MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

6.02 Introduction to EECS II
Spring 2011

Quiz 2

Name SOLUTIONS	Score

☐ 10a Devavrat Shah 24-402

☐ 11a Devavrat Shah 24-402

☐ 12n Fabian Lim 38-166

☐ 1p John Sun 38-166

☐ 2p John Sun 38-166

Please write your answers legibly in the spaces provided.   You can use the backs of the pages if you need extra room for your answer or scratch work. Make sure we can find your answer!

You can use a calculator and one 8.5" x 11" cribsheet.

Partial credit will only be given in cases where you show your work and (very briefly) explain your approach.

Prob. #1 (20 pts)	Prob. #2 (20 pts)	Prob. #3 (20 pts)	Prob. #4 (20 pts)	Prob. #5 (20 pts)

Problem 1. Gaussian Noise and Bit Error Rates (20 points)

Consider the figure below, which shows the step response for a particular transmission channel along with the eye diagram for channel response when transmitting 4 samples/bit.
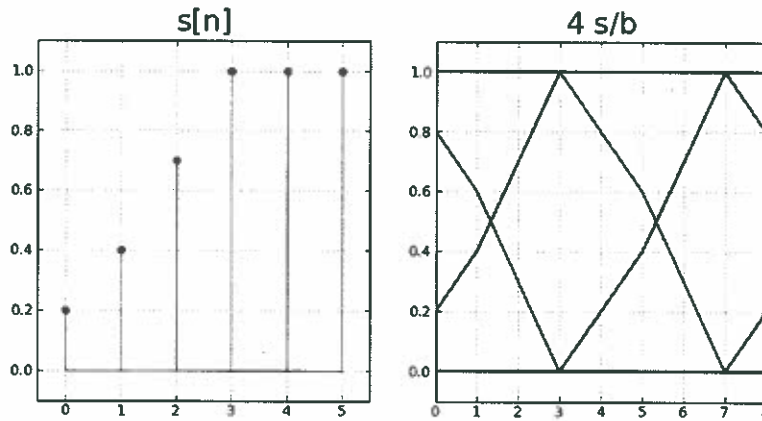
x	Φ(x)
-3.4	0.0003
-3.3	0.0005
-3.2	0.0007
-3.1	0.0010
-3.0	0.0013
-2.9	0.0019
-2.8	0.0026
-2.7	0.0035
-2.6	0.0047
-2.5	0.0062
-2.4	0.0082
-2.3	0.0107
-2.2	0.0139
-2.1	0.0179
-2.0	0.0228
-1.9	0.0287
-1.8	0.0359
-1.7	0.0446
-1.6	0.0548
-1.5	0.0668
-1.4	0.0808
-1.3	0.0968
-1.2	0.1151
-1.1	0.1357
-1.0	0.1587
-0.9	0.1841
-0.8	0.2119
-0.7	0.2420
-0.6	0.2743
-0.5	0.3085
-0.4	0.3446
-0.3	0.3821
-0.2	0.4207
-0.1	0.4602
-0.0	0.5000

Suppose there is additive Gaussian noise on this channel that sometimes causes a transmitted bit to be misidentified at the receiver. In answering the questions below, please assume that
  - the receiver uses the optimal detection sample for each bit (corresponding to the "center" of the eye)
  - the receiver uses a detection threshold of 0.5V
  - 0 and 1 bits are transmitted with probability of 0.5
  - the additive noise is independent of the bit being transmitted and has a Gaussian distribution with zero mean and standard deviation σ

(A) (8 points) When sending 4 samples/bit, if the bit error rate is measured to be .0062, what is the approximate value for σ, the standard deviation for the additive Gaussian noise? The table to the right shows $\Phi(x)$, the cumulative distribution function for the normal (Gaussian) distribution. Please show your work.

