MIT 6.02 **DRAFT** *Lecture Notes*
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CHAPTER 12
Frequency Response of LTI Systems

Sinusoids—and their close relatives, the complex exponentials—play a distinguished role in the study of LTI systems. The reason is that, for an LTI system, a sinusoidal input gives rise to a *sinusoidal output* again, and at the *same frequency* as the input. This property is not obvious from anything we have said so far about LTI systems. Only the amplitude and phase of the sinusoid might be, and generally are, modified from input to output, in a way that is captured by the *frequency response* of the system, which we introduce in this chapter.

■ 12.1 Sinusoidal Inputs

Before focusing on sinusoidal inputs, consider an input that is *periodic* but not necessarily sinusoidal. A signal $x[n]$ is periodic if

$$x[n+P] = x[n] \quad \text{for all } n \ ,$$

where P is some fixed positive integer. The smallest positive integer P for which this condition holds is referred to as the *period* of the signal (though the term is also used at times for positive integer multiples of P), and the signal is called P-periodic.

While it may not be obvious that sinusoidal inputs to LTI systems give rise to sinusoidal outputs, it's not hard to see that *periodic* inputs to LTI systems give rise to periodic outputs of the same period (or an integral fraction of the input period). The reason is that if the P-periodic input $x[.]$ produces the output $y[.]$, then time-invariance of the system means that shifting the input by P will shift the output by P. But shifting the input by P leaves the input unchanged, because it is P-periodic, and therefore must leave the output unchanged, which means the output must be P-periodic. (This argument actually leaves open the possibility that the period of the output is P/K for some integer K, rather than actually P-periodic, but in any case we will have $y[n+P] = y[n]$ for all n.)

12.1.1 Discrete-Time Sinusoids

A discrete-time (DT) sinusoid takes the form

$$x[n] = \cos(\Omega_0 n + \theta_0) \ , \tag{12.1}$$

We refer to Ω_0 as the *angular frequency* of the sinusoid, measured in radians/sample; Ω_0 is the number of radians by which the argument of the cosine increases when n increases by 1. (It should be clear that we can replace the cos with a sin in Equation (12.1), because cos and sin are essentially equivalent except for a $pi/2$ phase shift.)

Note that the *lowest* rate of variation possible for a DT signal is when it is constant, and this corresponds, in the case of a sinusoidal signal, to setting the frequency Ω_0 to 0. At the other extreme, the *highest* rate of variation possible for a DT signal is when it alternates signs at each time step, as in $(-1)^n$. A sinusoid with this property is obtained by taking $\Omega_0 = \pm\pi$, because $\cos(\pm\pi n) = (-1)^n$. Thus **all the action of interest with DT sinusoids happens in the frequency range** $[-\pi, \pi]$. Outside of this interval, everything repeats periodically in Ω_0, precisely because adding any integer multiple of 2π to Ω_0 does not change the value of the cosine in Equation (12.1).

It can be helpful to consider this DT sinusoid as derived from an underlying *continuous-time* (CT) sinusoid $\cos(\omega_0 t + \theta_0)$ of period $2\pi/\omega_0$, by sampling it at times $t = nT$ that are integer multiples of some sampling interval T. Writing

$$\cos(\Omega_0 n + \theta_0) = \cos(\omega_0 nT + \theta_0)$$

then yields the relation $\Omega_0 = \omega_0 T$ (with the constraint $|\omega_0| \leq \pi/T$, to reflect $|\Omega_0| \leq \pi$). It is now natural to think of $2\pi/(\omega_0 T) = 2\pi/\Omega_0$ as the *period* of the DT sinusoid, measured in samples. However, $2\pi/\Omega_0$ may not be an integer!

Nevertheless, if $2\pi/\Omega_0 = P/Q$ for some integers P and Q, i.e., if $2\pi/\Omega_0$ is *rational*, then indeed $x[n+P] = x[n]$ for the signal in Equation (12.1), as you can verify quite easily. On the other hand, if $2\pi/\Omega_0$ is irrational, the DT sequence in Equation (12.1) will not actually be periodic: there will be no integer P such that $x[n+P] = x[n]$ for all n. For example, $\cos(3\pi n/4)$ has frequency $3\pi/4$ radians/sample and a period of 8, because $2\pi/3\pi/4 = 8/3 = P/Q$, so the period, P, is 8. On the other hand, $\cos(3n/4)$ has frequency 3/4 radians/sample, and is not periodic as a *discrete-time* sequence because $2\pi/3/4 = 8\pi/3$ is irrational. We could still refer to $8\pi/3$ as its "period", because we can think of the sequence as arising from sampling the periodic *continuous-time* signal $\cos(3t/4)$ at integral values of t.

With all that said, it turns out that the response of an LTI system to a sinusoid of the form in Equation (12.1) is a sinusoid of the same (angular) frequency Ω_0, whether or not the sinusoid is actually DT periodic. The easiest way to demonstrate this fact is to rewrite sinusoids in terms of **complex exponentials**.

12.1.2 Complex Exponentials

The relation between complex exponentials and sinusoids is captured by Euler's famous identity:

$$e^{j\phi} = \cos\phi + j\sin\phi \ . \tag{12.2}$$

where $j = \sqrt{-1}$. $e^{j\phi}$ represents a complex number (or a point in the complex plane) that has a real component of $\cos\phi$ and an imaginary component of $\sin\phi$. It therefore has magnitude 1 (because $\cos^2\phi + \sin^2\phi = 1$), and makes an angle of ϕ with the positive real axis. In other words, $e^{j\phi}$ is the point on the unit circle in the complex plane (i.e., at radius 1 from the origin) and at an angle of ϕ relative to the positive real axis.

A short refresher on complex numbers may be worthwhile.

The complex number $c = a + jb$ can be thought of as the point (a, b) in the plane, and accordingly has *magnitude* $|c| = \sqrt{a^2 + b^2}$ and *angle* with the positive real axis of $\angle c = \arctan(b/a)$. Note that $a = |c|\cos(\angle c)$ and $b = |c|\sin(\angle c)$. Hence, in view of Euler's identity, we can also write the complex number in so-called *polar* form, $c = |c|.e^{j\angle c}$; this represents a point at distance $|c|$ from the origin, at an angle of $\angle c$.

The extra thing you can do with complex numbers, which you cannot do with just points in the plane, is *multiply* them. And the polar representation shows that the product of two complex numbers c_1 and c_2 is

$$c_1.c_2 = |c_1|.e^{j\angle c_1}.|c_2|.e^{j\angle c_2} = |c_1|.|c_2|.e^{j(\angle c_1 + \angle c_2)} ,$$

i.e., the magnitude of the product is the product of the individual magnitudes, and the angle of the product is the *sum* of the individual angles. It also follows that the *inverse* of a complex number c has magnitude $1/|c|$ and angle $-\angle c$.

Several other identities follow from Euler's identity above. Most importantly,

$$\cos\phi = \frac{1}{2}\left(e^{j\phi} + e^{-j\phi}\right) \qquad \sin\phi = \frac{1}{2j}\left(e^{j\phi} - e^{-j\phi}\right) = \frac{j}{2}\left(e^{-j\phi} - e^{j\phi}\right) . \tag{12.3}$$

Also, writing

$$e^{jA}e^{jB} = e^{j(A+B)} ,$$

and then using Euler's identity to rewrite all three of these complex exponentials, and finally multiplying out the left hand side, generates various useful identities, of which we only list two:

$$\begin{aligned} \cos(A)\cos(B) &= \tfrac{1}{2}\Big(\cos(A+B) + \cos(A-B)\Big) ; \\ \cos(A \mp B) &= \cos(A)\cos(B) \pm \sin(A)\sin(B) . \end{aligned} \tag{12.4}$$

■ 12.2 Frequency Response

We are now in a position to determine what an LTI system does to a sinusoidal input. The streamlined approach to this analysis involves considering a *complex* input of the form $x[n] = e^{j(\Omega_0 n + \theta_0)}$ rather than $x[n] = \cos(\Omega_0 n + \theta_0)$. The reasoning and mathematical calculations associated with convolution work as well for complex signals as they do for real signals, but the complex exponential turns out to be somewhat easier to work with (once you are comfortable working with complex numbers!)—and the results for the real sinusoidal signals we are interested in can then be extracted using identities such as those in Equation (12.3).

