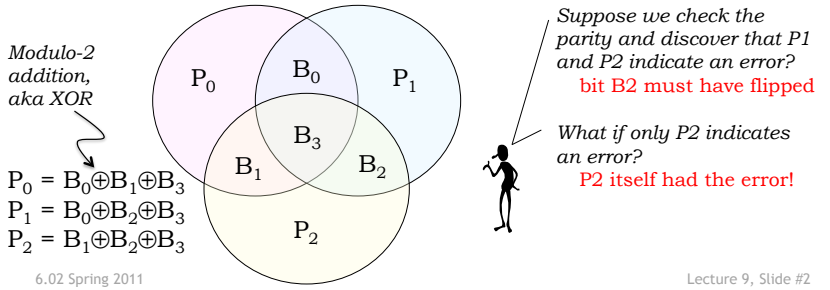
6.02 Spring 2011 Lecture #9

- How many parity bits?
- Dealing with burst errors
- Reed-Solomon codes

Single Error Correcting Codes (SECC)

Basic idea:

- Use multiple parity bits, each covering a subset of the data bits.
- No two message bits belong to exactly the same subsets, so a single error will generate a unique set of parity check errors.



Checking the parity

- Transmit: Compute the parity bits and send them along with the message bits
- Receive: After receiving the (possibly corrupted) message, compute a syndrome bit (E_i) for each parity bit. For the code on previous slide:

$$E_0 = B_0 \oplus B_1 \oplus B_3 \oplus P_0$$
$$E_1 = B_0 \oplus B_2 \oplus B_3 \oplus P_1$$
$$E_2 = B_1 \oplus B_2 \oplus B_3 \oplus P_2$$

- If all the E_i are zero: no errors!
- Otherwise the particular combination of the E_i can be used to figure out which bit to correct.

Using the Syndrome to Correct Errors

Continuing example from previous slides: there are three syndrome bits, giving us a total of 8 encodings.

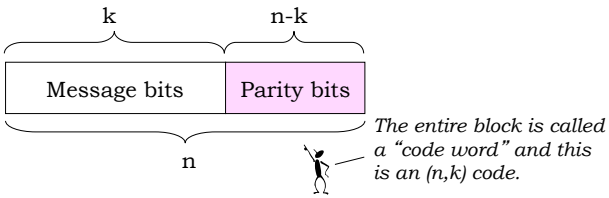
$E_2E_1E_0$	Single Error Correction
0 0 0	No errors
0 0 1	P0 has an error, flip to correct
0 1 0	P1 has an error, flip to correct
0 1 1	B0 has an error, flip to correct
1 0 0	P2 has an error, flip to correct
1 0 1	B1 has an error, flip to correct
1 1 0	B2 has an error, flip to correct
1 1 1	B3 has an error, flip to correct

What happens if there is more than one error?

The 8 encodings indicate the 8 possible correction actions: no errors, error in one of 4 data bits, error in one of 3 parity bits

(n,k,d) Systematic Block Codes

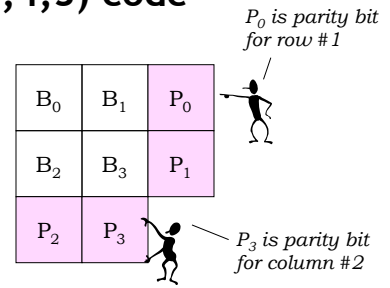
- Split message into *k*-bit blocks
- Add (*n*-*k*) parity bits to each block, making each block *n* bits long.



- Often we'll use the notation (n,k,d) where d is the minimum Hamming distance between code words.
- The ratio k/n is called the *code rate* and is a measure of the code's overhead (always ≤ 1, larger is better).

A simple (8,4,3) code

Idea: start with rectangular array of data bits, add parity checks for each row and column. Single-bit error in data will show up as parity errors in a particular row and column, pinpointing the bit that has the error.



0 1 1
1 1 0
1 0

Parity for each row and column is correct ⇒ no errors

0 1 1
1 0 0
1 0

Parity check fails for row #2 and column #2 ⇒ bit B_3 is incorrect

0 1 1
1 1 1
1 0

Parity check only fails for row #2 ⇒ bit P_1 is incorrect

Can you verify this code has a Hamming distance of 3?