Value for σ: 0.2

$$BER = .0062 = \frac{1}{2}\Phi\left(\frac{.5-1}{\sigma}\right) + \frac{1}{2}\left(1 - \Phi\left(\frac{.5-0}{\sigma}\right)\right)$$

$$= \frac{1}{2}\Phi\left(\frac{-.5}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{-.5}{\sigma}\right) =$$

$$= \Phi\left(\frac{-.5}{\sigma}\right)$$

$$\Rightarrow \frac{-.5}{\sigma} = -2.5 \quad \text{so} \quad \sigma = \frac{-.5}{-2.5} = .2$$

When transmitting at 3 bits/sample down the same channel, the designers are trying to decide between two alternatives for the transmitter design:

1. the transmitter sends 0V for a 0 bit and 1V for a 1 bit, resulting in the eye diagram shown on the left in the figure below. The possible sample voltages at the center of the eye are 0V, 0.2V, 0.8V and 1.0V.

2. the transmitter sends 0V for a 0 bit and 1.5V for a 1 bit, resulting in the eye diagram shown on the right in the figure below. The possible sample voltages at the center of the eye are 0V, 0.3V, 1.2V and 1.5V. The designers are hypothesizing that the increased transmission range will lower the bit error rate.

In either alternative *the receiver threshold is unchanged at 0.5V.*
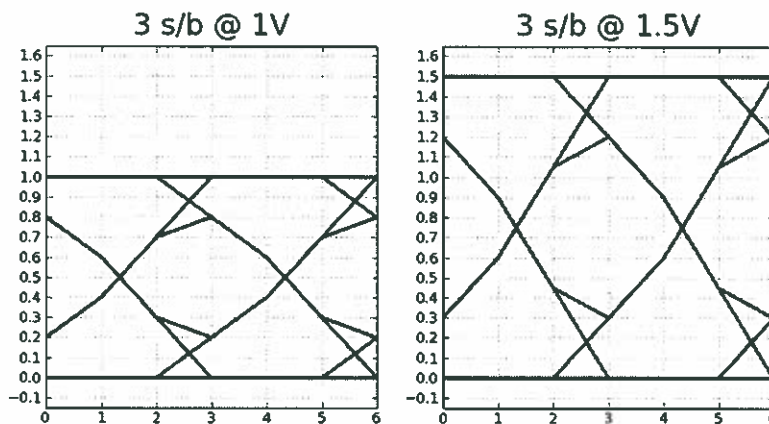
(B) (12 points) Assume that the Gaussian noise has $\sigma = 0.5$. For each alternative compute an estimate for the bit error rate using all four possible sample voltages. Indicate which alternative results in the fewest bit errors at the receiver. Please show your work.

BER for alternative 1: <u>0.2165</u>

BER for alternative 2: <u>0.1517</u>

Alternative with fewest bit errors: <u>ALT #2</u>

ALT #1: $BER = \frac{1}{4}\left[\Phi\left(\frac{.5-1}{.5}\right) + \Phi\left(\frac{.5-.8}{.5}\right) + \left(1-\Phi\left(\frac{.5-.2}{.5}\right)\right) + \left(1-\Phi\left(\frac{.5-0}{.5}\right)\right)\right]$

$= \frac{1}{4}\left[\Phi(-1) + \Phi(-.6) + \Phi(-.6) + \Phi(-1)\right]$

$= \frac{1}{4}\left[.1587 + .2743 + .2743 + .1587\right] = .2165$

ALT #2: $BER = \frac{1}{4}\left[\Phi\left(\frac{.5-1.5}{.5}\right) + \Phi\left(\frac{.5-1.2}{.5}\right) + \left(1-\Phi\left(\frac{.5-.3}{.5}\right)\right) + \left(1-\Phi\left(\frac{.5-0}{.5}\right)\right)\right]$

$= \frac{1}{4}\left[\Phi(-2) + \Phi(-1.4) + \Phi(-.4) + \Phi(-1)\right] = .1517$

$\quad\quad .0228 \quad\quad .0808 \quad .3446 \quad\quad .1587$

Problem 2. Error Detection and Correction (20 points)

(A) (3 points) Suppose p is the probability of a bit error. What is the probability that a codeword of N bits experiences more than 1 bit error?

Give expression

$$P(> 1 \text{ error}) = 1 - p(\text{no error}) - p(\text{exactly 1 error})$$

$$= 1 - (1-p)^N - Np(1-p)^{N-1}$$

(B) (3 points) Alice wants to transmit 4-bit messages along with any additional parity bits needed to perform single error correction at the receiver. What is the minimum number of bits needed in the transmitted codewords (message bits + parity bits) in order to ensure that single error correction is possible?