It may be helpful, however, to first just plough in and do the computations directly, substituting the real sinusoidal $x[n]$ from Equation (12.1) into the convolution expression from the previous chapter, and making use of Equation (12.4). The purpose of doing this is to (i) convince you that it can be done entirely with calculations involving real signals; and (ii) help you appreciate the efficiency of the calculations with complex exponentials when we get to them.

The direct approach mentioned above yields

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \sum_{m=-\infty}^{\infty} h[m]\cos\Big(\Omega_0(n-m)+\theta_0\Big) \\ &= \Big(\sum_{m=-\infty}^{\infty} h[m]\cos(\Omega_0 m)\Big)\cos(\Omega_0 n+\theta_0) \\ &\quad +\Big(\sum_{m=-\infty}^{\infty} h[m]\sin(\Omega_0 m)\Big)\sin(\Omega_0 n+\theta_0) \\ &= \mathcal{C}(\Omega_0)\cos(\Omega_0 n+\theta_0)+\mathcal{S}(\Omega_0)\sin(\Omega_0 n+\theta_0)\ , \end{aligned} \tag{12.5}$$

where we have introduced the notation

$$\mathcal{C}(\Omega)=\sum_{m=-\infty}^{\infty} h[m]\cos(\Omega m)\ , \qquad \mathcal{S}(\Omega)=\sum_{m=-\infty}^{\infty} h[m]\sin(\Omega m)\ . \tag{12.6}$$

Now define the complex quantity

$$H(\Omega)=\mathcal{C}(\Omega)-j\mathcal{S}(\Omega)=|H(\Omega)|.\exp\{j\angle H(\Omega)\}\ , \tag{12.7}$$

which we will call the **frequency response** of the system, for a reason that will emerge immediately below. Then the result in Equation (12.5) can be rewritten, using the second identity in Equation (12.4), as

$$\begin{aligned} y[n] &= |H(\Omega_0)|.\Big[\cos\angle H(\Omega_0).\cos(\Omega_0 n+\theta_0)-\sin\angle H(\Omega_0)\sin(\Omega_0 n+\theta_0)\Big] \\ &= |H(\Omega_0)|.\cos\Big(\Omega_0 n+\theta_0+\angle H(\Omega_0)\Big)\ . \end{aligned} \tag{12.8}$$

The result in Equation (12.8) is *fundamental and important*! It states that the entire effect of an LTI system on a sinusoidal input at frequency Ω_0 can be deduced from the (complex) frequency response evaluated at the frequency Ω_0. The amplitude or magnitude of the sinusoidal input gets scaled by the magnitude of the frequency response at the input frequency, and the phase gets augmented by the angle or phase of the frequency response at this frequency.

Now consider the same calculation as earlier, but this time with complex exponentials. Suppose

$$x[n]=A_0 e^{j(\Omega_0 n+\theta_0)} \quad \text{for all } n\ . \tag{12.9}$$

Convolution then yields

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \sum_{m=-\infty}^{\infty} h[m]A_0 e^{j\left(\Omega_0(n-m)+\theta_0\right)} \\ &= \left(\sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega_0 m}\right) A_0 e^{j(\Omega_0 n+\theta_0)} \ . \end{aligned} \tag{12.10}$$

Thus the output of the system, when the input is the (everlasting) exponential in Equation (12.9), is the same exponential, except multiplied by the following quantity evaluated at $\Omega = \Omega_0$:

$$\sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} = \mathcal{C}(\Omega) - j\mathcal{S}(\Omega) = H(\Omega) \ . \tag{12.11}$$

The first equality above comes from using Euler's equality to write $e^{-j\Omega m} = \cos(\Omega m) - j\sin(\Omega m)$, and then using the definitions in Equation (12.6). The second equality is simply the result of recognizing the frequency response from the definition in Equation (12.7).

To now determine was happens to a sinusoidal input of the form in Equation (12.1), use Equation (12.3) to rewrite it as

$$A_0\cos(\Omega_0 n + \theta_0) = \frac{A_0}{2}\left(e^{j(\Omega_0 n+\theta_0)} + e^{-j(\Omega_0 n+\theta_0)}\right) ,$$

and then superpose the responses to the individual exponentials (we can do that because of linearity), using the result in Equation (12.10). The result (after algebraic simplification) will again be the expression in Equation (12.8), except scaled now by an additional A_0, because we scaled our input by this additional factor in the current derivation.

To succinctly summarize the frequency response result explained above:

> If the input to an LTI system is a complex exponential, $e^{j\Omega n}$, then the output is $H(\Omega)e^{j\Omega n}$, where $H(\Omega)$ is the *frequency response* of the LTI system.

Example 1 (Moving-Average Filter) Consider an LTI system with unit sample response

$$h[n] = h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2] \ .$$

By convolving this $h[\cdot]$ with the input signal $x[\cdot]$, we see that

$$y[n] = (h * x)[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \ . \tag{12.12}$$

The system therefore produces an output signal that is the "3-point **weighted moving average**" of the input. The example in Figure 12-1 is of this form, with equal weights of $h[0] = h[1] = h[2] = 1/3$, producing the actual (moving) average.

The frequency response of the system, from the definition in Equation (12.11), is thus

$$H(\Omega) = h[0] + h[1]e^{-j\Omega} + h[2]e^{-j2\Omega} \ .$$

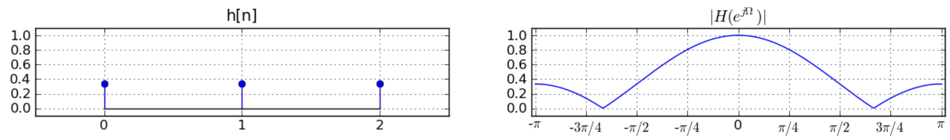

Figure 12-1: Three-point weighted moving average: h **and the frequency response,** H.

Considering the case where $h[0] = h[1] = h[2] = 1/3$, the frequency response can be rewritten as

$$\begin{aligned} H(\Omega) &= \frac{1}{3}e^{-j\Omega}\left(e^{j\Omega} + 1 + e^{-j\Omega}\right) \\ &= \frac{1}{3}e^{-j\Omega}(1 + 2\cos\Omega) \ . \end{aligned} \qquad (12.13)$$

Noting that $|e^{-j\Omega}| = 1$, it follows from the preceding equation that the magnitude of $H(\Omega)$ is

$$|H(\Omega)| = \frac{1}{3}|1 + 2\cos\Omega| \ ,$$

which is consistent with the plot on the right in Figure 12-1: it takes the value 1 at $\Omega = 0$, the value 0 at $\Omega = \arccos(-\frac{1}{2}) = \frac{2\pi}{3}$, and the value $\frac{1}{3}$ at $\Omega = \pm\pi$. The frequencies at which $|H(\Omega)| = 0$ are referred to as the *zeros* of the frequency response; in this moving-average example, they are at $\Omega = \pm\arccos(-\frac{1}{2}) = \pm\frac{2\pi}{3}$.

