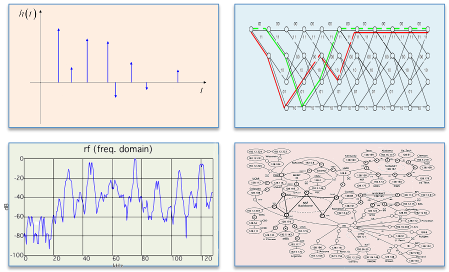


INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Spring 2011
Lecture #15

- frequency response
- LTI systems with "zeros"
- filters

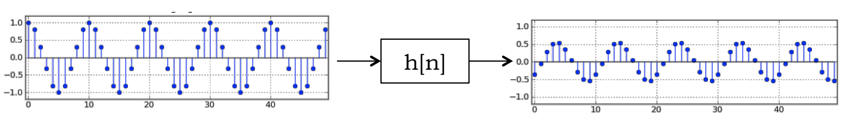
From Last Time

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$ Synthesis equation

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$ Analysis equation

- $x[n]$ and a_k are both periodic with period N
- $2\pi/N$ radians/sample is the fundamental frequency. Complex exponentials in Fourier series equations have frequencies which are some harmonic of $2\pi/N$
- If $x[n]$ is real, $a_{-k} = a_k^*$ (i.e., they are complex conjugates)
- a_0 is the average of the $x[n]$

Complex Exponentials and LTI Systems



Frequency division multiplexing depends on an interesting property of LTI channels:

> if the channel input x[n] is a complex exponential of a given amplitude, frequency and phase, the response will be a complex exponential *at the same frequency*, although the amplitude and phase may be altered. As we'll see, the change in amplitude and phase will, in general, depend on the frequency of the input.

Let's prove this to be true…

Frequency Response

$$Ae^{j\Omega n} \rightarrow \boxed{h[n]} \rightarrow y[n]$$

Using the convolution sum we can compute the system's response to a complex exponential as input:

$$\begin{aligned} y[n] &= \sum_m h[m]x[n-m] \\ &= \sum_m h[m]Ae^{j\Omega(n-m)} \\ &= Ae^{j\Omega n}\sum_m h[m]e^{-j\Omega m} \\ &= x[n]\cdot H(e^{j\Omega}) \end{aligned}$$

where we've defined the *frequency response* of a system as

$$\boxed{H(e^{j\Omega}) \equiv \sum_m h[m]e^{-j\Omega m}}$$

Reminds us it's the response for complex exponentials

Another Way to Characterize LTI Systems

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow y[n] = \sum_{k=\langle N \rangle} a_k H(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

The frequency response tells us how the system will affect each of the spectral coefficients that determine the input. As you can see from the equation above

$$b_k = a_k H(e^{jk\frac{2\pi}{N}})$$

are the spectral coefficients for y[n]. The frequency response completely characterizes an LTI system in the frequency domain, just as the unit sample response completely characterizes the system in the time domain.

Unit Sample and Response

$$\delta[n] \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow h[n]$$

We can compute the (periodic) spectral coefficients for the (periodic) unit sample:

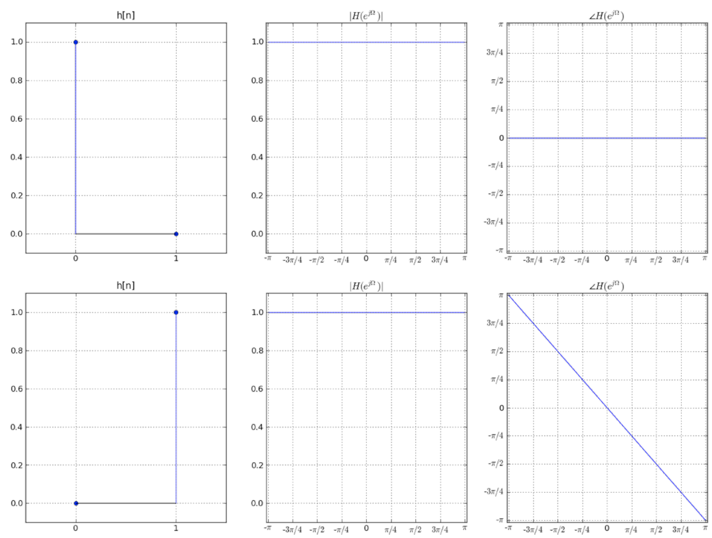
$$a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}x[0]e^{-jk\frac{2\pi}{N}0} = \frac{1}{N}$$

