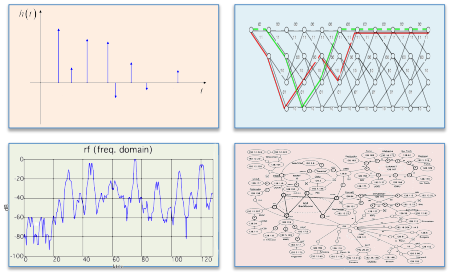


INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2011
Lecture #14

- complex exponentials
- discrete-time Fourier series
- spectral coefficients
- band-limited signals

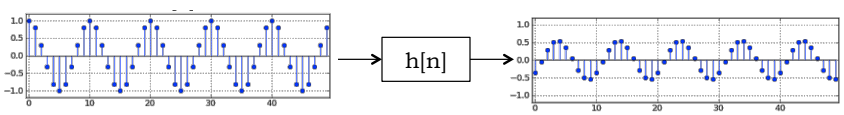
Frequency Division Multiplexing

To engineer the sharing of a channel through frequency division multiplexing we'll need a new set tools that will let us understand the behavior of signals and systems in the frequency domain. Plan:

– This week
- Analyze the frequency content of signals using the discrete-time Fourier series
- Determine what happens when we *band-limit* a signal
- Characterize LTI systems by their frequency response
- Introduce *filters*: LTI systems that eliminate a region of frequencies from a signal

– Next week
- Using modulation to position band-limited signals in different regions of the frequency spectrum
- Receiving a particular signal from a shared spectrum

Sinusoids and LTI Systems



Frequency division multiplexing depends on an interesting property of LTI channels:

> if the channel input x[n] is a sinusoid of a given amplitude, frequency and phase, the response will be a sinusoid *at the same frequency*, although the amplitude and phase may be altered. As we'll see, the change in amplitude and phase may depend on the frequency of the input.

The same property holds when the inputs are complex exponentials, which are closely related to sines and cosines (and, perhaps surprisingly, are much easier to analyze!).

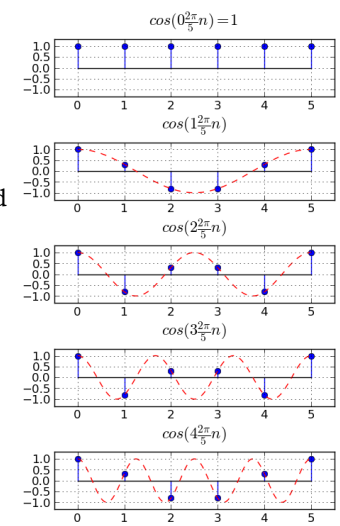
Periodic Sequences

A sequence x[n] is said to be periodic with a period of N samples ("periodic with period N") if

$$x[n] = x[n+N]$$

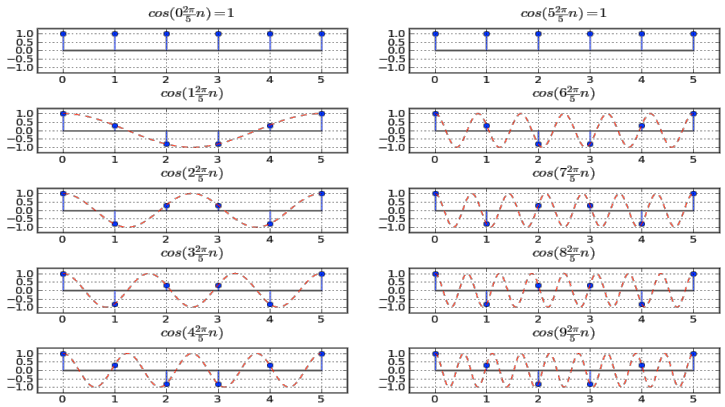
A sequence that is periodic with period N is also periodic with period 2N, 3N, …, and so on.

A sinusoidal sequence that is periodic with period N can only have one of a finite number of frequencies: all the harmonics of the fundamental frequency 2π/N radians/sample, i.e., frequencies of the form k·(2π/N) for some integer k.