From Equation (12.13), we see that the angle of $H(\Omega)$ is $-\Omega$ for those values of Ω where $1 + 2\cos\Omega > 0$; this is the angle contributed by the term $e^{-j\Omega}$. For frequencies where $1 + 2\cos\Omega < 0$, we need to add or subtract (it doesn't matter which) π radians to $-\Omega$, because $-1 = e^{\pm j\pi}$. Thus

$$\angle H(\Omega) = \begin{cases} -\Omega & \text{for } |\Omega| < 2\pi/3 \\ -\Omega \pm \pi & \text{for } (2\pi/3) < |\Omega| < \pi \end{cases}$$

Example 2 (The Effect of a Time Shift) What does shifting $h[n]$ in time do to the frequency response $H(\Omega)$? Specifically, suppose

$$h_D[n] = h[n - D] \ ,$$

so $h_D[n]$ is a time-shifted version of $h[n]$. How does the associated frequency response $H_D(\Omega)$ relate to $H(\Omega)$?

From the definition of frequency response in Equation (12.11), we have

$$H_D(\Omega) = \sum_{m=-\infty}^{\infty} h_D[m]e^{-j\Omega m} = \sum_{m=-\infty}^{\infty} h[m - D]e^{-j\Omega m} = e^{-j\Omega D}\sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \ ,$$

where the last equality is simply the result of the change of variables $m - D = n$, so $m = n + D$. It follows that

$$H_D(\Omega) = e^{-j\Omega D}H(\Omega) \ .$$

Equivalently,

$$|H_D(\Omega)| = |H(\Omega)|$$

and

$$\angle H_D(\Omega) = -\Omega D + \angle H(\Omega) \ ,$$

so the frequency response magnitude is unchanged, and the phase is modified by an additive term that is linear in Ω, with slope $-D$.

Although we have introduced the notion of a frequency response in the context of what an LTI system does to a single sinusoidal input, superposition will now allow us to use the frequency response to describe what an LTI system does to any input made up of a *linear combination of sinusoids at different frequencies*. You compute the (sinusoidal) response to each sinusoid in the input, using the frequency response *at the frequency of that sinusoid*. The system output will then be the same linear combination of the individual sinusoidal responses.

As we shall see in the next chapter, when we use Fourier analysis to introduce the notion of the *spectral content* or *frequency content* of a signal, the class of signals that can be represented as a linear combination of sinusoids at assorted frequencies is very large. So this superposition idea ends up being extremely powerful.

Example 3 (Response to Weighted Sum of Two Sinusoids) Consider an LTI system with frequency response $H(\Omega)$, and assume its input is the signal

$$x[n] = 5\sin(\frac{\pi}{4}n + 0.2) + 11\cos(\frac{\pi}{7}n - 0.4) \ .$$

The system output is then

$$y[n] = |H(\frac{\pi}{4})|.5\sin\Big(\frac{\pi}{4}n + 0.2 + \angle H(\frac{\pi}{4})\Big) + |H(\frac{\pi}{7})|.11\cos\Big(\frac{\pi}{7}n - 0.4 + \angle H(\frac{\pi}{7})\Big) \ .$$

■ 12.2.1 Properties of the Frequency Response

Existence The definition of the frequency response in terms of $h[m]$ and sines and cosines in Equation (12.7), or equivalently in terms of $h[m]$ and complex exponentials in Equation (12.11), generally involves summing an infinite number of terms, so again (just as with convolution) one needs conditions to guarantee that the sum is well-behaved. One case, of course, is where $h[m]$ is nonzero at only a finite number of time instants, in which case there is no problem with the sum. Another case is when the function $h[\cdot]$ is absolutely summable,

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq \mu < \infty \ ,$$

as this ensures that the sum defining the frequency response is itself absolutely summable. The absolute summability of $h[\cdot]$ is the condition for bounded-input bounded-output (BIBO) stability of an LTI system that we obtained in the previous chapter. It turns out that under this condition the frequency response is actually a *continuous* function of Ω.

Various other important properties of the frequency response follow quickly from the definition.

Periodicity in Ω Note first that $H(\Omega)$ repeats periodically on the frequency (Ω) axis, with period 2π, because a sinusoidal or complex exponential input of the form in Equation (12.1) or (12.9) is unchanged when its frequency is increased by any integer multiple of 2π. This can also be seen from Equation (12.11), the defining equation for the frequency response. It follows that only the interval $|\Omega| \leq \pi$ is of interest.

Lowest Frequency An input at the frequency $\Omega = 0$ corresponds to a constant (or "DC", which stands for *direct current*, but in this context just means "constant") input, so

$$H(0) = \sum_{n=-\infty}^{\infty} h[n] \tag{12.14}$$

is the *DC gain* of the system, i.e., the gain for constant inputs.

Highest Frequency At the other extreme, a frequency of $\Omega = \pm\pi$ corresponds to an input of the form $(-1)^n$, which is the highest-frequency variation possible for a discrete-time signal, so

$$H(\pi) = H(-\pi) = \sum_{n=-\infty}^{\infty} (-1)^n h[n] \tag{12.15}$$

is the *high-frequency gain* of the system.

Symmetry Properties for Real $h[n]$ We will only be interested in the case where the unit sample response $h[\cdot]$ is a real (rather than complex) function. Under this condition, the definition of the frequency response in Equations (12.7), (12.6) shows that the *real part of the frequency response*, namely $\mathcal{C}(\Omega)$, is an *even function of frequency*, i.e., has the same value when Ω is replaced by $-\Omega$. This is because each cosine term in the sum that defines $\mathcal{C}(\Omega)$ is an even function of Ω.

Similarly, for real $h[n]$, the *imaginary part of the frequency response*, namely $-\mathcal{S}(\Omega)$, is an *odd function of frequency*, i.e., gets multiplied by -1 when Ω is replaced by $-\Omega$. This is because each sine term in the sum that defines $\mathcal{S}(\Omega)$ is an odd function of Ω.

In this discussion, we have used the property that $h[\cdot]$ is real, so $\mathcal{C}$ and $\mathcal{S}$ are also both real, and correspond to the real and imaginary parts of the frequency response, respectively.

It follows from the above facts that for a real $h[n]$ the *magnitude* $|H(\Omega)|$ of the frequency response is an *even* function of Ω, and the *angle* $\angle H(\Omega)$ is an *odd* function of Ω.

You should verify that the claimed symmetry properties indeed hold for the $h[\cdot]$ in Example 1 above.

Real and Even $h[n]$ Equations (12.7) and (12.6) also directly show that if the real unit sample response $h[n]$ is an *even function of time*, i.e., if $h[-n] = h[n]$, then the associated frequency response must be purely *real*. The reason is that the summation defining $\mathcal{S}(\Omega)$, which yields the imaginary part of $H(\Omega)$, involves the product of the even function $h[m]$ with the odd function $\sin(\Omega m)$, which is thus an odd function of m, and hence sums to 0.

Real and Odd $h[n]$ Similarly if the real unit sample response $h[n]$ is an *odd function of time*, i.e., if $h[-n] = -h[n]$, then the associated frequency response must be purely *imaginary*.

Frequency Response of LTI Systems in Series We have already seen that a cascade or series combination of two LTI systems, the first with unit sample response $h_1[\cdot]$ and the second with unit sample response $h_2[\cdot]$, results in an overall system that is LTI, with unit sample response $(h_2 * h_1)[\cdot] = (h_1 * h_2)[\cdot]$.