Now use our new formula for the (periodic) system response from the previous slide:

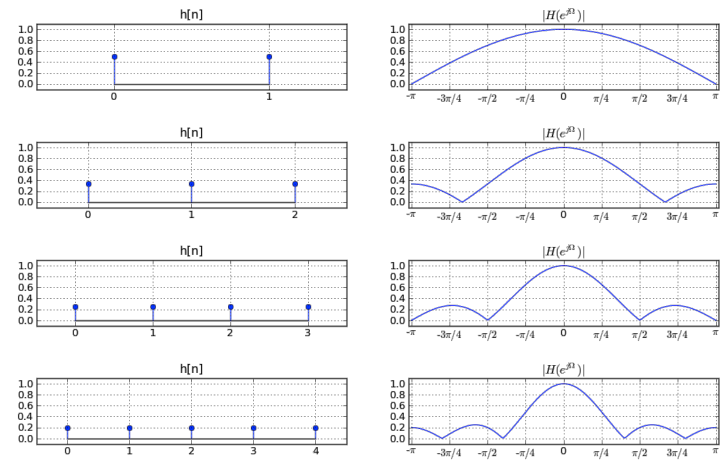
$$h[n] = \sum_{k=\langle N \rangle} \frac{1}{N} H(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

This is the synthesis equation!

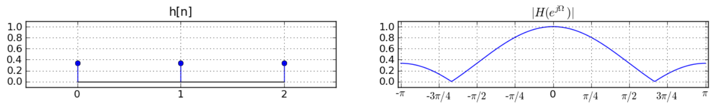
Example h[n] and $H(e^{j\Omega})$



Moving Averages



$H(e^{j\Omega})$ with Zeros



$$H(e^{j\Omega}) = \sum_m h[m]e^{-j\Omega m} = h[0]e^{-j\Omega 0} + h[1]e^{-j\Omega 1} + h[2]e^{-j\Omega 2}$$
$$= h[0] + h[1](e^{-j\Omega}) + h[2](e^{-j\Omega})^2$$

Hmm. A quadratic equation with two roots at frequency ±φ:

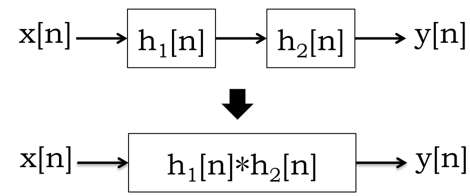
$$H(e^{j\Omega}) = (e^{j\Omega} - e^{j\varphi})(e^{j\Omega} - e^{-j\varphi})$$
$$= (e^{j\Omega})^2 - (e^{j\varphi} + e^{-j\varphi})(e^{j\Omega}) + e^{j\varphi}e^{-j\varphi}$$
$$= (e^{j\Omega})^2 - 2\cos(\varphi)(e^{j\Omega}) + 1$$

Matching terms in the two equations, we see that an LTI system would have a frequency response that went to zero at ±φ if

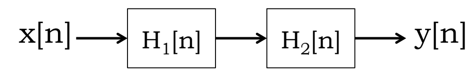
h[0]=1, h[1]=-2cos(φ) and h[2] = 1.