Frequencies k·(2π/N) when k ≥ N

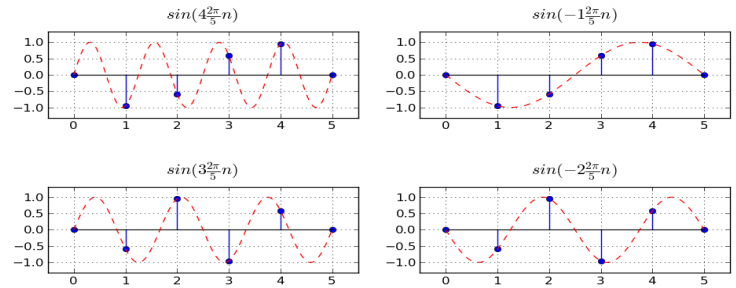
Frequency k·(2π/N) yields the same sequence as (k mod N)·(2π/N)



Unique sequences: k = 0, 1, 2, …, N-1

Highest frequency: (N-1)·(2π/N) < 2π

Negative Frequencies



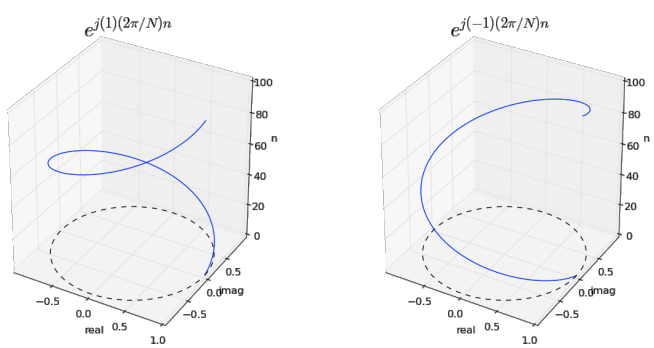
Sequences of frequency k·(2π/N), i.e., frequencies between 0 and 2π, are identical to sequences of frequency -(N-k)·(2π/N), i.e., between -2π and 0.

In 6.02, our convention will be to specify frequencies in the range -π and π, corresponding to k's in the range –(N/2) to (N/2).

Complex Exponentials

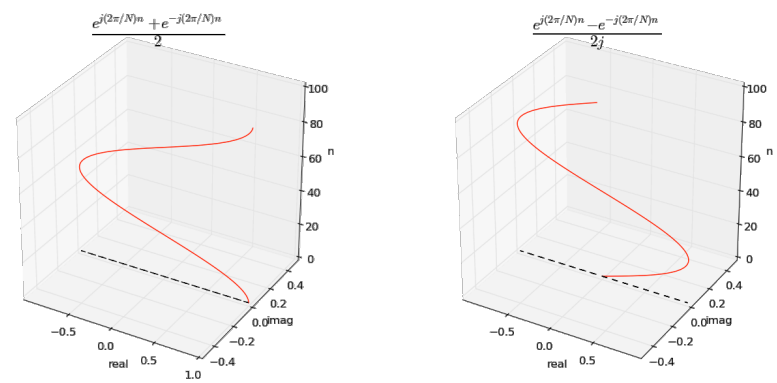
A complex exponential is a complex-valued function of a single argument – an angle measured in radians. Euler's formula shows the relation between complex exponentials and our usual trig functions:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$



Sine and Cosine and $e^{j\varphi}$

$$\cos(\varphi) = \frac{1}{2}e^{j\varphi} + \frac{1}{2}e^{-j\varphi} \qquad \sin(\varphi) = \frac{1}{2j}e^{j\varphi} - \frac{1}{2j}e^{-j\varphi}$$



Useful Properties of $e^{j\varphi}$

When $\varphi = 0$:

$$e^{j0} = 1$$

When $\varphi = \pm\pi$:

$$e^{j\pi} = e^{-j\pi} = -1$$

$$e^{j\pi n} = e^{-j\pi n} = (-1)^n$$

Summing samples over one period:

$$\sum_{n=\langle N\rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

n ranges over any N consecutive integers, e.g., n = 0, 1, …, N-1

Discrete-time Fourier Series

If x[n] is periodic with period N, it can be expressed as the sum of scaled periodic complex exponentials:

Complex exponential with period N and fundamental frequency 2π/N.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

k ranges over any N consecutive integers. Two common choices:

- k starts at 0 (0 ≤ freq ≤ 2π)
- k starts at –N/2 (-π ≤ freq ≤ π)

The *spectral coefficients* a_k for each of the discrete frequencies are, in general, complex, changing both the amplitude and phase of the associated complex exponential. If x[n] is real, $a_{-k} = a_k^*$.