To determine the overall frequency response of the system, imagine applying an (everlasting) exponential input of the form $x[n] = Ae^{j\Omega n}$ to the first subsystem. Its output will then be $w[n] = H_1(\Omega) \cdot Ae^{j\Omega n}$, which is again an exponential of the same form, just scaled by the frequency response of the first system. Now with $w[n]$ as the input to the second system, the output of the second system will be $y[n] = H_2(\Omega) \cdot H_1(\Omega) \cdot Ae^{j\Omega n}$. It follows that the overall frequency response $H(\Omega)$ is given by

$$H(\Omega) = H_2(\Omega)H_1(\Omega) = H_1(\Omega)H_2(\Omega) \ .$$

This is the first hint of a more general result, namely that **convolution in time corresponds to multiplication in frequency**:

$$h[n] = (h_1 * h_2)[n] \longleftrightarrow H(\Omega) = H_1(\Omega)H_2(\Omega) \ . \tag{12.16}$$

This result makes frequency-domain methods compelling in the analysis of LTI systems—simple multiplication, frequency by frequency, replaces the more complicated convolution of two complete signals in the time-domain. We will see this in more detail in the next chapter, after we introduce Fourier analysis methods to describe the spectral content of signals.

Frequency Response of LTI Systems in Parallel Using the same sort of argument as in the previous paragraph, the frequency response of the system obtained by placing the two LTI systems above in parallel rather than in series results in an overall system with frequency response $H(\Omega) = H_1(\Omega) + H_2(\Omega)$, so

$$h[n] = (h_1 + h_2)[n] \longleftrightarrow H(\Omega) = H_1(\Omega) + H_2(\Omega) \ . \tag{12.17}$$

Getting $h[n]$ From $H(\Omega)$ As a final point, we examine how $h[n]$ can be determined from $H(\Omega)$. The relationship we obtain here is crucial to designing filters with a desired or specified frequency response. It also points the way to the results we develop in the next chapter, showing how time-domain signals — in this case $h[\cdot]$ — can be represented as weighted combinations of exponentials, the key idea in Fourier analysis.

Begin with Equation (12.11), which defines the frequency response $H(\Omega)$ in terms of the signal $h[\cdot]$:

$$H(\Omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} .$$

Multiply both sides of this equation by $e^{j\Omega n}$, and integrate the result over Ω from $-\pi$ to π:

$$\int_{-\pi}^{\pi} H(\Omega)e^{j\Omega n}\, d\Omega = \sum_{m=-\infty}^{\infty} h[m]\Big(\int_{-\pi}^{\pi} e^{-j\Omega(m-n)}\, d\Omega\Big)$$

where we have assumed $h[\cdot]$ is sufficiently well-behaved to allow interchange of the summation and integration operations.

The integrals above can be reduced to ordinary real integrals by rewriting each complex exponential $e^{jk\Omega}$ as $\cos(k\Omega) + j\sin(k\Omega)$, which shows that the result of each integration will in general be a complex number that has a real and imaginary part. However, for all $k \neq 0$, the integral of $\cos(k\Omega)$ or $\sin(k\Omega)$ from $-\pi$ to π will yield 0, because it is the integral over an integer number of periods. For $k = 0$, the integral of $\cos(k\Omega)$ from $-\pi$ to π yields 2π, while the integral of $\sin(k\Omega)$ from $-\pi$ to π yields 0. Thus every term for which $m \neq n$ on the right side of the preceding equation will evaluate to 0. The only term that survives is the one for which $n = m$, so the right side simplifies to just $2\pi h[n]$. Rearranging the resulting equation, we get

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega)e^{j\Omega n}\, d\Omega . \qquad (12.18)$$

Since the integrand on the right is periodic with period 2π, we can actually compute the integral over *any* contiguous interval of length 2π, which we indicate by writing

$$h[n] = \frac{1}{2\pi}\int_{<2\pi>} H(\Omega)e^{j\Omega n}\, d\Omega . \qquad (12.19)$$

Note that this equation can be interpreted as representing the signal $h[n]$ as a weighted combination of a continuum of exponentials of the form $e^{j\Omega n}$, with frequencies Ω in a 2π range, and associated weights $H(\Omega)\, d\Omega$.

■ 12.2.2 Illustrative Examples

Example 4 (More Moving-Average Filters) The unit sample responses in Figure 12-2 all correspond to causal moving-average LTI filters, and have the form

$$h_L[n] = \frac{1}{L}\Big(\delta[n] + \delta[n-1] + \cdots + \delta[n-(L-1)]\Big) .$$

The corresponding frequency response, directly from the definition in Equation (12.11), is given by

$$H_L(\Omega) = \frac{1}{L}\Big(1 + e^{-j\Omega} + \cdots + e^{-j(L-1)\Omega}\Big) .$$

To examine the magnitude and phase of $H_L(\Omega)$ as we did in the special case of $L = 3$ in Example 1, it is helpful to rewrite the preceding expression. In the case of odd L, we can

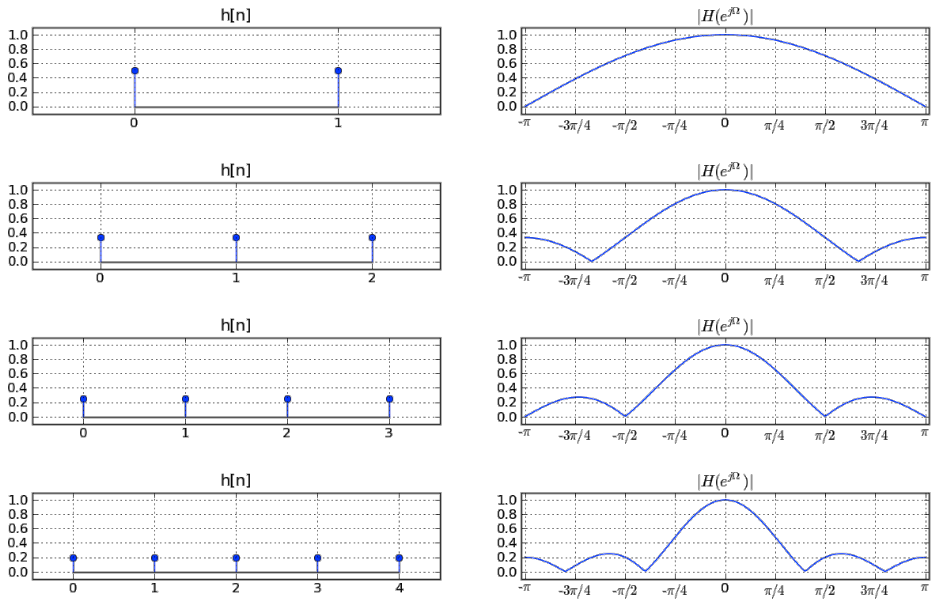

Figure 12-2: Unit sample response and frequency response of different moving average filters.

write

$$
\begin{aligned}
H_L(\Omega) &= \frac{1}{L}e^{-j(L-1)\Omega/2}\Big(e^{j(L-1)\Omega/2} + e^{j(L-3)\Omega/2} + \cdots + e^{-j(L-1)\Omega/2}\Big) \\
&= \frac{2}{L}e^{-j(L-1)\Omega/2}\Big(\frac{1}{2} + \cos(\Omega) + \cos(2\Omega) + \cdots + \cos((L-1)\Omega/2)\Big)\ .
\end{aligned}
$$

For even L, we get a similar expression:

$$
\begin{aligned}
H_L(\Omega) &= \frac{1}{L}e^{-j(L-1)\Omega/2}\Big(e^{j(L-1)\Omega/2} + e^{j(L-3)\Omega/2} + \cdots + e^{-j(L-1)\Omega/2}\Big) \\
&= \frac{2}{L}e^{-j(L-1)\Omega/2}\Big(\cos(\Omega/2) + \cos(3\Omega/2) + \cdots + \cos((L-1)\Omega/2)\Big)\ .
\end{aligned}
$$

For both even and odd L, the single complex exponential in front of the parentheses contributes $-(L-1)\Omega/2$ to the phase, but its magnitude is 1 for all Ω. For both even and odd cases, the sum of cosines in parentheses is purely real, and is either positive or negative at any specific Ω, hence contributing only 0 or $\pm\pi$ to the phase. So the magnitude of the frequency response, which is plotted on Slide 13.12 for these various examples, is simply the magnitude of the sum of cosines given in the above expressions.