Minimum number of bits in codeword: $\underline{7}$

$$n \le 2^{n-4} - 1 \implies \text{smallest } n \text{ is } 7.$$

i.e. the Hamming $(7,4,3)$ code.

Bob is designing a channel coding scheme for a new spacecraft that will be sending back pictures from Io, one of the moons of Jupiter. He has chosen a (9,1,9) replication code, i.e., the two codewords are 000000000 and 111111111.

(C) (3 points) What is the largest number of bit errors in a single code word that can be corrected using the (9,1,9) code?

$$\# \text{ corrected} = \left\lfloor \frac{HD - 1}{2} \right\rfloor = \left\lfloor \frac{9-1}{2} \right\rfloor$$

Largest number of correctable bit errors: $\underline{4}$

(D) (3 points) Bob is particularly worried about burst errors, i.e., multiple bit errors that occur in successive bits. Suppose Bob would like correct error bursts of up to 10 bits by interleaving a block of (9,1,9) code words. Given your answer to part C, what is the minimum number of words that can be interleaved to handle burst errors up to 10 bits?

Minimum value for I: $\underline{3}$

with 3-way interleaving, a 10-bit burst error will be
deinterleaved into:
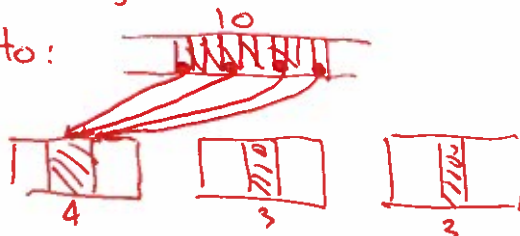
An Internet Sudoku gaming site transmits messages containing nine data bits and seven parity bits, arranged in a rectangle as follows:

D_{00}	D_{01}	D_{02}	P_{0x}
D_{10}	D_{11}	D_{12}	P_{1x}
D_{20}	D_{21}	D_{22}	P_{2x}
P_{x0}	P_{x1}	P_{x2}	P_{xx}

Each D_{ij} in the above diagram indicates a data bit, equally likely to be a 0 or 1. Each P_{ix} and P_{xj} is an even parity bit chosen to make the total number of 1s in the ith row or j^{th} column, respectively, even. P_{xx} is an even parity bit chosen to make the total number of 1s in the entire transmission even. Thus in an error-free transmission, the total number of 1s in 4-bit columns 0 thru 2 and 4-bit rows 0 thru 2, as well as in the entire 16-bit transmission, is even.

(E) (4 points) Suppose two nine-bit data words have a Hamming distance of 1. What is the Hamming distance between the 16-bit transmissions resulting from these two data words?

changing D ⇒ change row & col P bits

Hamming distance between transmissions: ___4___

∴ 3 bits have changed → change P_{xx}

Each of the following represents a transmission received, with at most a single-bit error. For each message, circle the bit that was changed due to a transmission error, or write NO ERROR if no errors are detected (1 point each).

Circle bit in error or write NO ERROR

(F)   *NO ERROR*

1	0	1	0
0	1	1	0
1	1	0	0
0	0	0	0

(G)

1	0	1	0
1	1	0	0
0	1	1	0
0	(1)	0	0

(H)

(0)	1	0	0
0	0	1	1
1	1	0	0
0	0	1	1

(I)

0	1	0	1
1	0	1	0
0	1	1	0
1	0	0	(0)

Problem 3. Convolutional Codes (20 points)

Consider the following trellis showing the operation of the Viterbi algorithm using a <u>hard branch metric</u> at the receiver as it processes a message encoded with a convolutional code C. Most of the path metrics have been filled in for each state at each time and the predecessor states determined by the Viterbi algorithm are shown by a solid transition arrow.