Solving for the a_k

Start with:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Multiply both sides by $e^{-jr(2\pi/N)n}$ and sum over N terms:

$$\sum_{n=\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}}$$

$$= a_r N$$

From slide 9: N if k=r, 0 otherwise

$$\boxed{a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}}$$

Discrete-time Fourier Series Pair

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$ Synthesis equation

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$ Analysis equation

If we have N samples of a periodic waveform of period N, we can find the waveform's spectral coefficients using the analysis equation.

If we have the spectral coefficients, we can reconstruct the original time-domain waveform using the synthesis equation.

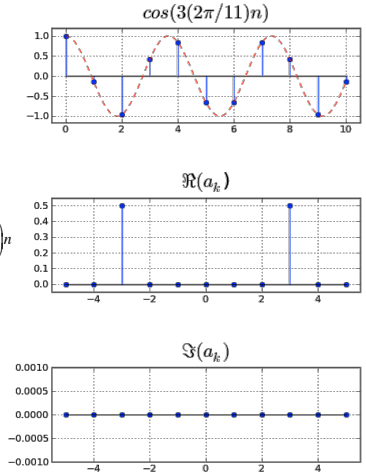
$$x[n] = \cos(r\frac{2\pi}{N}n)$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle}\cos(r\frac{2\pi}{N}n)e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{2N}\sum_{n=\langle N\rangle}\left[e^{jr\left(\frac{2\pi}{N}\right)n} + e^{-jr\left(\frac{2\pi}{N}\right)n}\right]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{2N}\sum_{n=\langle N\rangle}e^{j(r-k)\left(\frac{2\pi}{N}\right)n} + \frac{1}{2N}\sum_{n=\langle N\rangle}e^{-j(r+k)\left(\frac{2\pi}{N}\right)n}$$

$$= \begin{cases} \frac{1}{2} & k = \pm r \\ 0 & \text{otherwise} \end{cases}$$



$$x[n] = \sin(r\frac{2\pi}{N}n)$$

This time let's do it "by inspection". First rewrite x[n] (see slide #8):

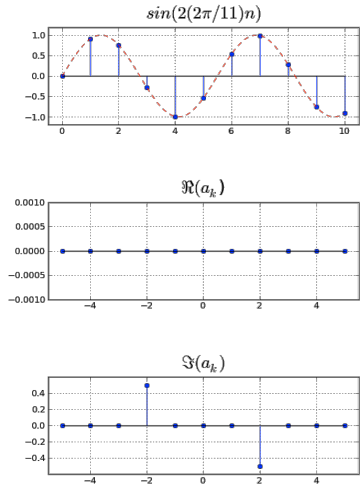
$$x[n] = \frac{1}{2j}e^{jr\frac{2\pi}{N}n} - \frac{1}{2j}e^{j(-r)\frac{2\pi}{N}n}$$

Now x[n] is a sum of complex exponentials and we can determine the a_k directly from the equation;

$$a_r = \frac{1}{2j} = -\frac{j}{2}$$

$$a_{-r} = -\frac{1}{2j} = \frac{j}{2}$$

$$a_k = 0 \quad \text{otherwise}$$



$$x[n] = 1 + 2\cos(3\frac{2\pi}{11}n) - 3\sin(5\frac{2\pi}{11}n)$$

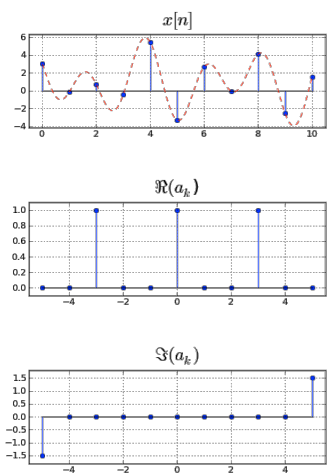
Again, by inspection: since the cos and sin are different frequencies, we can analyze them separately.