Example 5 (Cascaded Filter Sections) We saw in Example 1 that a 3-point moving average filter ended up having frequency-response zeros at $\Omega = \arccos(-\frac{1}{2}) = \pm 2\pi/3$. Reviewing the derivation there, you might notice that a simple way to adjust the location of the zeros is to allow $h[1]$ to be different from $h[0] = h[2]$. Take, for instance, $h[0] = h[2] = 1$ and $h[1] = \alpha$. Then

$$
H(\Omega) = 1 + \alpha e^{-j\Omega} + e^{-j2\Omega} = e^{-j\Omega}\Big(\alpha + 2\cos(\Omega)\Big)\ .
$$

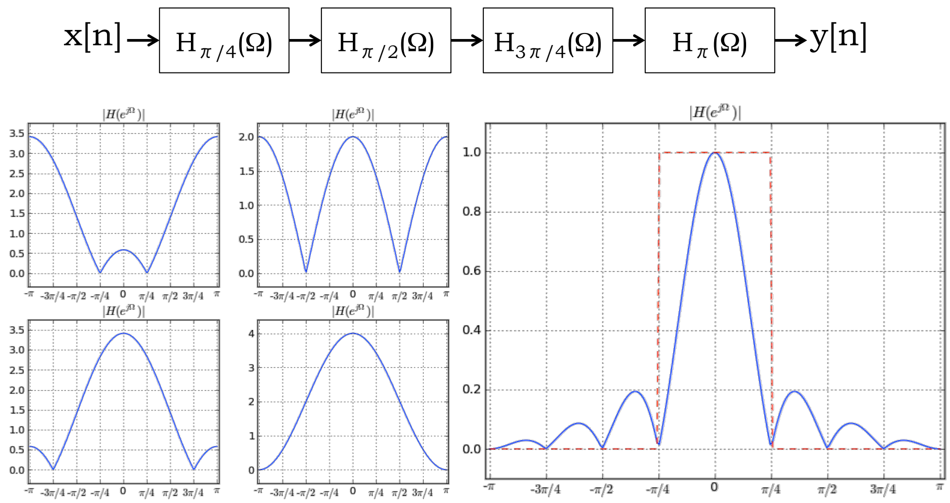

Figure 12-3: A "10-cent" low-pass filter obtained by cascading a few single-zero-pair filters.

It follows that

$$|H(\Omega)| = |\alpha + 2\cos(\Omega)| \ ,$$

and the zeros of this occur at $\Omega = \pm\arccos(-\alpha/2)$. In order to have the zeros at the pair of frequencies $\Omega = \pm\phi_o$, we would pick $h[1] = \alpha = -2\cos(\phi_o)$.

If we now cascade several such single-zero-pair filter sections, as in the top part of Figure 12-3, the overall frequency response is the product of the individual ones, as noted in Equation (12.16). Thus, the overall frequency response will have zero pairs at those frequencies where *any* of the individual sections has a zero pair, and therefore will have *all* the zero-pairs of the constituent sections. This is evident in curves on Figure 12-3, where the zeros have been selected to produce a filter that passes low frequencies (approximately in the range $|\Omega| \leq \pi/8$) preferentially to higher frequencies.

Example 6 (Nearly Ideal Low-Pass Filter) Figure 12-4 shows the unit sample response and frequency response of an LTI filter that is much closer to being an ideal low-pass filter. Such a filter would have $H(\Omega) = 1$ in the band $|\Omega| < \Omega_c$, and $H(\Omega) = 0$ for $\Omega_c < |\Omega| \leq \pi$; here Ω_c is referred to as the *cut-off* (or *cutoff*) frequency. Equation (12.18) shows that the corresponding $h[n]$ must then be given by

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\, d\Omega \\ &= \begin{cases} \frac{\sin(\Omega_c n)}{\pi n} & \text{for } n \neq 0 \\ \frac{\Omega_c}{\pi} & \text{for } n = 0 \end{cases} \end{aligned}$$

This unit sample response is plotted on the left curve in Figure 12-4, for n ranging from -300 to 300. The fact that $H(\Omega)$ is real should have prepared us for the fact that $h[n]$ is an even function of $h[n]$, i.e., $h[-n] = h[n]$. The slow decay of this unit sample response, falling off as $1/n$, is evident in the plot. In fact, it turns out that the ideal lowpass filter is *not*

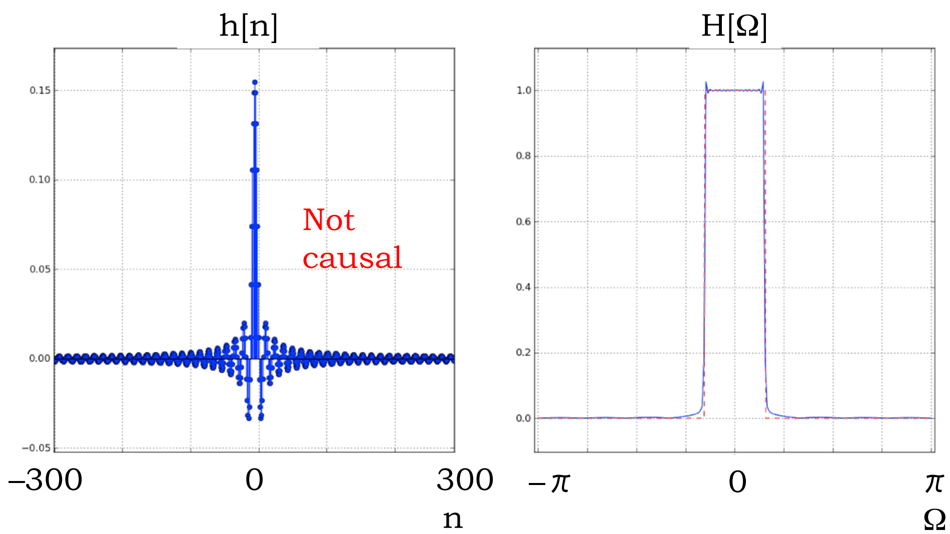

Figure 12-4: A more sophisticated low-pass filter that passes low frequencies $\leq \pi/8$ and blocks higher frequencies.

bounded-input bounded-output stable, because its unit sample response is not absolutely summable.

The frequency response plot on the right in Figure 12-4 actually shows *two* different frequency responses: one is the ideal lowpass characteristic that we used in determining $h[n]$, and the other is the frequency response corresponding to the truncated $h[n]$, i.e., the one given by using Equation (12.20) for $|n| \leq 300$, and setting $h[n] = 0$ for $|n| > 300$. To compute the latter frequency response, we simply substitute the truncated unit sample response in the expression that defines the frequency response, namely Equation (12.11); the resulting frequency response is again purely real. The plots of frequency response show that truncation still yields a frequency response characteristic that is close to ideal.

One problem with the truncated $h[n]$ above is that it corresponds to a *noncausal* system. To obtain a causal system, we can simply shift $h[n]$ forward by 300 steps. We have already seen in Example 2 that such shifting does not affect the magnitude of the frequency response. The shifting does change the phase from being 0 at all frequencies to being linear in Ω, taking the value -300Ω.