Time:	1	2	3	4	5	6	7	8	9
Rcvd:	111	001	011	111	???	000	001	111	110
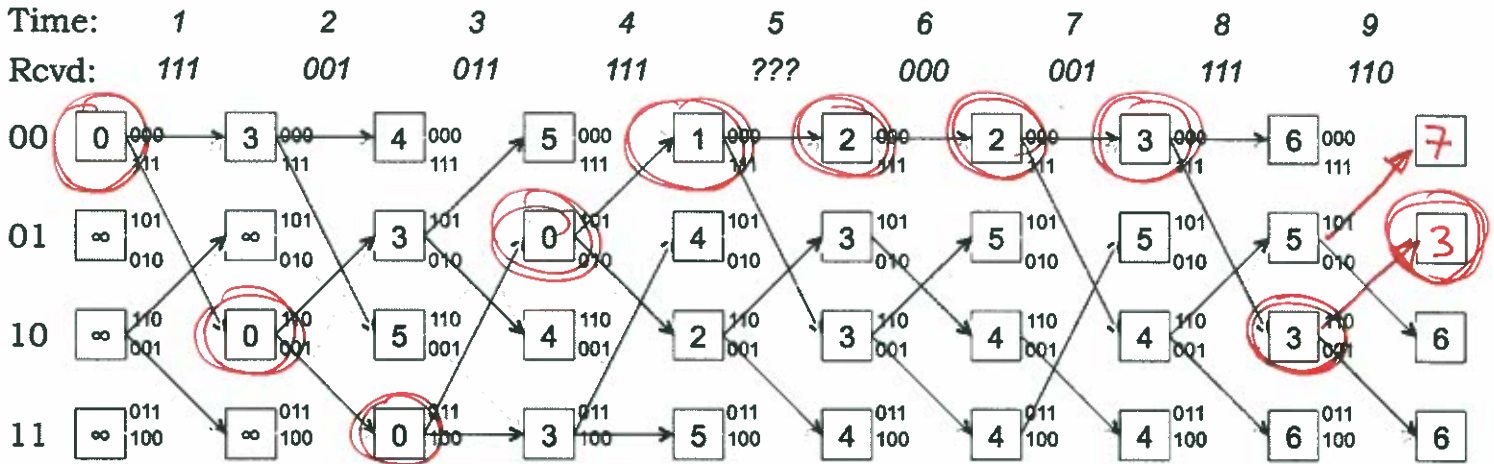
(A) (1 point) What is the code rate and constraint length of the convolutional code C?

3 parity bits sent
for each message
bit.

Code rate: __1/3__   Constraint length: __3__

(B) (2 points) What bits would be transmitted if the message "1011" were encoded using C?
Note this is not the message being decoding in the trellis above.

Transmitted bits: __111 110 010 001__

1   0   1   1

(C) (2 points) Please compute the missing path metrics in the <u>top two boxes of rightmost column</u> and enter their value in the appropriate boxes in the trellis diagram. Remember to draw the solid transition arrow showing the predecessor state for each metric you compute.

Compute missing path metrics, indicate predecessor state

for state 00: PM = min (6+2, 5+2) = 7
        from 00   01

for state 01: PM = min (3+0, 6+2) = 3
        from 10   11

(D) (4 points) The received parity bits for time 5 are missing from the trellis diagram. What values for the parity bits are consistent with the other information in the trellis? Note that there may be more than one set of such values.

Possible values for received parity bits at time 5: <u>010, 100</u>

using PMs!

Ⓐ $HD(???, 000) = 1$

Ⓑ $HD(???, 111) = 2$

Ⓒ $HD(???, 110) = 1$

Ⓓ $HD(???, 001) = 2$

???	
000	✗ violates Ⓐ
001	✗ violates Ⓒ
010	✓
011	✗ violates Ⓓ
100	✓
101	✗ violates Ⓐ
110	✗ violates Ⓑ
111	✗ violates Ⓐ

(E) (7 points) In the trellis diagram on the previous page, circle the states along the most-likely path through the trellis. Determine the decoded message that corresponds to that most-likely path.

Circle states on most-likely path

Decoded message: <u>110 000 010</u>

(6.02 in octal!)

(F) (4 points) Based on your answer to part E, how many bit errors were detected in the received transmission and at what time(s) did those error(s) occur?