Series Interconnection of LTI Systems

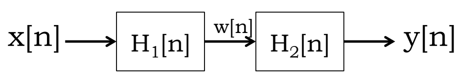
From Lecture 4:



How about:

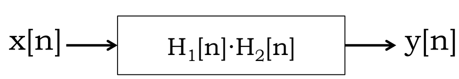


Series Interconnection of LTI Systems



$$w[n] = \sum_{k=\langle N\rangle} a_k H_1(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n} \equiv \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n}$$

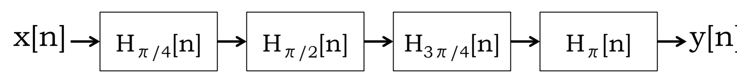
$$y[n] = \sum_{k=\langle N\rangle} b_k H_2(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n} = \sum_{k=\langle N\rangle} a_k H_1(e^{jk\frac{2\pi}{N}}) H_2(e^{jk\frac{2\pi}{N}})\, e^{jk\frac{2\pi}{N}n}$$

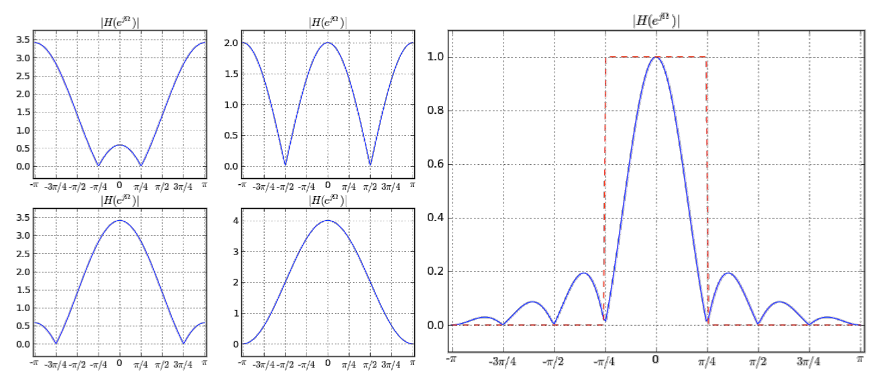


Frequency response of two LTI systems in series is the term-by-term product of the individual frequency responses.

A Poor Man's Low-pass Filter

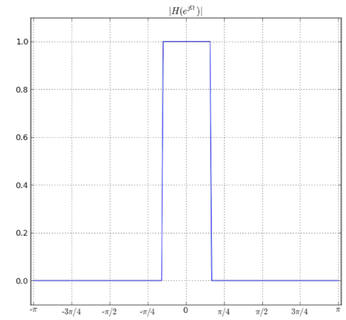
Suppose we wanted a low-pass filter with a cutoff frequency of π/4?

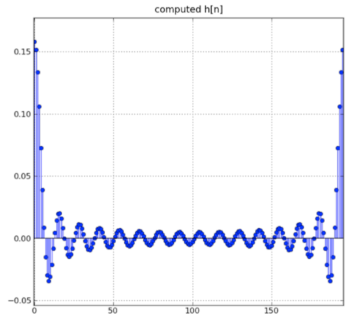




Wait! Maybe This Will Work Better…

Suppose we draw the frequency response we want and then use the equation on slide 6 to compute h[n] from the proposed $H(e^{j\Omega})$?

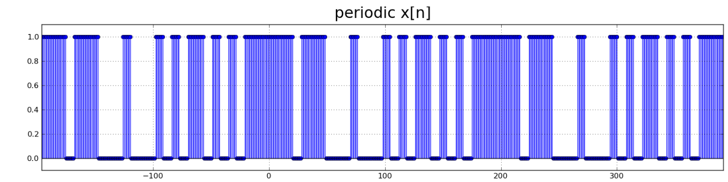




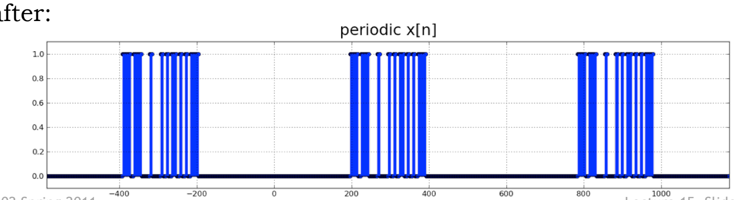
Example from previous lecture: N=196, cutoff at ±k=15.

Dealing With Periodicity Issues

Remembering that everything is periodic with period N, is this the signal we want to filter?