We see in Figure 12-5 the frequency response magnitudes and unit sample responses of some other near-ideal filters. A good starting point for the design of the unit sample responses of these filters is again Equation (12.18) to generate the ideal versions of the filters. Subsequent truncation and time-shifting of the corresponding unit sample responses yields causal LTI systems that are good approximations to the desired frequency responses.

Example 7 (Autoregressive Filters) Figure 12-6 shows the unit sample responses and frequency response magnitudes of some other LTI filters. These can all be obtained as the input-output behavior of causal systems whose output at time n depends on some previous values of $y[\cdot]$, along with the input value $x[n]$ at time n; these are termed *autoregressive*

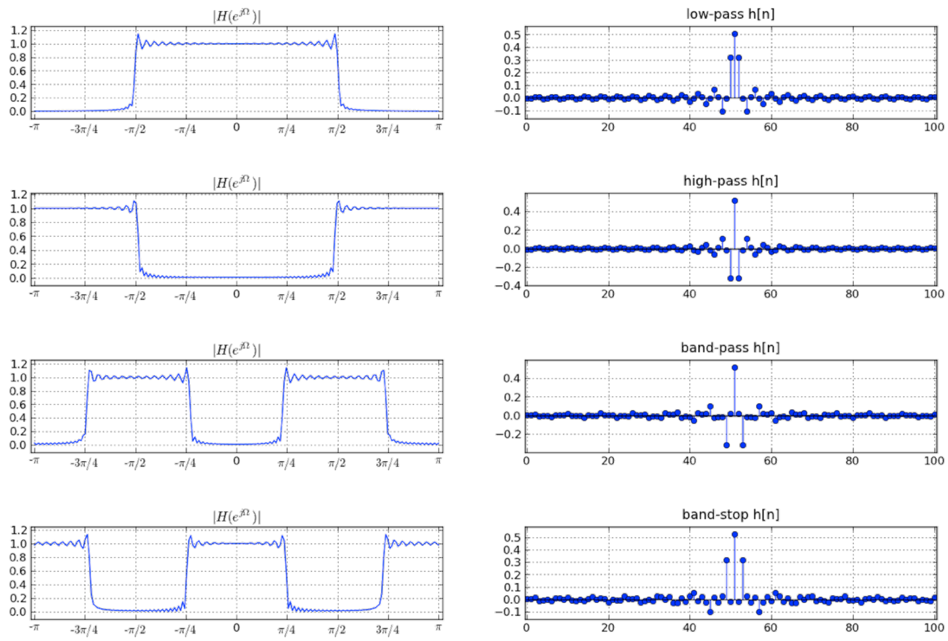

Figure 12-5: The frequency response and $h[\cdot]$ for some useful near-ideal filters.

systems. The simplest example is a causal system whose output and input are related by

$$y[n] = \lambda y[n-1] + \beta x[n] \tag{12.20}$$

for some constant parameters λ and β. This is termed a first-order autoregressive model, because $y[n]$ depends on the value of $y[\cdot]$ just one time step earlier. The unit sample response associated with this system is

$$h[n] = \beta\lambda^n u[n] \,, \tag{12.21}$$

where $u[n]$ is the unit step function. To deduce this result, set $x[n] = \delta[n]$ with $y[k] = 0$ for $k < 0$ since the system is causal (and therefore cannot tell the difference between an all-zero input and the unit sample input till it gets to time $k = 0$), then iteratively use Equation (12.20) to compute $y[n]$ for $n \geq 0$. This $y[n]$ will be the unit sample response, $h[n]$.

For a system with the above unit sample response to be bounded-input bounded-output (BIBO) stable, i.e., for $h[n]$ to be absolutely summable, we require $|\lambda| < 1$. If $0 < \lambda < 1$, the unit sample has the form shown in the top left plot in Figure 12-6. The associated frequency response in the BIBO-stable case, from the definition in Equation (12.11), is

$$H(\Omega) = \beta \sum_{m=0}^{\infty} \lambda^m e^{-j\Omega m} = \frac{\beta}{1 - \lambda e^{-j\Omega}} \,. \tag{12.22}$$

The magnitude of this is what is shown in the top right plot in Figure 12-6, for the case $0 < \lambda < 1$.

Another way to derive the unit sample response and frequency response is to start with the frequency domain. Suppose that the system in Equation (12.20) gets the input $x[n] =$

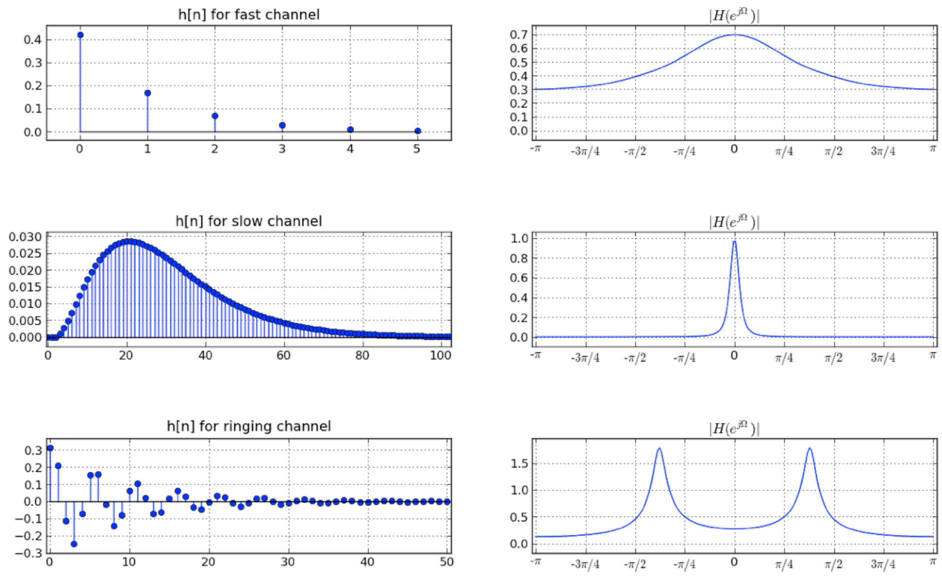

Figure 12-6: $h[\cdot]$ **and the frequency response for some other useful ideal** ***auto-regressive*** **filters.**

$e^{j\Omega n}$. Then, by the definition of the frequency response, the output is $y[n] = H(\Omega)e^{j\Omega n}$. Substituting $e^{j\Omega n}$ for $x[n]$ and $H(\Omega)e^{j\Omega n}$ for $y[n]$ in Equation (12.20), we get

$$H(\Omega)e^{j\Omega n} = \lambda H(\Omega)e^{j\Omega(n-1)} + \beta e^{j\Omega n}.$$

Moving the $H(\Omega)$ terms to one side and canceling out the $e^{j\Omega n}$ factor on both sides (we can do that because $e^{j\Omega n}$ is on the unit circle in the complex plane and cannot be equal to 0), we get

$$H(\Omega) = \frac{\beta}{1 - \lambda e^{-j\Omega}}.$$

This is the same answer as in Equation (12.22).

To obtain h, one can then expand $\frac{\beta}{1-\lambda e^{-j\Omega}}$ as a power series, using the property that $\frac{1}{1-z} = 1 + z + z^2 + \ldots$. The expansion has terms of the form $e^{-j\Omega}, e^{-j2\Omega}, e^{-j3\Omega}, \ldots$, and their coefficients form the unit sample response sequence.

Whether one starts with the time-domain, setting $x[n] = \delta[n]$, or the frequency-domain, setting $x[n] = e^{j\Omega n}$, depends on one's preference and the problem at hand. Both methods are generally equivalent, though in some cases one approach may be mathematically less cumbersome than the other.