PM of final state ⇒ **Number of bit errors detected:** <u>3</u>

Time(s) at which bit errors occurred: <u>4, 5, 7</u>

↳ where PM increments along most-likely path

Problem 4. MAC Protocols (20 points)

(A) (2 points each) Which of the following statements are <u>always</u> true for networks with $N > 1$ nodes using correctly implemented versions of unslotted Aloha, slotted Aloha, Time Division Multiple Access (TDMA) and Carrier Sense Multiple Access (CSMA)? Unless otherwise stated, assume that the slotted and unslotted versions of Aloha are stabilized and use the same stabilization method and parameters.

Circle TRUE or FALSE, then briefly explain or give counter example

(TRUE) FALSE   There exists some offered load pattern for which TDMA has lower throughput than slotted Aloha.

Explain: suppose 1 node is backlogged, others aren't. TDMA throughput = $\frac{1}{N}$, slotted Aloha thruput $\simeq 1$.

TRUE **(FALSE)**   In a slotted Aloha MAC protocol using binary exponential backoff, the probability of transmission will always eventually converge to some value p, and all nodes will eventually transmit with probability p.

Explain: in a stabilized Aloha protocol, p is always changing with successes & failures. $\Rightarrow$ does not converge to a particular value

TRUE **(FALSE)**   Suppose nodes I, II and III use a fixed probability of $p = 1/3$ when transmitting on a 3-node slotted Aloha network. If all the nodes are backlogged then over time the utilization averages out to $1/e \approx 37\%$.

Explain: $U = 3\left(\frac{1}{3}\right)\left(1-\frac{1}{3}\right)^2 = \frac{4}{9} \neq \frac{1}{e}$

TRUE **(FALSE)**   When the number of nodes, N, is large in a stabilized slotted Aloha network, setting $p_{max} = p_{min} = 1/N$ will achieve the same utilization as a TDMA network if all the nodes are backlogged.

Explain: TDMA can achieve full utilization but Aloha achieves $\sim \frac{1}{e}$

TRUE **(FALSE)**   Using contention windows with a CSMA implementation guarantees that a packet will be transmitted <u>successfully</u> (i.e., without collisions) within some bounded time.

Explain: contention windows guarantee a transmission <u>attempt</u> within bounded time, but there's no guarantee of <u>success</u>

Suppose that there are three nodes – A, B and C – seeking access to a shared medium using slotted Aloha, each using some fixed probability of transmission, where each packet takes one slot to transmit. Assume that the nodes are always backlogged, and that node A has half the probability of transmission as the other two, i.e., $p_A = p$ and $p_B = p_C = 2p$.

(B) (4 points) If $p_A = .3$, compute the average utilization of the network. Please show your work.

Average utilization of network: $\underline{0.384}$

$$U = (p)(1-2p)^2 + 2(2p)(1-p)(1-2p)$$

$$= (.3)(.4)^2 + 2(.6)(.7)(.4)$$

$$= 0.048 + .336 = 0.384$$

(C) (6 points) What value of p_A maximizes the average utilization of the network and what is the corresponding maximum utilization? Please show your work.

value of p_A that maximizes utilization: $\underline{0.202}$

corresponding maximum utilization: $\underline{0.456}$

Note: $p < .5$ since p_B & p_C must be <1.

$$U = p(1-2p)^2 + 2(2p)(1-p)(1-2p)$$

$$= p(1-4p+4p^2) + 4p(1-3p+2p^2)$$

$$= p - 4p^2 + 4p^3 + 4p - 12p^2 + 8p^3$$

$$= 5p - 16p^2 + 12p^3$$

max when $\dfrac{dU}{dp} = 0 = 5 - 32p + 36p^2$

roots p $\dfrac{32 \pm \sqrt{1024-720}}{72} = .687, .202$ $\overset{<.5}{\checkmark}$

$\Rightarrow U_{max} = 5(.202) - 16(.202)^2 + 12(.202)^3 = .456$

Problem 5. Fourier Series and Frequency Response (20 points)

The spectral coefficients, a_k, for a real-valued periodic signal $x[n]$ are plotted below using $N = 25$. Since the spectral coefficients are complex numbers, the plots show the real part and imaginary part for each index k, $-12 \leq k \leq 12$. The fundamental frequency of this periodic signal is $2\pi/N = 0.08\pi$ radians/timestep.