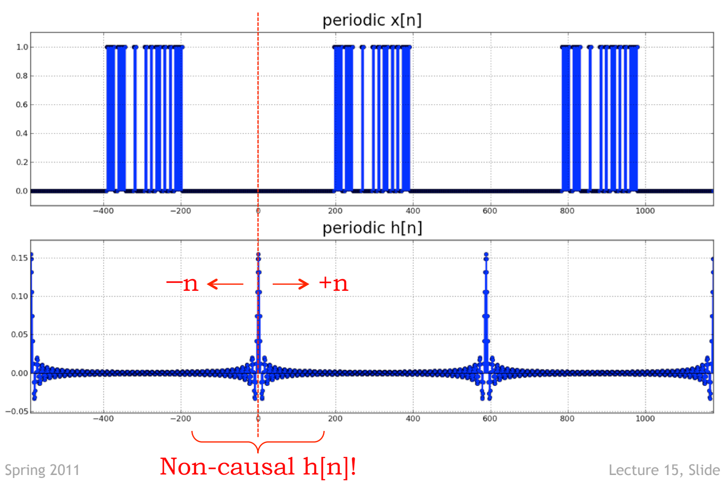


If we really want to see what happens when we filter a transmission that starts and stops, we want zeros before and after:



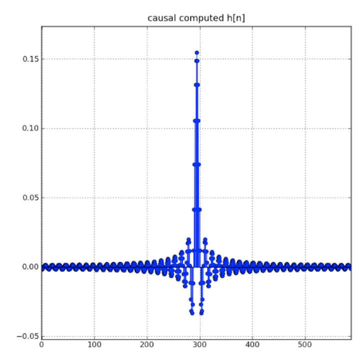
Updated Filtering Plan

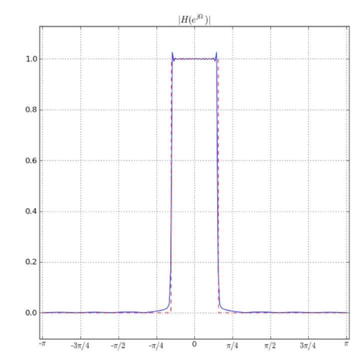
Now that N has grown because of the zero padding, we have to recompute our h[n] using the larger N:



Computed h[n] for Low-pass Filter

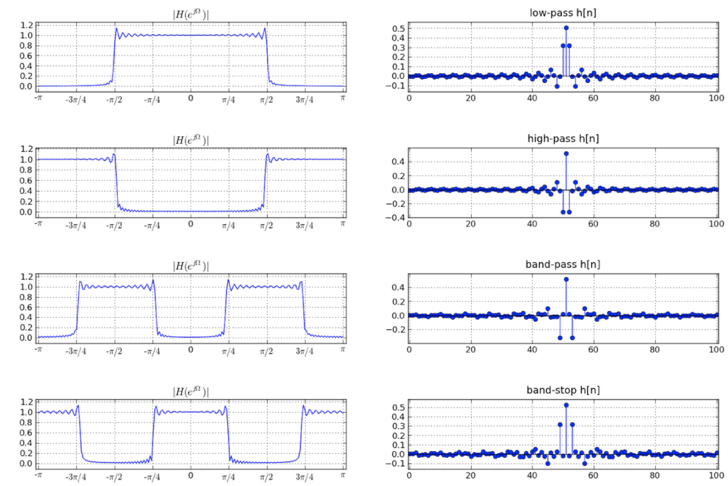
If you're hung up on causality (which admittedly is useful for real-time signal processing), we can make a causal h[n] by adding an N/2 sample delay.





google "FIR filter"

Useful Filters



The Need for Speed

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Computing these series involves $O(N^2)$ operations – when N gets large, the computations get very s l o w....

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka FFT, IFFT), rediscovering a technique known to Gauss. This method takes O(N log N) operations.

$N = 1000$, $N^2 = 1000000$ $N\log N \approx 10000$

Caveat: scaling is different for the FFT: the spectral coefficients *aren't* scaled by 1/N – that scaling happens on the inverse transform back to the time domain.

Frequency Response of Channels