The other two systems in Figure 12-6 correspond to *second*-order autoregressive models, for which the defining difference equation is

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + bx[n] \quad (12.23)$$

for some constants a_1, a_2 and b.

To take one concrete example, consider the system whose output and input are related according to

$$y[n] = 6y[n-1] - 8y[n-2] + x[n] \quad (12.24)$$

We want to determine $h[\cdot]$ and $H(\Omega)$. We can approach this task either by first setting $x[n] = \delta[n]$, finding $h[\cdot]$, and then applying Equation (12.11) to find $H(\Omega)$, or by first calculating $H(\Omega)$. Let us consider the latter approach here.

Setting $x[n] = e^{j\Omega n}$ in Equation (12.24), we get

$$e^{j\Omega n}H(\Omega) = 6e^{j\Omega(n-1)}H(\Omega) - 8e^{j\Omega(n-2)}H(\Omega) + e^{j\Omega n}.$$

Solving this equation for $H(\Omega)$ yields

$$H(\Omega) = \frac{1}{(1 - 2e^{-j\Omega})(1 - 4e^{-j\Omega})}.$$

One can now work out h by expanding H as a power series of terms involving various powers of $e^{-j\Omega}$, and also derive conditions on BIBO-stability and conditions under which $H(\Omega)$ is well-defined.

Coming back to the general second-order auto-regressive model, it can be shown (following a development analogous to what you may be familiar with from the analysis of LTI *differential* equations) that in this case the unit sample response takes the form

$$h[n] = (\beta_1\lambda_1^n + \beta_2\lambda_2^n)u[n] \ ,$$

where λ_1 and λ_2 are the roots of the *characteristic polynomial* associated with this system:

$$a(\lambda) = \lambda^2 + a_1\lambda + a_2 \ ,$$

and β_1, β_2 are some constants. The second row of plots of Figure 12-6 corresponds to the case where both λ_1 and λ_2 are real, positive, and less than 1 in magnitude. The third row corresponds to the case where these roots form a complex conjugate pair, $\lambda_2 = \lambda_1^*$ (and correspondingly $\beta_2 = \beta_1^*$), and have magnitude less than 1, i.e., lie within the unit circle in the complex plane.

Example 8 (Deconvolution Revisited) Consider the LTI system with unit sample response

$$h_1[n] = \delta[n] + 0.8\delta[n-1]$$

from the previous chapter. As noted there, you might think of this channel as being ideal, which would imply a unit sample response of $\delta[n]$, apart from a one-step-delayed echo, which accounts for the additional $0.8\delta[n-1]$. The corresponding frequency response is

$$H_1(\Omega) = 1 + 0.8e^{-j\Omega} \ ,$$

immediately from the definition of frequency response, Equation (12.11).

We introduced deconvolution in the last chapter as aimed at undoing—at the receiver—the convolution carried out on the input signal $x[\cdot]$ by the channel. Thus, from the channel output $y[\cdot]$, we wish to reconstruct the input $x[\cdot]$ using an LTI deconvolution filter with unit sample response $h_2[n]$ and associated frequency response $H_2(\Omega)$. We want the output $z[n]$ of the deconvolution filter at each time n to equal the channel input $x[n]$ at that time.[1]

[1]We might also be content to have $z[n] = x[n-D]$ for some integer $D > 0$, but this does not change anything

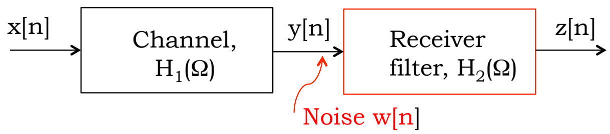

Figure 12-7: Noise at the channel output.

Therefore, the overall unit sample response of the channel followed by the deconvolution filter must be $\delta[n]$, so

$$(h_2 * h_1)[n] = \delta[n] .$$

We saw in the last chapter how to use this relationship to determine $h_2[n]$ for all n, given $h_1[\cdot]$. Here $h_2[\cdot]$ serves as the "convolutional inverse" to $h_1[\cdot]$.

In the frequency domain, the analysis is much simpler. We require the frequency response of the cascade combination of channel and deconvolution filter, $H_2(\Omega)H_1(\Omega)$ to be 1. This condition immediately yields the frequency response of the deconvolution filter as

$$H_2(\Omega) = 1/H_1(\Omega) , \tag{12.25}$$

so in the frequency domain deconvolution is simple multiplicative inversion, frequency by frequency. We thus refer to the deconvolution filter as the *inverse system* for the channel. For our example, therefore,

$$H_2(\Omega) = 1/(1 + 0.8e^{-j\Omega}) .$$

This is identical to the form seen in Equation (12.22) in Example 7, from which we find that

$$h_2[n] = (-0.8)^n u[n] ,$$

in agreement with our time-domain analysis in the previous chapter.

The frequency-domain treatment of deconvolution brings out an important point that is much more hidden in the time-domain analysis. From Equation (12.25), we note that $|H_2(\Omega)| = 1/|H_1(\Omega)|$ so the deconvolution filter has *high* frequency response magnitude in precisely those frequency ranges where the channel has *low* frequency response magnitude. In the presence of the inevitable noise at the channel output (Figure 12-7), we would normally and reasonably want to *discount* these frequency ranges, as the channel input $x[n]$ produces little effect at the output in these frequency ranges, relative to the noise power at the output in these frequency ranges. However, the deconvolution filter does the exact opposite of what is reasonable here: it *emphasizes* and amplifies the channel output in these frequency ranges. Deconvolution is therefore not a good approach to determine the channel input in the presence of noise.

■ Acknowledgments

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essential in the following development.

■ Problems and Questions

1. Ben Bitdiddle designs a simple causal LTI system characterized by the following unit sample response:

$$\begin{aligned} h[0] &= 1 \\ h[1] &= 2 \\ h[2] &= 1 \\ h[n] &= 0 \; \forall n > 2 \end{aligned}$$

   (a) What is the frequency response, $H(\Omega)$?

   (b) What is the magnitude of H at $\Omega = 0, \pi/2, \pi$?

   (c) If this LTI system is used as a filter, what is the set of frequencies that are removed?

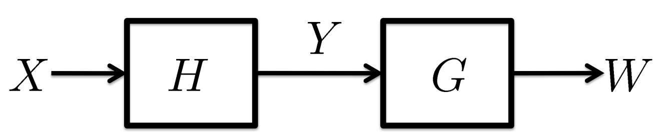
2. Suppose a causal linear time invariant (LTI) system with frequency response H is described by the following difference equation relating input $x[\cdot]$ to output $y[\cdot]$:

$$y[n] = x[n] + \alpha x[n-1] + \beta x[n-2] + \gamma x[n-3]. \quad (12.26)$$

   Here, α, β, and γ are constants independent of Ω.

   (a) Determine the values of α, β and γ so that the frequency response of system H is $H(\Omega) = 1 - 0.5e^{-j2\Omega}\cos\Omega$.

   (b) Suppose that $y[\cdot]$, the output of the LTI system with frequency response H, is used as the input to a second causal LTI system with frequency response G, producing W, as shown below.



   (c) If $H(e^{j\Omega}) = 1 - 0.5e^{-j2\Omega}\cos\Omega$, what should the frequency response, $G(e^{j\Omega})$, be so that $w[n] = x[n]$ for all n?

   (d) Suppose $\alpha = 1$ and $\gamma = 1$ in the above equation for an H with a **different** frequency response than the one you obtained in Part (a) above. For this different H, you are told that $y[n] = A(-1)^n$ when $x[n] = 1.0 + 0.5(-1)^n$ for all n. Using this information, determine the value of β in Eq. (12.26) and the value of A in the formula for $y[n]$.