$\Re(a_k)$

$\Im(a_k)$
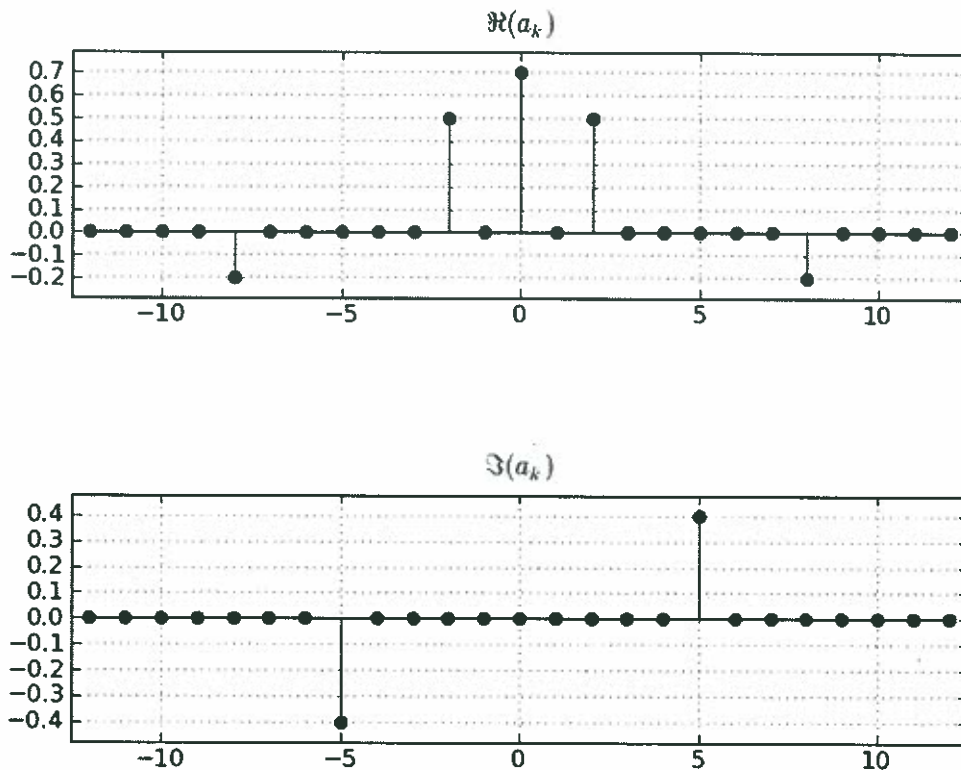
(A) (5 points) Please give an equation for $x[n]$ in terms of constants and appropriately scaled sines and cosines that are harmonics of the fundamental frequency. Hint: your equation should be the sum of 4 terms, each involving a different frequency.

Equation for x[n]: $x[n] = .7 + \cos(0.16\pi n) - .8\sin(0.4\pi n) - .4\cos(.64\pi n)$

- $a_0 = .7 \Rightarrow$ constant term $= .7$

- $a_2 = a_{-2} = 0.5 \Rightarrow$ term of $\cos(2 \cdot \frac{2\pi}{25} n)$

- $a_5 = .4j$ $a_{-5} = -.4j \Rightarrow$ term of. $-.8\sin(5 \cdot \frac{2\pi}{25} n)$

- $a_8 = a_{-8} = -.2 \Rightarrow$ term of $-.4\cos(8 \cdot \frac{2\pi}{25} n)$

(B) (5 points) Ben designs a simple LTI system characterized by the following unit sample response:

    h[0] = 1
    h[1] = -2
    h[2] = 1
    h[n] = 0 otherwise

Please give the equation for the frequency response of the system, $H(e^{j\Omega})$, and evaluate the magnitude of the frequency response at frequencies 0, $\pi/2$ and π. If this system is used as filter, what frequency or frequencies are removed?