How many parity bits are needed?

- Suppose we want to do single-bit error correction
  - Need unique combination of syndrome bits for each possible single bit error + no errors
  - n-bit blocks → n possible single bit errors
  - Syndrome bits all zero → no errors
- Assume n-k parity bits (out of n total bits)
  - Hence there are n-k syndrome bits
  - $2^{n-k} - 1$ non-zero combinations of n-k syndrome bits
- So, at a minimum, we need $n \le 2^{n-k} - 1$
  - Given k, use constraint to determine minimum n needed to ensure single error correction is possible
  - (n,k) Hamming SECC codes: (7,4) (15,11) (31,26)

The (7,4) Hamming SECC code is shown on slide 19, see the Notes for details on constructing the Hamming codes. The clever construction makes the syndrome bits into the index needing correction.

Error-Correcting Codes

- Parity is a (n+1,n,2) code
  - Good code rate, but only 1-bit error detection
- Replicating each bit r times is a (r,1,r) code
  - Simple way to get great error correction; poor code rate
  - Handy for solving quiz problems!
  - Number of parity bits grows linearly with size of message
- "Rectangular" codes with row/column parity
  - Easy to visualize how multiple parity bits can be used to triangulate location of 1-bit error
  - Number of parity bits grows as square root of message size
- Hamming single error correcting codes (SECC) are (n,n-p,3) where $n = 2^p - 1$ for $p > 1$
  - See Wikipedia article for details
  - Number of parity bits grows as $\log_2$ of message size

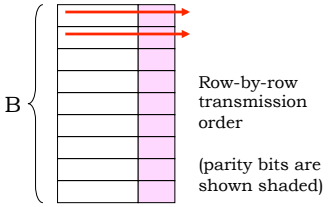
Noise models

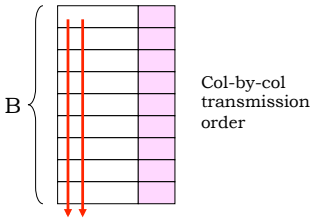
- Gaussian noise
  - Equal chance of noise at each sample
  - Gaussian PDF: low probability of large amplitude
  - Good for modeling total effect of many small, random noise sources
- Impulse noise
  - Infrequent bursts of high-amplitude noise, e.g., on a wireless channel
  - Some number of consecutive bits lost, bounded by some burst length B
  - Single-bit error correction seems like it's useless for dealing with impulse noise…
    or is it???

Correcting single-bit errors is nice, but in many situations errors come in bursts many bits long (e.g., damage to storage media, burst of interference on wireless channel, …). How does single-bit error correction help with that?

Dealing with Burst Errors

Well, can we think of a way to turn a B-bit error burst into B single-bit errors?

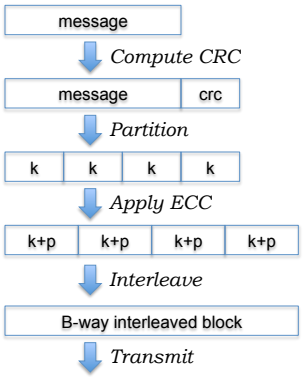


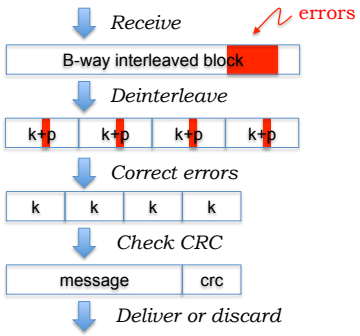


Problem: Bits from a particular code word are transmitted sequentially, so a B-bit burst produces multi-bit errors.

Solution: interleave bits from B different code words. Now a B-bit burst produces 1-bit errors in B different code words.