3. Consider an LTI filter with input signal $x[n]$, output signal $y[n]$, and unit sample response

$$h[n] = a\delta[n] + b\delta[n-1] + b\delta[n-2] + a\delta[n-3] \ ,$$

where a and b are *positive* parameters, with $b > a > 0$. Thus $h[0] = h[3] = a$ and $h[1] = h[2] = b$, while $h[n]$ at all other times is 0. Your answers in this problem should be in terms of a and b.

   (a) Determine the frequency response $H(\Omega)$ of the filter.

   (b) Suppose $x[n] = (-1)^n$ for all integers n from $-\infty$ to ∞. Use your expression for $H(\Omega)$ in Part (a) above to determine $y[n]$ at *all* times n.

   (c) As a time-domain check on your answer from Part (a), use **convolution** to determine the values of $y[5]$ and $y[6]$ when $x[n] = (-1)^n$ for all integers n from $-\infty$ to ∞.

   (d) The frequency response $H(\Omega) = |H(\Omega)|e^{j\angle H(\Omega)}$ that you found in Part (a) for this filter can be written in the form

$$H(\Omega) = G(\Omega)e^{-j3\Omega/2} \ ,$$

   where $G(\Omega)$ is a **real** function of Ω that can be positive or negative, depending on the values of a, b, and Ω. Determine $G(\Omega)$, writing it in a form that makes clear it is a real function of Ω.

   (e) Suppose the input to the filter is $x[n] = (-1)^n + \cos(\frac{\pi}{2}n + \theta_0)$ for all n from $-\infty$ to ∞, where θ_0 is some constant. Use the **frequency response** $H(\Omega)$ to determine the output $y[n]$ of the filter (writing it in terms of a, b, and θ_0).

   Depending on how you solve the problem, it may help you to recall that $\cos(\pi/4) = 1/\sqrt{2}$ and $\cos(3\pi/4) = -1/\sqrt{2}$. Also keep in mind our assumption that $b > a > 0$.

4. Consider the following three plots of the magnitude of three frequency responses, $|H_I(e^{j\Omega})|$, $|H_{II}(e^{j\Omega})|$, and $|H_{III}(e^{j\Omega})|$, shown in Figure 12-8.

   Suppose a linear time-invariant system has a frequency response $H_A(e^{j\Omega})$ given by the formula

$$H_A(e^{j\Omega}) = \frac{1}{\left(1 - 0.95e^{-j(\Omega - \frac{\pi}{2})}\right)\left(1 - 0.95e^{-j(\Omega + \frac{\pi}{2})}\right)}$$

   (a) Which frequency response plot (I, II, or III) best corresponds to $H_A(e^{j\Omega})$ above? What is the numerical value of M in the plot you selected?

   (b) For what values of a_1 and a_2 will the system described by the difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n]$$

   have a frequency response given by $H_A(e^{j\Omega})$ above?

5. Suppose the input to a linear time invariant system is the sequence

$$x[n] = 2 + \cos\frac{5\pi}{6}n + \cos\frac{\pi}{6}n + 3(-1)^n$$

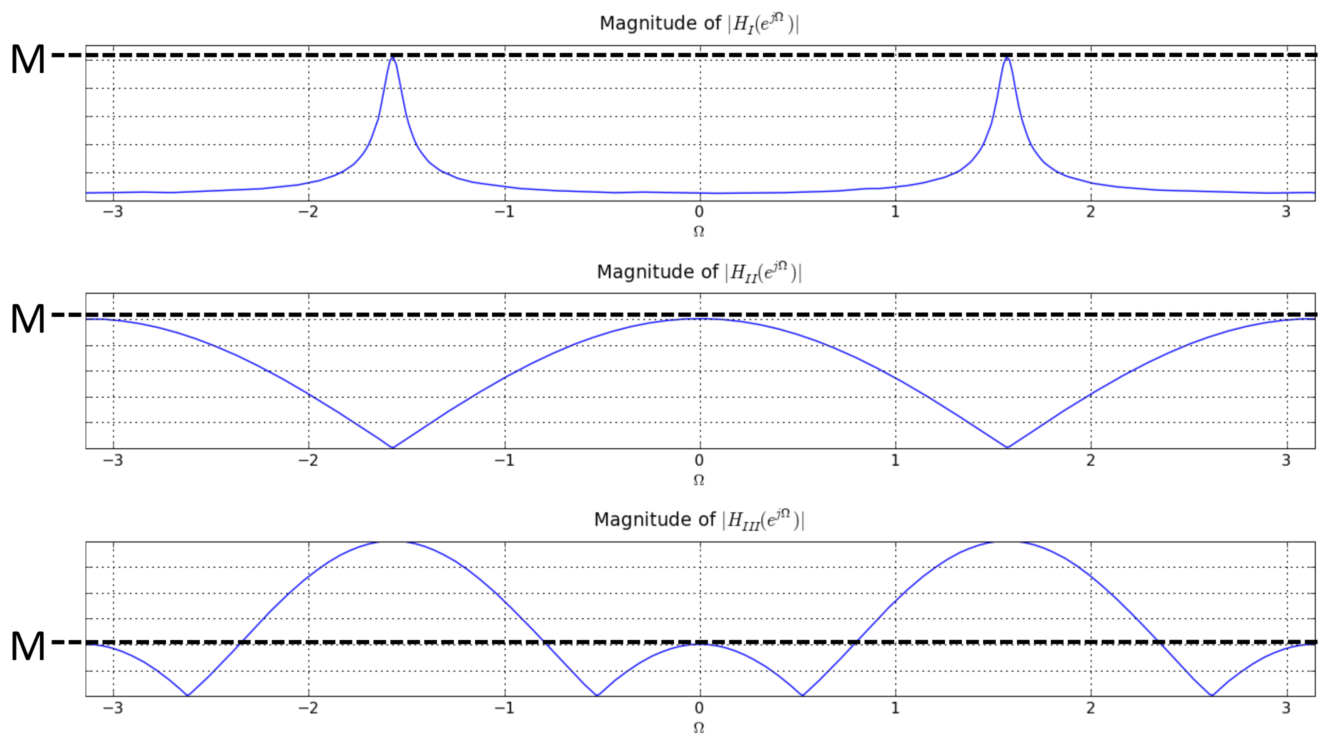


Figure 12-8: Channel frequency response curves for Problems 4 through 6.

(a) What is the maximum value of the sequence x, and what is the smallest positive value of n for which x achieves its maximum?

(b) Suppose the above sequence x is the input to a linear time invariant system described by one of the three frequency response plots in Figure 12-8 (I, II, or III). If y is the resulting output and is given by

$$y[n] = 8 + 12(-1)^n,$$

which frequency response plot best describes the system? What is the value of M in the plot you selected?

6. Suppose the unit sample response of an LTI system has only three nonzero *real* values, $h[0], h[1]$, and $h[2]$. In addition, suppose these three real values satisfy these three equations:

$$\begin{aligned} h[0] + h[1] + h[2] &= 5 \\ h[0] + h[1]e^{-j\pi/2} + h[2]e^{-j2\pi/2} &= 0 \\ h[0] + h[1]e^{j\pi/2} + h[2]e^{j2\pi/2} &= 0 \end{aligned}$$

(a) Without doing any algebra, simply by inspection, you should be able to write down the frequency response $H(\Omega)$ for some frequencies. Which frequencies are these? And what is the value of H at each of these frequencies?

(b) Which of the above plots in Figure 12-8 (I, II, or III) is a plot of the magnitude of the frequency response of this system, and what is the value of M in the plot you selected? Be sure to justify your selection and your computation of M.

(c) Suppose the input to this LTI system is

$$x[n] = e^{j\pi/6n}.$$

What is the value of $y[n]/x[n]$?