Equation for $H(e^{j\Omega})$: $\underline{\ 1 - 2e^{-j\Omega} + e^{-j2\Omega}\ }$

magnitude at frequency 0 = | $H(e^{j0})$ |: $\underline{\ O\ }$

magnitude at frequency $\pi/2$ = | $H(e^{j\pi/2})$ |: $\underline{\ 2\ }$

magnitude at frequency π = | $H(e^{j\pi})$ |: $\underline{\ 4\ }$

frequency or frequencies removed by filter: $\underline{\ \Omega = 0\ }$

$H(e^{j0}) = 1 - 2 + 1 = 0$

$H(e^{j\pi/2}) = 1 - 2(\cos\tfrac{-\pi}{2} + j\sin\tfrac{-\pi}{2}) + (\cos-\pi + j\sin-\pi)$

$\qquad = 1 - 2(0 + j(-1)) + (-1 + j\cdot 0)$

$\qquad = 1 - 2j - 1 = -2j \qquad |H(e^{j\pi/2})| = |-2j| = 2$

$H(e^{j\pi}) = 1 + 2 + 1 = 4$

(C) (10 points) Design a LTI system that will filter the x[n] given at the start of this problem, removing all components except the sinusoid of frequency $5(2\pi/25) = 0.4\pi$ radians/sample. Note that the output sinusoid has some phase φ and amplitude A, with a zero average, i.e., it ranges in value between $\pm A$ about 0. The A associated with your system can have any non-zero value and φ can have any value between $-\pi$ and π.

x[n] ⟶ [h[n]] ⟶ 0
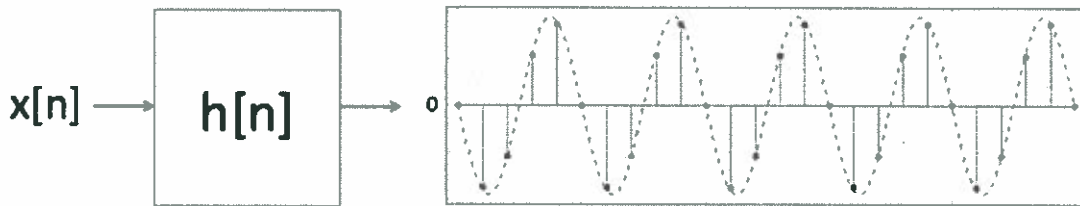
Please give the unit sample response, h[n], of your system. If your system is constructed by connecting simpler LTI systems in series, you can give the h[n] for each of the simpler systems and then give an expression for how their h[n] are combined to form the h[n] for the overall system.

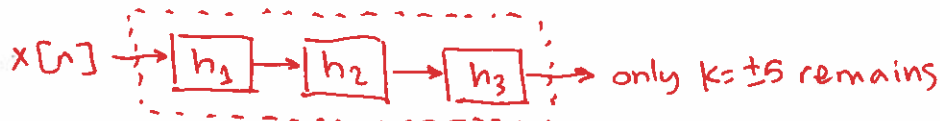In order to be eligible for partial credit, briefly explain what each component of your system is designed to do.

Give h[n] for system and explain your answer

STRAIGHT FORWARD:

h_1: remove $k=0$ ⟹ $h_1 = [1,-2,1]$ (from part B)

h_2: remove $k=\pm2$ ⟹ $h_2 = [1,-2\cos(.16\pi),1] = [1,-1.753,1]$

h_3: remove $k=\pm8$ ⟹ $h_3 = [1,-2\cos(.64\pi),1] = [1,+.852,1]$

x[n] ⟶ [h_1] ⟶ [h_2] ⟶ [h_3] ⟶ only $k=\pm5$ remains

$h[n] = h_1 * h_2 * h_3 = [1,-2.9,3.31,-2.82,3.31,-2.9,1]$

CLEVER:

WANT: x[n] ⟶ [$H(e^{j\Omega})$] ⟶
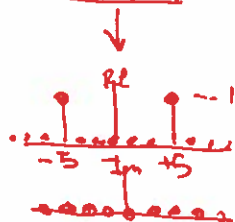
⟶ but this is just a cosine!

$h[n] = 2\cos\left(5\frac{2\pi}{25}n\right)$

END OF QUIZ 2!