Interleaving





Framing

- The receiver needs to know
  - the beginning of the B-way interleaved block in order to do deinterleaving
  - the beginning of each ECC block in order to do error correction.
  - Since the interleaved block is made up of B ECC blocks, knowing where the interleaved block begins automatically supplies the necessary start info for the ECC blocks
- 8b10b encoding provides what we need! Here's what gets transmitted
  - Prefix to help train clock recovery (alternating 0s/1s, …)
  - 8b10b sync symbol
  - Packet data: B ECC blocks recoded as 8b10b symbols (after 8b10b decoding and error correction we get {#,data,chk})
  - Suffix to ensure transmitter doesn't cutoff prematurely, receiver has time to process last packet before starting search for beginning of next packet
  - On some channels: idle time (no transmission)

Our Recipe (so far)

- Transmit
  - Packetize: split message into fixed-size blocks, add sequence numbers, checksum
  - SECC: split {#,data,chk} into k-bit blocks, add parity bits to create n-bit code words with min Hamming distance of 3, B-way interleaving
  - 8b10b encoding: provide synchronization info to locate start of packet and sufficient transitions for clock recovery
  - Convert each bit into samples_per_bit voltage samples
- Receive
  - Perform clock recovery using transitions, derive bit stream from voltage samples
  - 8b10b decoding: locate sync, decode
  - SECC: deinterleave to spread out burst errors, perform error correction on n-bit blocks producing k-bit blocks
  - Packetize: verify checksum and discard faulty packets. Keep track of received sequence numbers, ask for retransmit of missing packets. Reassemble packets into original message.

Remaining agenda items

- With B ECC blocks per message, we can correct somewhere between 1 and B errors depending on where in the message they occur.
  - Can we make an ECC that corrects up to B errors without any constraints where errors occur?
  - Yes! Reed-Solomon codes
- Framing is necessary, but the sync itself can't be protected by an ECC scheme that requires framing.
  - This makes life hard for channels with higher BERs
  - Is there an error correction scheme that works on un-framed bit streams?
  - Yes! Convolutional codes: encoding and the clever decoding scheme will be discussed next week.

In search of a better code

- Problem: information about a particular message unit (bit, byte, ..) is captured in just a few locations, i.e., the message unit and some number of parity units. So a small but unfortunate set of errors might wipe out all the locations where that info resides, causing us to lose the original message unit.
- Potential Solution: figure out a way to spread the info in each message unit throughout *all* the code words in a block. Require only some fraction good code words to recover the original message.

Thought experiment…

- Suppose you had two 8-bit values to communicate: A, B
- We'd like an encoding scheme where each transmitted value included information about both A and B
  - How about sending y = Ax + B for various values of x?
  - Standardize on a particular sequence for x, known to both the transmitter and receiver. That way, we don't have to actually send the x's – the receiver will know what they are. For example, x = 1, 2, 3, 4, …
  - How many values do you need to solve for A and B?
  - We'll send extra to provide for recovery from errors…

Example

- Suppose you received four values from the transmitter y = 73, 249, 321, 393, corresponding to x = 1, 2, 3 and 4
  - 4 Eqns: A·1+B=73, A·2+B=249, A·3+B=321, A·4+B=393
- We need two of these equations to solve for A and B; there are six possible choices for which two to use
- Take each pair and solve for A and B

$A \cdot 1 + B = 73$	$A \cdot 1 + B = 73$	$A \cdot 1 + B = 73$
$A \cdot 2 + B = 249$	$A \cdot 3 + B = 321$	$A \cdot 4 + B = 393$
A=175, B=-102	*A=124, B=-51*	*A=106.6, B=-33.6*
$A \cdot 2 + B = 249$	$A \cdot 2 + B = 249$	$A \cdot 3 + B = 321$
$A \cdot 3 + B = 321$	$A \cdot 4 + B = 393$	$A \cdot 4 + B = 393$
A=72, B=105	*A=72, B=105*	*A=72, B=105*

- Majority rules: A=72, B=105
  - The received value 73 had an error
  - If no errors: all six solutions for A and B would have matched

Spreading the wealth…

- Generalize this idea: oversampled polynomials. Let

  $P(x) = m_0 + m_1x + m_2x^2 + \ldots + m_{k-1}x^{k-1}$

  where $m_0, m_1, \ldots, m_{k-1}$ are the k message units to be encoded. Transmit value of polynomial at n different predetermined points $v_0, v_1, \ldots, v_{n-1}$:

  $P(v_0), P(v_1), P(v_2), \ldots, P(v_{n-1})$

  Use any k of the received values to construct a linear system of k equations which can then be solved for k unknowns $m_0, m_1, \ldots, m_{k-1}$. Each transmitted value contains info about all m_i.
- Note that using integer arithmetic, the P(v) values are numerically greater than the m_i and so require more bits to represent than the m_i. In general the encoded message would require a lot more bits to send than the original message!