a_0 = average value = 1

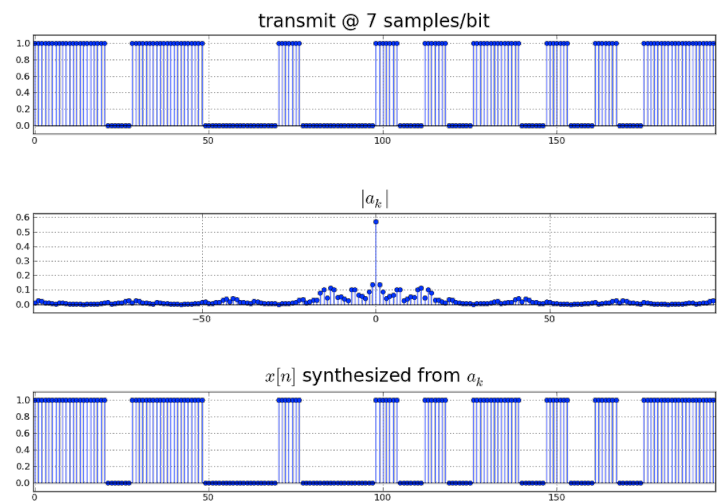
$a_{\pm 3}$ = 2(1/2) = 1 [from cos term]

a_{-5} = -3(j/2) = -1.5j [from sin term]
a_5 = -3(-j/2) = 1.5j

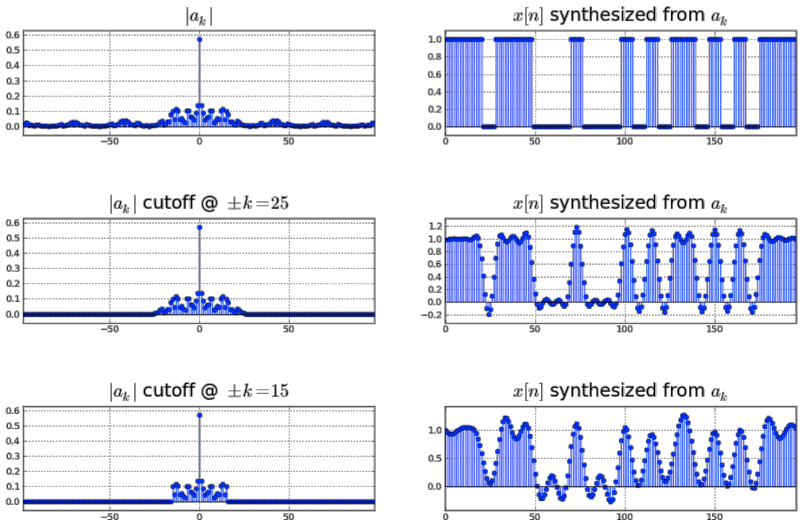
a_k = 0 otherwise



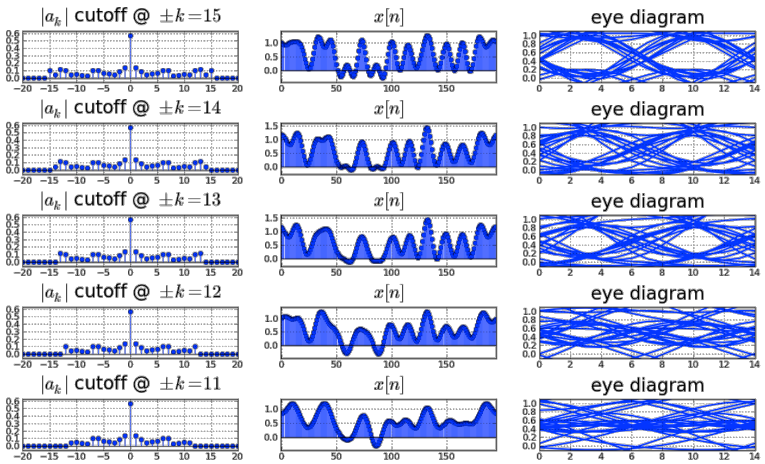
Spectrum of Digital Transmissions



Effect of Band-limiting a Transmission



How Low Can We Go?



7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14