Solving for the m_i

- Solving k *linearly independent* equations for the k unknowns (i.e., the m_i):

$$\begin{pmatrix} 1 & v_0 & v_0^2 & \cdots & v_0^{k-1} \\ 1 & v_1 & v_1^2 & \cdots & v_1^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_{k-1} & v_{k-1}^2 & \cdots & v_{k-1}^{k-1} \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{k-1} \end{pmatrix} = \begin{pmatrix} P(v_0) \\ P(v_1) \\ \vdots \\ P(v_{k-1}) \end{pmatrix}$$

- Solving a set of linear equations using Gaussian Elimination (multiplying rows, switching rows, adding multiples of rows to other rows) requires add, subtract, multiply and divide operations.

- These operations (in particular division) are only well defined over *fields*, e.g., rational numbers, real numbers, complex numbers -- not at all convenient to implement in hardware.

Finite Fields to the Rescue

- Reed's & Solomon's idea: do all the arithmetic using a finite field (also called a Galois field). If the m_i have B bits, then use a finite field with order 2^B so that there will be a field element corresponding to each possible value for m_i.

- For example with B = 2, here are the tables for the various arithmetic operations for a finite field with 4 elements. Note that every operation yields an element in the field, i.e., the result is the same size as the operands.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

A	-A	A^{-1}
0	0	0
1	1	1
2	2	3
3	3	2

A + (-A) = 0 A * (A^{-1}) = 1

How many values to send?

- Note that in a Galois field of order 2^B there are at most 2^B unique values v we can use to generate the P(v)
  - if we send more than 2^B values, some of the equations we might use when solving for the m_i will not be linearly independent and we won't have enough information to find a unique solution for the m_i.
  - Sending P(0) isn't very interesting (only involves m_0)
- Reed-Solomon codes use $n = 2^B-1$ (n is the number of P(v) values we generate and send).
  - For many applications B = 8, so n = 255
  - A popular R-S code is (255,223), i.e., a code block consisting of 223 8-bit data bytes + 32 check bytes

Use for error correction

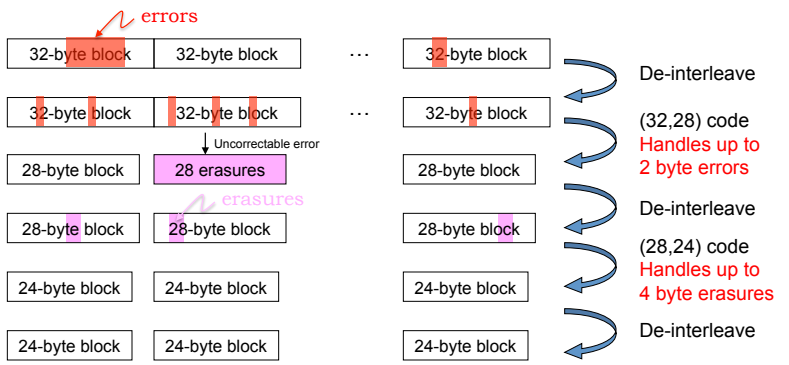
- If one of the $P(v_i)$ is received incorrectly, if it's used to solve for the m_i, we'll get the wrong result.
- So try all possible (n choose k) subsets of values and use each subset to solve for m_i. Choose solution set that gets the majority of votes.
  - No winner? Uncorrectable error… throw away block.
- (n,k) code can correct up to (n-k)/2 errors since we need enough good values to ensure that the correct solution set gets a majority of the votes.
  - R-S (255,223) code can correct up to 16 symbol errors; good for error bursts: 16 consecutive symbols = 128 bits!

Erasures are special

- If a particular received value is known to be erroneous (an "erasure"), don't use it all!
  - How to tell when received value is erroneous? Sometimes there's channel information, e.g., carrier disappears.
  - See next slide for clever idea based on concatenated R-S codes
- (n,k) R-S code can correct n-k erasures since we only need k equations to solve for the k unknowns.
- Any combination of E errors and S erasures can be corrected so long as $2E + S \le n-k$.

Example: CD error correction

- On a CD: two concatenated R-S codes



Result: correct up to 3500-bit error bursts (2.4mm on CD surface)